TINGLEY'S PROBLEM ON FUNCTION SPACES

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1. Introduction

Tingley's problem, introduced by Tingley in 1987 [32], asks whether every surjective isometry between the unit spheres of Banach spaces is extended to a surjective isometry between the whole spaces. Various solutions have been proposed for Tingley's problem. Wang suggested the first solution [33], who worked on the space of all continuous functions that vanish at infinity on a locally compact Hausdorff space (see [34]). Many interesting results have shown that Tingley's problem has been solved in specific spaces, and no counterexample has been found.

According to [36, p.730], Ding was the first to consider Tingley's problem in different types of spaces [13]. Ding [14, Corollary 2] proved that the real Banach space consisting of all null sequences of real numbers satisfies the Mazur-Ulam property. Liu [20] also contributed early to Tingley's problem on different types of spaces. Later, Cheng and Dong [8] formally introduced the concept of the Mazur-Ulam property.

Definition 1.1. A real or complex Banach space B has the *Mazur-Ulam property* if any surjective isometry from the unit sphere of B onto the unit sphere of another real or complex Banach space B' admits an extension to a surjective real-linear isometry from B onto B'.

In their studies, Tan [26, 27, 28] demonstrated that the space $L^p(\mathbb{R})$ for σ-finite positive measure space has the Mazur-Ulam property. Additionally, Boyko, Kadets, Martín, and Werner introduced two concepts, C-richness [4, Definition 2.3] and lushness [4, Definition 2.1], for subspaces of continuous functions. They also proved that a C-rich subspace is lush [4, Theorem 2.4]. In another study, Tan, Huang and Liu [29] introduced the notion of local GL (generalized lush) spaces and showed that every local GL space has the Mazur-Ulam property.

Tanaka [31] introduced a new direction in investigating Tingley's problem by presenting a positive solution for the Banach algebra of complex matrices. Mori and Ozawa [22] demonstrated that the Mazur-Ulam property holds for unital C^*-algebras and real von Neumann algebras. Cueto-Avellaneda and Peralta [11] proved that the complex (or real) Banach space of all continuous maps taking values in a complex (or real) Hilbert space has the Mazur-Ulam property (cf. [12]). The results of Becerra-Guerrero, Cueto-Avellaneda, Fernández-Polo, and Peralta [2] and Kalenda and Peralta [19] demonstrated that any JBW*-triple has the Mazur-Ulam property. Peralta and Švarc [25] extended the results of Mori and Ozawa [22] for unital JB*-algebras.

The Mazur-Ulam property for a Banach space of dimension 2 remained unresolved for many years. The final solution was presented by the remarkable and outstanding advance of Banakh [1], who proved that any Banach space of dimension 2 has the Mazur-Ulam property. The problem regarding a Banach space of finite dimension greater than 2 remains open. The study of the Mazur-Ulam property is currently a challenging subject (cf. [5, 35]). Jiménez-Vargas, Morales-Compoy, Peralta, and Ramírez [18, Theorems 3.8, 3.9] likely provided the first example of complex Banach spaces that have the complex Mazur-Ulam property (cf. [24]). Hatori [16] formally introduced the concept of the complex Mazur-Ulam property.

Definition 1.2. A complex Banach space B is said to have the *complex Mazur-Ulam property*, emphasizing the term 'complex', if for any surjective isometry from the unit sphere of B onto the unit sphere of another complex Banach space B' admits an extension to a surjective real-linear isometry from B onto B'

Note that a complex Banach space has the complex Mazur-Ulam property provided that it has the Mazur-Ulam property as a real Banach space since a complex Banach space is a real Banach space simultaneously.

In the paper [16], the complex Mazur-Ulam property for uniform algebras is proved. It is shown that the existence of a unit in a uniform algebra is crucial for the proof of this property. The problem of the complex Mazur-Ulam property for a uniformly closed algebra on a locally compact Hausdorff space is discussed in the same paper.

In a recent paper by Cueto-Avellaneda, Hirota, Miura and Peralta [10], it is demonstrated that every surjective isometry between the unit spheres of two uniformly closed algebras on locally compact Hausdorff spaces, which separate the points without common zeros, can be extended to a surjective real linear isometry between these algebras. Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [6] have recently proved the complex Mazur-Ulam property for a commutative JB*-triple. Both results concern the spaces of continuous functions without constants. Peralta [23] gives the first example of an infinite-dimensional non-commutative C^*-algebra containing no unitaries and with the Mazur-Ulam property.

In this paper, we further study the problem of the complex Mazur-Ulam property. We introduce a separation condition named $(**)$ for a Banach space in section 4. We prove that a real (resp. complex) Banach space, which satisfies the condition $(**)$, has the (resp. complex) Mazur-Ulam property. An extremely C-regular subspace satisfies the condition $(*)$. Fleming and Jamison introduced it cite[Definition 2.3.9]fj1, which is a generalization of an extremely regular subspace coined by Cengiz [9].

2. Notations and terminologies

Throughout this paper, we will use the following notations: B, B_1, and B_2 will always refer to real or complex Banach spaces. For a real or complex Banach space B the unit sphere $\{a \in B : \|a\| = 1\}$ of B is denoted by $S(B)$ and the closed unit ball $\{a \in B : \|a\| \leq 1\}$ by $\mathrm{Ball}(B)$. The set of all maximal convex subsets of $S(B)$ is denoted by $\mathfrak{F}_B$. We denote by $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$) the set of all real (resp. complex) numbers. We denote the open unit disk in $\mathbb{C}$ by D, the closed unit disk by $\bar{D}$, and $\mathbb{T} = \{z \in \mathbb{K} : |z| = 1\}$. Throughout the paper, Y denotes a locally compact Hausdorff space, and X is a compact Hausdorff space. The space of all $\mathbb{K}$-valued continuous functions on Y, which vanish at infinity, is denoted by $C_0(Y, \mathbb{K})$. If Y is compact, then we simply denote $C(Y, \mathbb{K})$ instead of $C_0(Y, \mathbb{K})$. The supremum norm on a subset W of Y is denoted by $\|\cdot\|_{\infty(W)}$ or $\|\cdot\|_\infty$. For a function $f \in C_0(Y, \mathbb{K})$ and $S \subset Y$, we denote the restriction of f on S by $f|S$. For $A \subset C_0(Y, \mathbb{K})$ and $S \subset Y$, we denote $A|S = \{f|S \colon f \in A\}$.

We will also use $T \colon S(B_1) \to S(B_2)$ to refer to a surjective isometry without assuming any linearity. We write $\mathbb{T} = \{\lambda \in \mathbb{K} \colon |\lambda| = 1\}$ for a $\mathbb{K}$-Banach space, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We say that the set of all extreme points of the unit ball of the dual space B^* of B by the Choquet boundary for B, that is,

$$\mathrm{Ch}(B) = \{p \in B^* \colon p \text{ is an exteme point of } \mathrm{Ball}(B)\}.$$

This notation may not be familiar, but we can assume that every Banach space can be viewed as a closed space of $C_0(\mathrm{Ball}(B^*) \setminus \{0\})$. This means that the set of all extreme points of the closed unit ball of the dual space can be considered the Choquet boundary for B. In the following, we suppose that a $\mathbb{K}$-Banach space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is a closed subspace of $C_0(\mathrm{Ball}(B^*) \setminus \{0\}, \mathbb{K})$.

3. The homogeneous extension is the unique candidate

The homogeneous extension $\widetilde{T}: B_1 \to B_2$ of T is ;

$$\widetilde{T}(f) = \begin{cases} \|f\| T\left(\frac{f}{\|f\|}\right), & f \neq 0 \\ 0, & f = 0 \end{cases}$$

$\widetilde{T}$ is a surjective function that preserves the norm. Thus, to solve Tingley's problem, we need to prove that $\widetilde{T}$ is additive; that is, $\widetilde{T}(f+g) = \widetilde{T}(f) + \widetilde{T}(g)$ for all $f, g \in B_1$. Any subset of $S(B_j)$, which is a singleton, is convex. By Zorn's lemma, we have a non-empty family $\mathfrak{F}_B$ consisting of all maximal convex subsets of $S(B_j)$ such that $\bigcup_{F \in \mathfrak{F}_B} F = S(B)$. Each $F \in \mathfrak{F}_B$ can be indexed by $\mathrm{Ch}(B) \times \mathbb{T}$, but it needs not to be unique.

Lemma 3.1. *For every $F \in \mathfrak{F}_B$, there exists a $(p, \lambda) \in \mathrm{Ch}(B) \times \mathbb{T}$ such that $F = \{f \in S(B) \colon f(p) = \lambda\}$.*

By the axiom of choice, there exists $P \subset \mathrm{Ch}(B)$ such that

$$\begin{array}{ccc} L_B : P \times \mathbb{T} & \longrightarrow & \mathfrak{F}_B \\ \Cup & & \Cup \\ (p, \lambda) & \longrightarrow & F_{p,\lambda} \end{array}$$

is a bijection. Here we denote $F_{p,\lambda} = \{f \in S(B) \colon f(p) = \lambda\}$. We call P a set of representatives for $\mathfrak{F}_B$. Note that a set of representatives does not need to be unique. A key gradient is

Theorem 3.2. *T preserves maximal convex subsets in both directions.*

$$F \in \mathfrak{F}_{B_1} \longleftrightarrow T(F) \in \mathfrak{F}_{B_2}$$

This is originally exhibited by Cheng and Dong [8]. But a crystal clear proof is given by Tanaka [30]. It induces the bijection

$$\mathbf{T} \colon \mathfrak{F}_{B_1} \to \mathfrak{F}_{B_2},$$

which defines the lift Φ of $T \colon S(B_1) \to S(B_2)$ by

$$\begin{array}{ccc} P_1 \times \mathbb{T} & \xrightarrow{\Phi} & P_2 \times \mathbb{T} \\ {\scriptstyle L_{B_1}} \downarrow & \circlearrowright & \downarrow {\scriptstyle L_{B_2}} \\ \mathfrak{F}_{B_1} & \xrightarrow[\mathbf{T}]{} & \mathfrak{F}_{B_2} \end{array}$$

Denoting

$$\Phi(p, \lambda) = (\phi(p, \lambda), \tau(p, \lambda)), \quad (p, \lambda) \in P_1 \times \mathbb{T},$$

we have

$$\mathbf{T}(F_{p,\lambda}) = F_{\phi(p,\lambda),\tau(p,\lambda)}, \quad (p,\lambda) \in P_1 \times \mathbb{T}.$$

Rewriting it, we arrive at an important equality

$$(T(f))(\phi(p,\lambda)) = \tau(p,\lambda), \quad f \in F_{p,\lambda}. \tag{3.1}$$

Let $p \in P_1$ be fixed. Suppose that we can prove :

[1] $\phi(p,\lambda)$ does not depend on λ : say $\phi(p,\lambda) = \phi(p)$,
[2] $\tau(p,\lambda) = \tau(p,1)\lambda$ or $\tau(p,1)\overline{\lambda}$.

The equation (3.1) is as

$$(T(f))(\phi(p)) = \tau(p,1)\lambda \text{ or } \tau(p,1)\overline{\lambda}, \quad f \in F_{p,\lambda}$$

for any $\lambda \in \mathbb{T}$. As $\lambda = f(p)$ for $f \in F_{p,\lambda}$ we infer that
(3.2)

$$(T(f))(\phi(p)) = \tau(p,1)f(p) \text{ or } \tau(p,1)\overline{f(p)},\ f \in S(B_1) \text{ with } |f(p)| = 1$$

under the conditions

[1] $\phi(p,\lambda)$ does not depend on λ : say $\phi(p,\lambda) = \phi(p)$,
[2] $\tau(p,\lambda) = \tau(p,1)\lambda$ or $\tau(p,1)\overline{\lambda}$.

Suppose that (3.2) holds for any $f \in S(B_1)$. We will arrive at the final destination. Then, under the conditions

[1] $\phi(p,\lambda)$ does not depend on λ : say $\phi(p,\lambda) = \phi(p)$,
[2] $\tau(p,\lambda) = \tau(p,1)\lambda$ or $\tau(p,1)\overline{\lambda}$
[3] $(T(f))(\phi(p)) = \tau(p,1)f(p)$ or $\tau(p,1)\overline{f(p)}, \quad \forall f \in S(B_1)$

we infer that

$$(\widetilde{T}(f))(\phi(p)) = \tau(p,1)f(p) \text{ or } \tau(p,1)\overline{f(p)} \quad \forall f \in B_1 \tag{3.3}$$

Recall that the homogeneous extension $\widetilde{T} : B_1 \to B_2$ is defined by

$$\widetilde{T}(f) = \begin{cases} \|f\| T\left(\frac{f}{\|f\|}\right), & f \neq 0, \\ 0, & f = 0. \end{cases}$$

By (3.3) we have for $f, g \in B_1$

$$\begin{aligned} (\widetilde{T}(f+g))(\phi(p)) &= \tau(p,1)(f+g)(p) \text{ or } \tau(p,1)\overline{(f+g)}(p) \\ &= \tau(p,1)(f(p)+g(p)) \text{ or } \tau(p,1)\overline{f(p)+g(p)} \\ &= (\widetilde{T}(f))(\phi(p)) + (\widetilde{T}(g))(\phi(p)) \\ &= (\widetilde{T}(f) + \widetilde{T}(g))(\phi(p)). \end{aligned}$$

If the set of all $p \in P_1$, which satisfies the above equality, is a norming family for B_1, we have

$$\widetilde{T}(f+g) = \widetilde{T}(f) + \widetilde{T}(g), \quad f, g \in B_1. \tag{3.4}$$

We arrive at the final destination. We conclude that

Proposition 3.3. *Suppose that the set of all $p \in P_1$ which satisfies*

[1] $\phi(p,\lambda)$ *does not depend on* λ *: say* $\phi(p,\lambda) = \phi(p)$*,*

[2] $\tau(p,\lambda) = \tau(p,1)\lambda$ *or* $\tau(p,1)\overline{\lambda}$*,*

[3] $(T(f))(\phi(p)) = \tau(p,1)f(p)$ *or* $\tau(p,1)\overline{f(p)}$*,* $\quad \forall f \in S(B_1)$

is a norming family of B_1*. Then, the homogeneous extension* $\widetilde{T}$ *of* T *is additive, which means that* B_1 *has the Mazur-Ulam property.*

4. Looking forward a neat, sufficient condition for [1], [2] and [3]

In [17], we consider the condition $(*)$.

Definition 4.1 (Definition 5.1 in [17])**.** We say that B satisfies the condition $(*)$ whenever there exists a set of representative P for $\mathfrak{F}_B$ with the condition : for every $p \in P$,$\varepsilon > 0$, and a closed subset F of P with respect to the relative topology induced by the weak*-topology on B^* such that $p \notin F$, there exists $a \in S(B)$ such that $p(a) = 1$ and $|q(a)| \leq \varepsilon$ for all $q \in F$.

Fleming and Jamison [15, Definition 2.3.9] introduced the extremely C-regular space, which is a generalization of an extremely regular space defined by Cengiz [9]. We denote the supremum norm of $f \in C_0(Y,\mathbb{K})$ by $\|f\|_\infty$.

Definition 4.2 (Definition 2.3.9 in [15], [9])**.** A $\mathbb{K}$-linear subspace E of $C_0(Y,\mathbb{K})$ for a locally compact Hausdorff space Y is called an extremely C-regular space (resp. regular) if for each $x \in \mathrm{Ch}(E)$ (resp. $x \in Y$) satisfies the condition that for each $\varepsilon > 0$ and each open neighborhood U of x, there exists $f \in E$ such that $f(x) = 1 = \|f\|_\infty$, and $|f| < \varepsilon$ on $Y \setminus U$.

Note that for a uniformly closed extremely C-regular $\mathbb{K}$-linear subspace E of $C_0(Y,\mathbb{K})$ which separates the point of Y, $x \in \mathrm{Ch}(E)$ if and only if x is a strong boundary point for E if and only if the representing measure for the point evaluation τ_x for x on E is only the Dirac measure at x (see [17, Theorem 3.11]). Note also that a uniformly closed subalgebra of $C_0(Y,\mathbb{K})$ which separates the point of Y is extremely C-regular.

Example 4.3 (Example 5.2 in [17])**.** Suppose that E is a uniformly closed C-regular $\mathbb{K}$-linear subspace of $C_0(Y,\mathbb{K})$ for a locally compact Hausdorff space Y which separates the point of Y. Suppose that $P = \{\tau_x \colon x \in \mathrm{Ch}(E)\}$. Then, P is a set of representatives, and the condition

$(*)$ holds with P. Thus a uniformly closed subalgebra of $C_0(Y, \mathbb{K})$ which separates the point of Y satisfies the condition $(*)$.

Theorem 4.4 (Theorem 6.3 in [17])**.** *Suppose that a real (resp. complex) Banach space satisfies the condition $(*)$. Then B has the (resp. complex) Mazur-Ulam property.*

We have the following from Theorem 4.4. Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [6] studies the complex Mazur-Ulam property for commutative JB*-triples. In particular, [6, Corollary 3.2] exhibits the same result for $C_0(Y, \mathbb{C})$. Recall that a uniform algebra A on a compact Hausdorff space X is a closed subalgebra of $C(X, \mathbb{C})$ which separates the points of X and contains constants.

Corollary 4.5 (Corollaries 6.4 in [17])**.** *A uniformly closed extremely C-regular real (resp. complex) space has the (resp. complex) Mazur-Ulam property. In particular, a closed subalgebra of $C_0(Y, \mathbb{C})$ and a uniform algebra have the complex Mazur-Ulam property.*

Suppose that $P(\bar{D})$ be a disk algebra on the closed unit disk $\bar{D}$ in the complex plane; i.e.,

$$P(\bar{D}) = \{f \in C(\bar{D}, \mathbb{C}) \colon f \text{ is analytic on the open unit disk } D\}.$$

Then $\overline{\operatorname{Re} P(\bar{D})}$ is a space of harmonic function on D. As the unit circle $\mathbb{T}$ is the Šilov boundary for $\overline{\operatorname{Re} P(\bar{D})}$ and $\overline{\operatorname{Re} P(\bar{D})}|\mathbb{T} = C(\mathbb{T}, \mathbb{R})$, we infer that $\overline{\operatorname{Re} P(\bar{D})}$ has the Mazur-Ulam property. In general, as a result, we see the following.

Corollary 4.6. *Let A be a uniform algebra. Then, the uniform closure $\overline{\operatorname{Re} A}$ of the real part $\operatorname{Re} A$ of A has the Mazur-Ulam property.*

For a compact subset K in the complex place, $R(K)$ (resp. $A(K)$) denotes the uniform algebra, which consists of complex-valued continuous functions that are uniformly approximated on K by rational functions with poles off K (resp. analytic on the interior of K). By Corollary 4.6, the space of harmonic functions on the interior of K, $\overline{\operatorname{Re} R(K)}$ and $\overline{\operatorname{Re} A(K)}$ have the Mazur-Ulam property. Inspired by Corollary 4.6, we may consider the following problem about the Mazur-Ulam property for space of harmonic functions.

Problem 4.7. *Let K be a non-empty compact subset of the complex plane. Let*

(i) $H_0(K) = \{u \in C(K, \mathbb{R}) \colon u \textit{ is harmonic on the interior of } K\}$,
(ii) $H(K) = \{u \in C(K, \mathbb{R}) \colon u \textit{ is uniformly approximated on } K$ *by harmonic functions on open sets which include* $K\}$.

Do $H_0(K)$ and $H(K)$ have the Mazur-Ulam property?

Note that

$$\overline{\mathrm{Re}}P(\bar{D}) = H_0(\bar{D}) = H(\bar{D}).$$

It is known that $H_0(K)$ nor $H(K)$ do not satisfy the condition $(*)$. We version up the condition $(*)$. In general, it is known that if every boundary point of K is a peak point for $R(K)$, then $H(K)|\partial K = C(\partial K, \mathbb{R})$, where ∂K denotes the boundary of K. In this case, $H(K)$, hence and $H_0(K)$ have the Mazur-Ulam property. In particular, if $\mathbb{C}\setminus K$ has a finite number of components, then every boundary point of K is a peak point; hence, $R(K)$ has the Mazur-Ulam property. The necessary and sufficient condition for K such that every boundary point of K is a peak point for $R(K)$ is known as a theorem of Melnikov (see. [37]).

Definition 4.8. Suppose that P is a set of representatives for $\mathfrak{F}_B$. Let $Q \subset P$. We say that $p \in Q$ is a strong boundary point for Q in the sense of Fleming and Jamison if the following holds: for any open neighborhood U of p and positive ε, there exists $f \in S(B)$ such that $f(p) = 1$, $|f| < \varepsilon$ on $Q \setminus U$.

Definition 4.9. B satisfies the condition $(**)$ if and only if there exists $\emptyset \neq Q \subset P$ such that every $p \in Q$ is a strong boundary point for Q in the sense of Fleming and Jamison, and Q is a norming family for B; i.e. $\|f\| = \sup_{p\in Q} |f(p)|$ for every $f \in B$.

In general, if B satisfies the condition $(*)$, then it satisfies the condition $(**)$ with $Q = P$. A closed subalgebra of $C_0(Y)$, which separates the points of Y, particularly a uniform algebra, is an extremely C-regular space. Hence, it satisfies $(**)$. A space of harmonic functions

$$\{u \in C(K, \mathbb{R}) \colon u \text{ is harmonic in the interior of } K\}$$

for a plane comapcta K need not be an extremely C-regular space, but satisfies $(**)$.

Theorem 4.10. *Suppose that a real (resp. complex) Banach space satisfies the condition $(**)$. Then B has the (resp. complex) Mazur-Ulam property.*

We can prove Theorem 4.10 in a similar way as Theorem 4.4. We omit proof. It is known that the set of all peak points for $R(K)$ is dense in the boundary ∂K of K. Hence $H(K)$ and $H_0(K)$ satisfies the condition $(**)$. We have

Corollary 4.11. *The spaces $H(K)$ and $H_0(K)$ have the Mazur-Ulam property.*

5. Appendix

In this section, A denotes a uniform algebra on a compact Hausdorff space X. We present an idea for proving that a uniform algebra has the complex Mazur-Ulam property. We focus on the Hausdorff distance between the maximal convex sets. The Hausdorff distance $d_H(\cdot,\cdot)$ between these sets is crucial for proving [1] and [2]. Recall that

[1] $\phi(p,\lambda)$ does not depend on λ : say $\phi(p,\lambda)=\phi(p)$,

[2] $\tau(p,\lambda)=\tau(p,1)\lambda$ or $\tau(p,1)\overline{\lambda}$

The Hausdorff distance $d_H(\cdot,\cdot)$ between $F_{p,\lambda}$ and $F_{q,\mu}$ in $\mathfrak{F}_B$ is

$$d_H(F_{p,\lambda},F_{q,\mu})=\max\{\sup_{f\in F_{p,\lambda}}(\inf_{h\in F_{q,\mu}}\|f-h\|),\sup_{h\in F_{q,\mu}}(\inf_{f\in F_{p,\lambda}}\|h-f\|)\}$$

One can prove that any two-point-subset $\{p,q\}\subset \mathrm{Ch}(A)$ is a peak interpolation set for A. In particular, for any $\lambda,\mu\in\mathbb{T}$ there exists a function $f\in S(A)$ such that $f(p)=\lambda$ and $f(q)=\mu$. It follows that

$$d_H(F_{p,\lambda},F_{q,\mu})=2,\quad \lambda,\mu\in\mathbb{T} \tag{5.1}$$

for every pair of different points $p,q\in\mathrm{Ch}(A)$ Through a simple calculation

$$d_H(F_{p,\lambda},F_{p,\lambda'})=|\lambda-\lambda'|,\quad p\in P,\lambda,\lambda'\in\mathbb{T},$$

with which (5.1) ensures [1]. Suppose that [1] does not hold: There exists (p,λ) and (p,λ') such that $\phi(p,\lambda)\neq\phi(p,\lambda')$. We may assume that $|\lambda-\lambda'|<2$. By (5.1)

$$\sharp\{(q,\mu)\in P_1\times\mathbb{T}\colon d_H(F_{p,\lambda},F_{q,\mu})=|\lambda-\lambda'|\}=2,$$

say, $(q,\mu)=(p,\lambda')$, (p,μ) where μ is the symmetric point of λ' with respect to λ.

On the other hand, since $\mathbf{T}$ preserves the Hausdorff distance

$$\sharp\{(q,\mu)\in P_2\times\mathbb{T}\colon d_H(F_{\phi(p,\lambda),\tau(p,\lambda)},F_{q,\mu})=|\lambda-\lambda'|\}\geq 3.$$

In fact, at least $(q,\mu)=(\phi(p,\lambda'),\tau(p,\lambda'))$, $(\phi(p,\lambda),\lambda\overline{\lambda'}\tau(p,\lambda))$, and $(\phi(p,\lambda),\lambda''\tau(p,\lambda))$, where λ'' is the simmetric point of $\lambda\overline{\lambda'}\tau(p,\lambda)$ with respect to $\tau(p,\lambda)$. This is a contradiction because $\mathbb{T}$ preserves the Hausdorff distance. Furthermore

Proposition 5.1. *We have*

$$[1]\ \phi(p,\lambda)=\phi(p,\lambda') \textit{ for every } p\in P_1 \textit{ and } \lambda,\lambda'\in\mathbb{T}.$$

Letting

$$P_1^+=\{p\in P_1:\tau(p,i)=i\tau(p,1)\}$$

and

$$P_1^-=\{p\in P_1\tau(p,i)=-i\tau(p,1)\},$$

we have $P_1^+\cup P_1^-=P_1$ *and*

$$[2] \quad \begin{cases} \tau(p,\lambda) = \lambda\tau(p,1), & p \in P_1^+, \lambda \in \mathbb{T} \\ \tau(p,\lambda) = \overline{\lambda}\tau(p,1), & p \in P_1^-, \lambda \in \mathbb{T}. \end{cases}$$

We obtain a so-called the additive Bishop lemma, and we conclude that

$$\begin{aligned} &\{f \in S(A)\colon f(p) = \alpha\} \\ &\quad = \left\{ f \in S(A) : d\left(f, F_{p,\frac{\alpha}{|\alpha|}}\right) \le 1 - |\alpha|, d\left(f, F_{p,\frac{-\alpha}{|\alpha|}}\right) \le 1 + |\alpha| \right\}. \end{aligned}$$

for every $|\alpha| \le 1$, where

$$d(f, F_{p,\lambda}) = \inf\{\|f - g\|\colon g \in F_{p,\lambda}\}.$$

Then we have [3]. It follows by Proposition 3.3 that a uniform algebra has the Mazur-Ulam property.

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