Large deviation behavior of estimators for flattened distributions in a middle part

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Abstract

From the viewpoint of large deviation, the Bahadur efficiency based on the information inequality for the tail probability of estimators is well known in the asymptotic theory of estimation. On the other hand, the large deviation efficiency up to the second order was introduced in Akahira (2006, 2010) from a different viewpoint from the Bahadur efficiency. In this article, from the latter viewpoint, the lower bound for the large deviation probability for asymptotically median unbiased estimators is obtained for flattened distributions in a middle part, which do not belong to an exponential family. The influence of the flat part of distributions on the bound with its relation to the large deviation efficiency is investigated.

1 Introduction

The asymptotic efficiency of estimators up to the higher order has been studied under suitable regularity conditions from the viewpoint of the concentration probability around the true parameter (see e.g., Akahira and Takeuchi 1981, 2003, Ghosh 1994, Pfanzagl and Wefelmeyer 1985). Indeed, in such a case the Edgeworth expansion for estimators is useful, and the bias-adjusted maximum likelihood estimator (MLE) is shown to be third order asymptotically efficient in some class of estimators under the regularity conditions.

On the other hand, the large deviation efficiency of estimators is also considered up to the second order (see Akahira, 2006, 2010). In such a case the saddlepoint approximation plays an important role, and the MLE is shown to be second order large deviation efficient for an exponential family of distributions. The spirits are near to the above higher order asymptotics.

Historically, the Bahadur efficiency is well known as a concept considered from the viewpoint of large deviation. Indeed, for any consistent estimator $\hat{\theta}_n$ of an unknown real-valued parameter

θ and any $\varepsilon > 0$, the tail probability

$$\alpha\left(\hat{\theta}_n, \theta, \varepsilon\right) := P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > \varepsilon\right\}$$

tends to zero as $n \to \infty$. Under suitable conditions it is shown that the rate of convergence is exponential and has an asymptotic expansion of the form

$$\alpha\left(\hat{\theta}_n, \theta, \varepsilon\right) = \mathrm{e}^{-n\beta\left(\hat{\theta}_n, \theta, \varepsilon\right)}\left(c_0 + \frac{c_1}{n} + \cdots\right),$$

where $\beta\left(\hat{\theta}_n, \theta, \varepsilon\right)$ is positive and c_i's are constant. Here the constant $\beta\left(\hat{\theta}_n, \theta, \varepsilon\right)$ is called an exponential rate. Bahadur (1971) gave the upper bound for the exponential rate of consistent estimators by use of the amount of Kullback-Leibler (K-L) information and showed that the MLE attained the bound in the Laplace distribution case, which was called the Bahadur efficiency of the MLE (see also Fu, 1973). Using the asymptotic expansion of the amount of K-L information, the Bahadur type second order efficiency is also considered by Fu (1982).

In this article, from a different viewpoint from the Bahadur efficiency we give the lower bound for the large deviation probability of asymptotically median unbiased estimators for flattened normal and Laplace distributions in a middle part. We also investigate how the bound are affected by the flat part of each distribution and consider the large deviation efficiency of estimators.

2 Definitions and the lower bound for the large deviation probability for estimators

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables with a probability density function (p.d.f.) $f(x;\theta)$ with respect to a σ-finite measure, where $\theta \in \Theta$ and Θ is an open interval in $\mathbf{R}^1$. Put $\boldsymbol{X} = (X_1, \ldots, X_n)$. If an estimator $\hat{\theta} = \hat{\theta}(\boldsymbol{X})$ of θ satisfies

$$P_{\theta,n}\left\{\hat{\theta}_n \le \theta\right\} = \frac{1}{2} + o(1), \quad P_{\theta,n}\left\{\hat{\theta}_n \ge \theta\right\} = \frac{1}{2} + o(1)$$

as $n \to \infty$, then $\hat{\theta}_n$ is said to be weakly asymptotically median unbiased (wAMU) for θ. Let $\mathbb{A}$ be a class of the all wAMU estimators of θ.

Definition If there exists $\hat{\theta}^*_n = \hat{\theta}^*_n(\boldsymbol{X})$ in $\mathbb{A}$ such that for any $\hat{\theta}_n \in \mathbb{A}$, any $\theta \in \Theta$ and any $a > 0$

$$\begin{aligned} P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\} \ge& P_{\theta,n}\left\{\left|\hat{\theta}^*_n - \theta\right| > a\right\}\{1 + o(1)\} \\ =& B_n(a,\theta)\{1 + o(1)\} \quad \text{(say)} \end{aligned}$$

as $n \to \infty$, then $\hat{\theta}_n^*$ is said to be (first order) large deviation efficient (LDE). If there exists $\tilde{\theta}_n^* = \tilde{\theta}_n^*(\boldsymbol{X})$ in $\mathbb{A}$ such that

$$\lim_{a\to 0}\lim_{n\to\infty} \frac{P_{\theta,n}\left\{\left|\tilde{\theta}_n^* - \theta\right| > a\right\}}{B_n(a,\theta)} = 1,$$

then $\tilde{\theta}_n^*$ is said to be (first order) weakly large deviation efficient (wLDE).

It is clear that the LDE estimator is wLDE. Second order weakly large deviation efficiency can be also defined in a similar way to Akahira (2006, 2010). In order to get the lower bound $B_n(a,\theta)$, we take the following approach introduced by Akahira (2006, 2010). First, we assume that $\{x \mid f(x;\theta) > 0\}$ is independent of θ. Let θ_0 be any fixed in Θ and $a > 0$. Then we consider a problem of testing the hypothesis H: $\theta = \theta_0 + a$ against the alternative K: $\theta = \theta_0$, where $\theta_0 + a \in \Theta$. Let $\phi^*(\boldsymbol{X})$ be the most powerful (MP) test of level $1/2 + o(1)$ as $n \to \infty$ and $\hat{\theta}_n$ be any wAMU estimator. Putting

$$A_{\hat{\theta}_n} := \left\{\boldsymbol{x} \mid \hat{\theta}_n(\boldsymbol{x}) \le \theta_0 + a\right\},$$

we see that the indicator $\chi_{A_{\hat{\theta}_n}}(\boldsymbol{x})$ of $A_{\hat{\theta}_n}$ is a test of the level $1/2 + o(1)$ as $n \to \infty$, because of $\hat{\theta}_n \in \mathbb{A}$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$. Since

$$E_{\theta_0}(\phi^*) \ge E_{\theta_0}\left[\chi_{A_{\hat{\theta}_n}}\right] = P_{\theta_0,n}\left\{\hat{\theta}_n \le \theta_0 + a\right\},$$

it follows that

$$P_{\theta_0,n}\left\{\hat{\theta}_n - \theta_0 > a\right\} \ge 1 - E_{\theta_0}(\phi^*) \tag{2.1}$$

for large n. In order to obtain the lower bound, i.e., the right-hand side of (2.1), we need the power function of the MP test ϕ^*. Now, it is seen from the fundamental lemma of Neyman-Pearson that a test with rejection region of the form

$$\bar{Z}(\theta_0, a) := \frac{1}{n}\sum_{i=1}^{n} Z_i(\theta_0, a) > c \tag{2.2}$$

is MP, where

$$Z_i(\theta, a) := \log\left(f(X_i;\theta)/f(X_i;\theta+a)\right) \quad (i = 1, \ldots, n)$$

and c is a constant chosen such that the asymptotic level of the test is $1/2 + o(1)$ as $n \to \infty$. Note that $Z_1(\theta_0, a), \ldots, Z_n(\theta_0, a)$ are i.i.d. Letting $\mu(\theta_0, a) = E_{\theta_0+a}\left[Z_1(\theta_0, a)\right]$ and $\sigma^2(\theta_0, a) = V_{\theta_0+a}\left[Z_1(\theta_0, a)\right]$, we put

$$W_n := \frac{\sqrt{n}}{\sigma(\theta_0, a)}\left\{\bar{Z}(\theta_0, a) - \mu(\theta_0, a)\right\},$$

where $\sigma(\theta_0, a) = \sqrt{\sigma^2(\theta_0, a)}$. Since the MP test with the rejection region (2.2) is of the asymptotic level $1/2 + o(1)$, i.e.,

$$P_{\theta_0+a,n}\left\{W_n \le \frac{\sqrt{n}}{\sigma(\theta_0, a)}(c - \mu(\theta_0, a))\right\} = \frac{1}{2} + o(1)$$

as $n \to \infty$, by the central limit theorem we choose $\mu(\theta_0, a)$ as c. Since

$$E_{\theta_0}(\phi^*) = P_{\theta_0,n}\left\{\bar{Z}(\theta_0, a) > \mu(\theta_0, a)\right\},$$

it follows from (2.1) that

$$P_{\theta_0,n}\left\{\hat{\theta}_n - \theta_0 > a\right\} \ge P_{\theta_0,n}\left\{\bar{Z}(\theta_0, a) \le \mu(\theta_0, a)\right\} \tag{2.3}$$

for large n. In order to obtain the asymptotic expansion of the lower tail probability of $\bar{Z}(\theta_0, a)$ in (2.3), we use the saddlepoint approximation (see Jensen 1995, Barndorff-Nielsen and Cox 1989 and also Appendix mentioned later). Let

$$M_{Z_1(\theta_0,a),\theta_0}(t) := E_{\theta_0}\left[\exp\{tZ_1(\theta_0, a)\}\right], \quad K_{Z_1(\theta_0,a),\theta_0}(t) := \log M_{Z_1(\theta_0,a),\theta_0}(t) \tag{2.4}$$

for all t in some open interval involving the origin, that is, they are the moment generating function (m.g.f.) and the cumulant generating function (c.g.f.) of $Z_1(\theta_0, a)$, respectively. Let $\hat{t}(\theta_0, a)$ be a solution of t of the equation

$$K'_{Z_1(\theta_0,a),\theta_0}(t) = \mu(\theta_0, a). \tag{2.5}$$

Since

$$\begin{aligned} M_{Z_1(\theta_0,a),\theta_0+a}(t) =& E_{\theta_0+a}\left[\exp\left\{tZ_1(\theta_0, a)\right\}\right] = E_{\theta_0}\left[\exp\left\{(t-1)Z_1(\theta_0, a)\right\}\right] \\ =& M_{Z_1(\theta_0,a),\theta_0}(t-1), \end{aligned} \tag{2.6}$$

it follows from (2.4) that

$$K_{Z_1(\theta_0,a),\theta_0+a}(t) = K_{Z_1(\theta_0,a),\theta_0}(t-1), \tag{2.7}$$

which implies that

$$\mu(\theta_0, a) = E_{\theta_0+a}\left[Z_1(\theta_0, a)\right] = K'_{Z_1(\theta_0,a),\theta_0+a}(0) = K'_{Z_1(\theta_0,a),\theta_0}(-1).$$

From (2.5) we have $t = \hat{t}(\theta_0, a) = -1$, since $K''_{Z_1(\theta_0,a),\theta_0}(t) > 0$. It is noted that

$$\mu(\theta_0, a) = -E_{\theta_0+a}\left[\log\frac{f(X_1; \theta_0 + a)}{f(X_1; \theta_0)}\right] = -I^{KL}(\theta_0 + a, \theta_0) \quad \text{(say)}, \tag{2.8}$$

where I^{KL} is called the Kullback-Leibler information amount. Note that $\mu(\theta_0, a) < 0$. From (2.7) we have $K''_{Z(\theta_0,a),\theta_0}(-1) = \sigma^2(\theta_0, a)$.

In a similar way to the above, we can derive a lower bound for the tail probability $P_{\theta,n}\left\{\hat{\theta}_n - \theta < -a\right\}$ for any $\hat{\theta}_n \in \mathbb{A}$. Indeed, we consider the indicator $\chi_{\tilde{A}_{\hat{\theta}_n}}$ with $\tilde{A}_{\hat{\theta}_n} = \{\boldsymbol{x} \mid \hat{\theta}(\boldsymbol{x}) \geq \theta_0 - a\}$ as a test of the level $1/2 + o(1)$ in the problem of testing hypothesis H: $\theta = \theta_0 - a$ against the alternative K: $\theta = \theta_0$. Then we have for large n

$$P_{\theta_0,n}\left\{\hat{\theta}_n - \theta_0 < -a\right\} \geq 1 - E_{\theta_0}\left(\tilde{\phi}^*\right),$$

where $\tilde{\phi}^*$ is the MP test of the level $1/2 + o(1)$, which implies

$$P_{\theta_0,n}\left\{\hat{\theta}_n - \theta_0 < -a\right\} \geq 1 - P_{\theta_0,n}\left\{\bar{Z}(\theta_0, -a) > \mu(\theta_0, -a)\right\} = P_{\theta_0,n}\left\{\bar{Z}(\theta_0, -a) \leq \mu(\theta_0, -a)\right\}. \tag{2.9}$$

Since θ_0 is arbitrary, from (2.3) and (2.9) and the saddlepoint approximation we have the following results as θ instead of θ_0.

Theorem For any wAMU estimator $\hat{\theta}_n$ of $\theta(\in \Theta)$ and any $a(> 0)$ with $\theta + a \in \Theta$, it holds that for large n

$$P_{\theta,n}\left\{\hat{\theta}_n - \theta > a\right\} \geq \frac{1}{\lambda}\mathrm{e}^{n\mu(\theta,a)}\left\{B_0(\lambda) + O\left(\frac{1}{n}\right)\right\} \tag{2.10}$$

where $\lambda = \lambda(\theta, a) = \sqrt{n}\sigma(\theta, a)$ and

$$B_0(\lambda) = \lambda \mathrm{e}^{\lambda^2/2}\left\{1 - \Phi(\lambda)\right\}. \tag{2.11}$$

The proof follows from (2.3) and (6.2) in Appendix, since $M_{Z_1(\theta,a),\theta}(-1) = 1$ and $K'_{Z_1(\theta,a),\theta}(0) = E_\theta[Z_1(\theta, a)] = I^{KL}(\theta, \theta + a) > 0$ by (2.4), (2.6) and (2.8). For the case $a > 0$ with $\theta - a \in \Theta$, from (2.9) and (6.2) in Appendix, we also obtain a similar lower bound to (2.10) for the tail probability $P_{\theta,n}\left\{\hat{\theta}_n - \theta < -a\right\}$.

Corollary For any wAMU estimator $\hat{\theta}_n$ of $\theta(\in \Theta)$ and any $a(> 0)$ with $\theta + a \in \Theta$, it holds that for large n

$$\frac{P_{\theta,n}\left\{\hat{\theta}_n - \theta > a\right\}}{\mathrm{e}^{n\mu(\theta,a)}\Big/\sqrt{2\pi n\sigma^2(\theta, a)}} \geq 1 + O\left(\frac{1}{n}\right). \tag{2.12}$$

The proof is straightforward from (2.10) and (2.11), since for large n

$$\frac{1 - \Phi(\lambda)}{\phi(\lambda)} = \frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right) \tag{2.13}$$

by Mills' ratio, where $\Phi(\lambda) = \int_{-\infty}^{\lambda} \phi(t) \mathrm{d}t$ with $\phi(t) = \left(1/\sqrt{2\pi}\right) \mathrm{e}^{-t^2/2}$.

Remark For any wAMU estimator $\hat{\theta}_n$ of $\theta(\in \Theta)$ and any $a(> 0)$ with $\theta - a \in \Theta$ it holds that for large n

$$\frac{P_{\theta,n}\left\{\hat{\theta}_n - \theta < -a\right\}}{\mathrm{e}^{n\mu(\theta,-a)} \Big/ \sqrt{2\pi n \sigma^2(\theta,-a)}} \geq 1 + O\left(\frac{1}{n}\right). \tag{2.14}$$

Let $\mu = \mu(\theta, a)$, $\tilde{\mu} = \mu(\theta, -a)$, $\sigma = \sqrt{\sigma^2(\theta, a)}$ and $\tilde{\sigma} = \sqrt{\sigma^2(\theta, -a)}$. For the large deviation probability $P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\}$ for $\hat{\theta}_n$, it follows from (2.12) and (2.14) that for any $a(> 0)$ with $\theta \pm a \in \Theta$

$$\frac{P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\}}{\frac{1}{\sqrt{2\pi n}}\left(\frac{1}{\sigma}\mathrm{e}^{n\mu} + \frac{1}{\tilde{\sigma}}\mathrm{e}^{n\tilde{\mu}}\right)} \geq 1 + O\left(\frac{1}{n}\right), \tag{2.15}$$

for any $a(> 0)$ satisfying $\theta + a \in \Theta$ and $\theta - a \notin \Theta$

$$\frac{P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\}}{\mathrm{e}^{n\mu} \Big/ \left(\sqrt{2\pi n}\sigma\right)} \geq 1 + O\left(\frac{1}{n}\right),$$

and for any $a(> 0)$ satisfying $\theta + a \notin \Theta$ and $\theta - a \in \Theta$

$$\frac{P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\}}{\mathrm{e}^{n\tilde{\mu}} \Big/ \left(\sqrt{2\pi n}\tilde{\sigma}\right)} \geq 1 + O\left(\frac{1}{n}\right),$$

for large n. It is noted that for any wAMU estimator $\hat{\theta}_n$ of $\theta(\in \Theta)$ and any $a(> 0)$ with $\theta \pm a \notin \Theta$, $P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\} \geq 0$. Here, (2.15) is also represented as

$$P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\} \geq \frac{1}{\sqrt{2\pi n}}\left(\frac{1}{\sigma}\mathrm{e}^{n\mu} + \frac{1}{\tilde{\sigma}}\mathrm{e}^{n\tilde{\mu}}\right)\left(1 + O\left(\frac{1}{n}\right)\right) \tag{2.16}$$

for large n and the right-hand side of (2.16) is the lower bound for the large deviation probability for wAMU estimators.

3 Flattened normal distribution in a middle part

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables with a p.d.f. $f_\varepsilon(x-\theta)$ with respect to the Lebesgue measure, where $x \in \mathbf{R}^1$ and $\theta \in \Theta = \mathbf{R}^1$. Let ε be any fixed positive number. Further we assume that

$$f_\varepsilon(x) = \begin{cases} c_\varepsilon \phi(\varepsilon) & \text{for } |x| \leq \varepsilon, \\ c_\varepsilon \phi(x) & \text{for } |x| > \varepsilon, \end{cases} \tag{3.1}$$

where $\phi(x) = \left(1/\sqrt{2\pi}\right)e^{-x^2/2}$ for $x \in \mathbf{R}^1$ with

$$c_\varepsilon = \frac{1}{2\left(\varepsilon\phi(\varepsilon) + 1 - \Phi(\varepsilon)\right)}.$$

A distribution with the p.d.f. of (3.1) is called a flattened normal distribution $\mathrm{fN}[-\varepsilon, \varepsilon]$.

Note that the distribution with the p.d.f. $f_\varepsilon(\cdot)$ does not belong to an exponential family of distributions. Put

$$z(\theta, a) = \log \frac{f_\varepsilon(x-\theta)}{f_\varepsilon(x-\theta-a)}. \tag{3.2}$$

Let $0 < a < 2\varepsilon$. From (3.1) we have

$$z(\theta, a) = \begin{cases} -a(x-\theta) + \frac{a^2}{2} & \text{for } x \le \theta - \varepsilon, \\ -\frac{\varepsilon^2}{2} + \frac{1}{2}(x-\theta-a)^2 & \text{for } \theta - \varepsilon < x \le \theta + a - \varepsilon, \\ 0 & \text{for } \theta + a - \varepsilon < x \le \theta + \varepsilon, \\ -\frac{1}{2}(x-\theta)^2 + \frac{\varepsilon^2}{2} & \text{for } \theta + \varepsilon < x \le \theta + a + \varepsilon, \\ -a(x-\theta) + \frac{a^2}{2} & \text{for } \theta + a + \varepsilon < x. \end{cases} \tag{3.3}$$

Let $Z_i(\theta, a) = \log\left(f_\varepsilon(X_i - \theta)/f_\varepsilon(X_i - \theta - a)\right)$ $(i = 1, 2, \ldots)$. From (3.3) we obtain for small $a > 0$

$$\begin{aligned} &\mu(\theta, a) \\ =&E_{\theta+a}\left[Z_1(\theta, a)\right] \\ =&c_\varepsilon \int_{-\infty}^{\theta-\varepsilon} \left\{-a(x-\theta) + \frac{a^2}{2}\right\} \phi(x-\theta-a)\mathrm{d}x \\ &+ c_\varepsilon \int_{\theta-\varepsilon}^{\theta+a-\varepsilon} \left\{-\frac{\varepsilon^2}{2} + \frac{1}{2}(x-\theta-a)^2\right\} \phi(x-\theta-a)\mathrm{d}x \\ &+ c_\varepsilon\phi(\varepsilon) \int_{\theta+\varepsilon}^{\theta+a+\varepsilon} \left\{-\frac{1}{2}(x-\theta)^2 + \frac{\varepsilon^2}{2}\right\} \mathrm{d}x + c_\varepsilon \int_{\theta+a+\varepsilon}^{\infty} \left\{-a(x-\theta) + \frac{a^2}{2}\right\} \phi(x-\theta-a)\mathrm{d}x \\ =&-\frac{a^2}{2} + O\left(a^3\right). \end{aligned} \tag{3.4}$$

We also have for small $a > 0$

$$E_{\theta+a}\left[Z_1^2(\theta, a)\right] = a^2 + O\left(a^3\right),$$

which yields

$$\sigma^2(\theta, a) = V_{\theta+a}\left(Z_1(\theta, a)\right) = a^2 + O\left(a^3\right). \tag{3.5}$$

Then it follows from (3.4), (3.5) and Corollary that for any $\hat{\theta}_n \in \mathbb{A}$ and small a with $0 < a < 2\varepsilon$,

$$\frac{P_{\theta,n}\left\{\hat{\theta}_n - \theta > a\right\}}{e^{-n\left\{(a^2/2)+O(a^3)\right\}} \Big/ \sqrt{2\pi n(a^2 + O(a^3))}} \ge 1 + O\left(\frac{1}{n}\right), \tag{3.6}$$

which yields

$$\frac{P_{\theta,n}\left\{\left|\hat{\theta}_n-\theta\right|>a\right\}}{\sqrt{2}\mathrm{e}^{-n\left\{(a^2/2)+O(a^3)\right\}}\Big/\sqrt{\pi n(a^2+O(a^3))}}\geq 1+O\left(\frac{1}{n}\right) \tag{3.7}$$

for large n, from (2.15). From (3.6) and (3.7) we have for any $\hat{\theta}_n\in\mathbb{A}$

$$\varliminf_{a\to 0}\varliminf_{n\to\infty}\frac{P_{\theta,n}\left\{\hat{\theta}_n-\theta>a\right\}}{\mathrm{e}^{-n\left\{(a^2/2)+O(a^3)\right\}}\Big/\sqrt{2\pi n(a^2+O(a^3))}}\geq 1$$

and

$$\varliminf_{a\to 0}\varliminf_{n\to\infty}\frac{P_{\theta,n}\left\{\left|\hat{\theta}_n-\theta\right|>a\right\}}{\sqrt{2}\mathrm{e}^{-n\left\{(a^2/2)+O(a^3)\right\}}\Big/\sqrt{\pi n(a^2+O(a^3))}}\geq 1. \tag{3.8}$$

It is noted that the denominator of the left-hand side of (3.8) is independent of ε.

Next, letting $Y_i=X_i-\theta$ $(i=1,2,\ldots)$, we have as the m.g.f. of Y_1

$$\begin{aligned}
M_{Y_1}(t)=&E\left[\mathrm{e}^{tY_1}\right]=\int_{-\infty}^{\infty}\mathrm{e}^{ty}f_\varepsilon(y)\mathrm{d}y\\
=&\frac{c_\varepsilon\phi(\varepsilon)}{t}\left(\mathrm{e}^{\varepsilon t}-\mathrm{e}^{-\varepsilon t}\right)+c_\varepsilon\mathrm{e}^{t^2/2}\left\{2-\Phi(\varepsilon+t)-\Phi(\varepsilon-t)\right\}\\
=&1+\left(\frac{1}{2}+\frac{c_\varepsilon\varepsilon^3}{3}\phi(\varepsilon)\right)t^2+O\left(t^4\right)\\
=&1+\frac{t^2}{2\beta_\varepsilon}+O\left(t^4\right)
\end{aligned} \tag{3.9}$$

for small $|t|$, where

$$\beta_\varepsilon=\left(1+\frac{2c_\varepsilon}{3}\varepsilon^3\phi(\varepsilon)\right)^{-1}.$$

It is noted that $0<\beta_\varepsilon<1$. From (3.9) we also obtain as the c.g.f. of Y_1

$$K_{Y_1}(t)=\log M_{Y_1}(t)=\frac{t^2}{2\beta_\varepsilon}+O(t^4)$$

for small $|t|$. If $K'_{Y_1}(t)=a$, then

$$t=\hat{t}:=\beta_\varepsilon a+O\left(a^3\right), \tag{3.10}$$

which yields

$$\sigma^2(\hat{t}):=K''_{Y_1}(\hat{t})=\frac{1}{\beta_\varepsilon}+O\left(a^2\right), \tag{3.11}$$

and from (3.9) and (3.10)

$$M^n_{Y_1}(\hat{t})=\exp\left\{\frac{1}{2}n\left(\beta_\varepsilon a^2+O\left(a^4\right)\right)\right\}. \tag{3.12}$$

It is noted from (2.11) and (2.13) that

$$B_0(\lambda) = \frac{1}{\sqrt{2\pi}} + O\left(\frac{1}{\lambda^2}\right) \tag{3.13}$$

for large λ (see also Jensen (1995, page 24)). Put $\bar{X} = (1/n)\sum_{i=1}^n X_i$ and $\bar{Y} = (1/n)\sum_{i=1}^n Y_i$. Since $\sqrt{n}(\bar{X} - \theta)$ converges in law to the normal distribution $\mathrm{N}(0, 1/\beta_\varepsilon)$ as $n \to \infty$, under $P_{\theta,n}$, it is seen that $\bar{X}$ is wAMU for θ. Letting $\hat{\lambda} = \sqrt{n}|\hat{t}|\sigma(\hat{t})$, from (3.10) – (3.13) and (6.1) in Appendix we have for small $a(> 0)$

$$\begin{aligned} P_{\theta,n}\left\{\bar{X} - \theta > a\right\} &= P_{0,n}\left\{\bar{Y} > a\right\} \\ &= \frac{M_{Y_1}^n(\hat{t})\mathrm{e}^{-n\hat{t}a}}{\sqrt{2\pi n}\hat{t}\sigma(\hat{t})}\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \frac{\mathrm{e}^{-n\left((\beta_\varepsilon a^2/2) + O\left(a^4\right)\right)}}{\sqrt{2\pi\beta_\varepsilon n}\; a\left(1 + O\left(a^2\right)\right)}\left(1 + O\left(\frac{1}{n}\right)\right) \end{aligned} \tag{3.14}$$

as $n \to \infty$.

Letting $K'_{Y_1}(t) = -a$, we have

$$t = \hat{t} := -\beta_\varepsilon a + O\left(a^3\right), \tag{3.15}$$

which yields

$$\sigma^2(\hat{t}) = K''_{Y_1}(\hat{t}) = \frac{1}{\beta_\varepsilon} + O\left(a^2\right), \tag{3.16}$$

and from (3.9) and (3.15)

$$M_{Y_1}^n(\hat{t}) = \exp\left\{\frac{1}{2}n\left(\beta_\varepsilon a^2 + O\left(a^4\right)\right)\right\}. \tag{3.17}$$

Letting $\hat{\lambda} = \sqrt{n}|\hat{t}|\sigma(\hat{t})$, from (3.13), (3.15) – (3.17) and (6.2) in Appendix we have for small $a(> 0)$

$$P_{\theta,n}\left\{\bar{X} - \theta < -a\right\} = P_{0,n}\left\{\bar{Y} < -a\right\} = \frac{\mathrm{e}^{-n\left((\beta_\varepsilon a^2/2) + O\left(a^4\right)\right)}}{\sqrt{2\pi\beta_\varepsilon n}\; a\left(1 + O\left(a^2\right)\right)}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{3.18}$$

as $n \to \infty$. Then it follows from (3.14) and (3.18) that for small $a(> 0)$

$$P_{\theta,n}\left\{|\bar{X} - \theta| > a\right\} = \frac{\sqrt{2}\mathrm{e}^{-n\left((\beta_\varepsilon a^2/2) + O\left(a^4\right)\right)}}{\sqrt{\pi\beta_\varepsilon n}\; a\left(1 + O\left(a^2\right)\right)}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{3.19}$$

as $n \to \infty$. By (3.8) and (3.19) we have

$$\begin{aligned} &\lim_{a\to 0}\lim_{n\to\infty} \frac{P_{\theta,n}\left\{|\bar{X} - \theta| > a\right\}}{\sqrt{2}\mathrm{e}^{-n\left\{(a^2/2) + O\left(a^3\right)\right\}} \Big/ \sqrt{\pi n\left(a^2 + O\left(a^3\right)\right)}}\left(1 + O\left(\frac{1}{n}\right)\right) \\ =&\lim_{a\to 0}\lim_{n\to\infty} \frac{1}{\sqrt{\beta_\varepsilon}}(1 + O(a))\left(1 + O\left(\frac{1}{n}\right)\right)\exp\left\{n\left(\frac{1 - \beta_\varepsilon}{2}a^2 + O\left(a^3\right)\right)\right\} = \infty, \end{aligned}$$

since $\beta_\varepsilon < 1$. Hence it is seen from (3.8) and (3.19) that $\bar{X}$ is not wLDE. If $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables according to the normal distribution $\mathrm{N}(\theta, 1)$, then $\bar{X}$ is LDE (see Akahira, 2006, 2010).

Next we consider the sample median. Let $0 < a < \varepsilon$ and $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics of i.i.d. random variables $X_1, \ldots, X_n$ according to $\mathrm{fN}[-\varepsilon, \varepsilon]$, i.e., $X_{(1)} \leq \cdots \leq X_{(n)}$. Assume that $n = 2k - 1$ for $k = 1, 2, \ldots$. Then the sample median is given by $X_{(k)}$. Letting

$$p := P\{X_1 > a\} = \frac{1}{2} - ac_\varepsilon \phi(\varepsilon), \tag{3.20}$$

and $S_a = \#\{i \mid X_i > a\}$, i.e., the number of i satisfying $X_i > a$, we see that S_a has a binomial distribution with parameters n and p. Using the approximation (2.4.11) of Jensen (1995, page 43), we obtain

$$\begin{aligned} P_{\theta,n}\left\{X_{(k)} - \theta > a\right\} &= P_{0,n}\left\{X_{(k)} > a\right\} = P_{0,n}\left\{\frac{1}{n}S_a \geq \frac{k}{n}\right\} \\ =&\frac{p^k q^{n-k}}{1 - \left(\frac{\hat{q}p}{\hat{p}q}\right)^{\mathrm{sgn}(\hat{p}-p)}} \cdot \frac{1}{\sqrt{n}\hat{p}^{k+(1/2)}\hat{q}^{n-k+(1/2)}}\left\{B_0(\lambda) + O\left(\frac{1}{n}\right)\right\} \end{aligned} \tag{3.21}$$

as $n \to \infty$, where $q = 1 - p$, $\hat{p} = k/n$, $\hat{q} = 1 - \hat{p}$ and by (3.13)

$$B_0(\lambda) = \lambda e^{\lambda^2/2}\{1 - \Phi(\lambda)\} = \frac{1}{\sqrt{2\pi}} + O\left(\frac{1}{\lambda^2}\right) \quad (\lambda \to \infty) \tag{3.22}$$

with

$$\lambda = \sqrt{n}\left|\log\frac{\hat{p}q}{\hat{q}p}\right|\sqrt{\hat{p}\hat{q}} \tag{3.23}$$

(see also Akahira 2006). Since $\hat{p} > p$ and

$$\sqrt{n\hat{p}\hat{q}} = \sqrt{n}\left(\frac{1}{2} + O\left(\frac{1}{n^2}\right)\right),$$

it follows from (3.23) that

$$\lambda = \sqrt{n}\left(2ac_\varepsilon\phi(\varepsilon) + O(a^2)\right). \tag{3.24}$$

From (3.21) – (3.24) we have

$$P_{\theta,n}\left\{X_{(k)} - \theta > a\right\} = \frac{1}{(\hat{p} - p)/(\hat{p}q)} \cdot \frac{1}{\sqrt{n\hat{p}\hat{q}}}\left(\frac{p}{\hat{p}}\right)^k\left(\frac{q}{\hat{q}}\right)^{n-k}\left\{\frac{1}{\sqrt{2\pi}} + O\left(\frac{1}{n}\right)\right\}. \tag{3.25}$$

Since

$$\begin{aligned} &\frac{\hat{p} - p}{\hat{p}q} = 4ac_\varepsilon\phi(\varepsilon) + O\left(a^2\right), \\ &\log\left(\frac{p}{\hat{p}}\right)^k\left(\frac{q}{\hat{q}}\right)^{n-k} = -n\left\{\frac{(\hat{p} - p)^2}{2pq} + O\left(a^3\right)\right\} = -n\left\{2a^2c_\varepsilon^2\phi^2(\varepsilon) + O\left(a^3\right)\right\}, \end{aligned}$$

it follows from (3.25) that

$$P_{\theta,n}\left\{X_{(k)}-\theta>a\right\}=\frac{1}{\sqrt{2\pi n}\left\{2ac_\varepsilon\phi(\varepsilon)+O\left(a^2\right)\right\}}\left[\exp\left\{-n\left(2a^2c_\varepsilon^2\phi^2(\varepsilon)+O\left(a^3\right)\right)\right\}\right] \cdot\left(1+O\left(\frac{1}{n}\right)\right). \tag{3.26}$$

Letting $X_i'=-X_i$ $(i=1,2,\ldots)$, we have

$$P_{\theta,n}\left\{X_{(k)}-\theta<-a\right\}=P_{0,n}\left\{-X_{(k)}>a\right\}=P_{0,n}\left\{X_{(k)}'>a\right\},$$

hence in a similar way to the above it is seen that $P_{\theta,n}\left\{X_{(k)}-\theta<-a\right\}$ is also given by the right-hand side of (3.26). This yields

$$\frac{P_{\theta,n}\left\{\left|X_{(k)}-\theta\right|>a\right\}}{\mathrm{e}^{-n\left\{2a^2c_\varepsilon^2\phi^2(\varepsilon)+O\left(a^3\right)\right\}}\Big/\left\{\sqrt{2\pi n}\,ac_\varepsilon\phi(\varepsilon)(1+O(a))\right\}}=1+O\left(\frac{1}{n}\right). \tag{3.27}$$

Since $2c_\varepsilon<1/\phi(\varepsilon)$ for $\varepsilon>0$, it follows from (3.7) and (3.27) that

$$\begin{aligned}&\lim_{a\to0}\lim_{n\to\infty}\frac{P_{\theta,n}\left\{\left|X_{(k)}-\theta\right|>a\right\}}{\sqrt{2}\mathrm{e}^{-n\left\{(a^2/2)+O\left(a^3\right)\right\}}\Big/\sqrt{\pi n\left(a^2+O\left(a^3\right)\right)}}\left(1+O\left(\frac{1}{n}\right)\right)\\&=\lim_{a\to0}\lim_{n\to\infty}\frac{1}{2c_\varepsilon\phi(\varepsilon)}\left[\exp\left\{n\left(\frac{a^2}{2}\left(1-4c_\varepsilon^2\phi^2(\varepsilon)\right)+O\left(a^3\right)\right)\right\}\right](1+O(a))\left(1+O\left(\frac{1}{n}\right)\right)\\&=\infty.\end{aligned} \tag{3.28}$$

On the other hand, since the limiting distribution of $\sqrt{n}\left(X_{(k)}-\theta\right)$ is known to be normal with mean 0 and variance $1/\left(4c_\varepsilon^2\phi^2(\varepsilon)\right)$, $X_{(k)}$ is seen to be wAMU for θ. Hence it is seen from (3.28) that $X_{(k)}$ is not wLDE.

In order to compare $\bar{X}$ and $X_{(k)}$ in terms of the large deviation probability, we obtain

$$\begin{aligned}&\lim_{a\to0}\lim_{n\to\infty}\frac{P_{\theta,n}\left\{\left|\bar{X}-\theta\right|>a\right\}}{P_{\theta,n}\left\{\left|X_{(k)}-\theta\right|>a\right\}}\\&=\lim_{a\to0}\lim_{n\to\infty}\left[\exp\left\{-\frac{n}{2}\left(a^2\left(\beta_\varepsilon-4c_\varepsilon^2\phi^2(\varepsilon)\right)+O\left(a^3\right)\right)\right\}\right]\frac{2c_\varepsilon}{\sqrt{\beta_\varepsilon}}\phi(\varepsilon)(1+O(a))\\&=0\end{aligned} \tag{3.29}$$

for both of small and large ε, that is, $\bar{X}$ has asymptotically smaller large deviation probability than $X_{(k)}$ for such an ε.

Next, we consider the case when $\varepsilon=2a$. Since $c_{2a}=1+O\left(a^3\right)$, it follows that $\beta_{2a}=1+O\left(a^3\right)$, which yields

$$\lim_{a\to0}\lim_{n\to\infty}\frac{P_{\theta,n}\left\{\left|\bar{X}-\theta\right|>a\right\}}{\sqrt{2}\mathrm{e}^{-n\left\{(a^2/2)+O\left(a^3\right)\right\}}\big/\left\{\sqrt{\pi n}\left(a+O\left(a^3\right)\right)\right\}}=1 \tag{3.30}$$

from (3.19). Then it is seen from (3.8) that $\bar{X}$ is wLDE, which is consistent with the result in the normal case $\mathrm{N}(\theta, 1)$ (see Akahira, 2006). But, it is seen from (3.29) and (3.30) that $X_{(k)}$ is not wLDE, since

$$1 - 4c_{2a}^2\phi^2(2a) = 1 - \frac{2}{\pi} + O\left(a^2\right) > 0$$

for sufficiently small a.

4 Flattened Laplace distribution in a middle part

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables with a p.d.f. $f_\varepsilon(x-\theta)$ with respect to the Lebesgue measure, where $x \in \mathbf{R}^1$ and $\theta \in \Theta = \mathbf{R}^1$. Let ε be any fixed positive number. Further we assume that

$$f_\varepsilon(x) = \begin{cases} c_\varepsilon \mathrm{e}^{-\varepsilon} & \text{for } |x| \le \varepsilon, \\ c_\varepsilon \mathrm{e}^{-|x|} & \text{for } |x| > \varepsilon, \end{cases} \tag{4.1}$$

where

$$c_\varepsilon = \frac{\mathrm{e}^\varepsilon}{2(\varepsilon+1)}.$$

A distribution with the p.d.f. of (4.1) is called a flattened Laplace distribution $\mathrm{fL}[-\varepsilon, \varepsilon]$.

Let $z(\theta, a)$ be defined by (3.2) and $0 < a < 2\varepsilon$. From (4.1) we have

$$z(\theta, a) = \begin{cases} a & \text{for } x \le \theta - \varepsilon, \\ -x + \theta + a - \varepsilon & \text{for } \theta - \varepsilon < x \le \theta + a - \varepsilon, \\ 0 & \text{for } \theta + a - \varepsilon < x \le \theta + \varepsilon, \\ -x + \theta + \varepsilon & \text{for } \theta + \varepsilon < x \le \theta + a + \varepsilon, \\ -a & \text{for } \theta + a + \varepsilon < x. \end{cases}$$

Let $Z_i(\theta, a) = \log\left(f_\varepsilon(X_i - a)/f_\varepsilon(X_i - \theta - a)\right)$ $(i = 1, 2, \ldots)$. In a similar way to the flattened normal case, we obtain for small $a(> 0)$

$$\mu(\theta, a) = E_{\theta+a}\left[Z_1(\theta, a)\right] = \frac{1}{2(\varepsilon+1)}\left\{-a^2 + \frac{a^3}{6} + O\left(a^4\right)\right\}, \tag{4.2}$$

and

$$E_{\theta+a}\left[Z_1^2(\theta, a)\right] = \frac{1}{\varepsilon+1}\left\{a^2 - \frac{a^3}{6} + O\left(a^4\right)\right\},$$

which yields

$$\sigma^2(\theta, a) = V_{\theta+a}(Z_1(\theta, a)) = \frac{1}{\varepsilon+1}\left\{a^2 - \frac{a^3}{6} + O\left(a^4\right)\right\}. \tag{4.3}$$

Then it follows from (4.2), (4.3) and Corollary that for any $\hat{\theta}_n \in \mathbb{A}$

$$\lim_{a\to 0}\lim_{n\to\infty} \frac{P_{\theta,n}\left\{\hat{\theta}_n - \theta > a\right\}}{\mathrm{e}^{-n\left[\left\{a^2/(2(\varepsilon+1))\right\}+O\left(a^3\right)\right]} \Big/ \left\{\sqrt{2\pi n}\left(\left(a/\sqrt{\varepsilon+1}\right)+O\left(a^2\right)\right)\right\}} \geq 1$$

and

$$\lim_{a\to 0}\lim_{n\to\infty} \frac{P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\}}{\sqrt{2}\mathrm{e}^{-n\left[\left\{a^2/(2(\varepsilon+1))\right\}+O\left(a^3\right)\right]} \Big/ \left\{\sqrt{\pi n}\left(\left(a/\sqrt{\varepsilon+1}\right)+O\left(a^2\right)\right)\right\}} \geq 1. \tag{4.4}$$

Next, letting $Y_i = X_i - \theta$ $(i = 1, 2, \ldots)$, we has as the m.g.f. of Y_1

$$M_{Y_1}(t) = E\left[\mathrm{e}^{tY_1}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{ty} f_\varepsilon(y)\mathrm{d}y = 1 + \frac{1}{2(\varepsilon+1)}\left\{2(\varepsilon+1)+\varepsilon^2+\frac{\varepsilon^3}{3}\right\}t^2 + O\left(t^4\right) \tag{4.5}$$

for small $|t|$, which yields

$$K_{Y_1}(t) = \log M_{Y_1}(t) = \frac{1}{2(\varepsilon+1)}\left\{2(\varepsilon+1)+\varepsilon^2+\frac{\varepsilon^3}{3}\right\}t^2 + O\left(t^4\right)$$

for small $|t|$. If $K'_{Y_1}(t) = a$, then

$$t = \hat{t} := \frac{(\varepsilon+1)a}{2(\varepsilon+1)+\varepsilon^2+(\varepsilon^3/3)} + O\left(a^3\right) = \gamma_\varepsilon a + O\left(a^3\right) \quad \text{(say)}, \tag{4.6}$$

which yields

$$\sigma^2(\hat{t}) = K''_{Y_1}(\hat{t}) = \frac{1}{\gamma_\varepsilon} + O\left(a^2\right) \tag{4.7}$$

and from (4.5) and (4.6)

$$M^n_{Y_1}(\hat{t}) = \exp\left\{\frac{1}{2}n\left(\gamma_\varepsilon a^2 + O\left(a^4\right)\right)\right\}. \tag{4.8}$$

Here, it follows from the central limit theorem that $\bar{X}$ is wAMU. In a similar way to the flattened normal case, from (4.6) – (4.8) and (6.1) in Appendix we have for small $a(>0)$

$$\begin{aligned} P_{\theta,n}\left\{\bar{X} - \theta > a\right\} &= P_{0,n}\left\{\bar{Y} > a\right\} \\ &= \frac{M^n_{Y_1}(\hat{t})\mathrm{e}^{-n\hat{t}a}}{\sqrt{2\pi n}\hat{t}\sigma(\hat{t})}\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \frac{\mathrm{e}^{-n\left\{(\gamma_\varepsilon a^2/2)+O\left(a^4\right)\right\}}}{\sqrt{2\pi\gamma_\varepsilon n}\left(a + O\left(a^3\right)\right)}\left(1 + O\left(\frac{1}{n}\right)\right) \end{aligned} \tag{4.9}$$

as $n \to \infty$. In a similar way to the derivation of (3.18), we also obtain for small $a(>0)$

$$P_{\theta,n}\left\{\bar{X} - \theta < -a\right\} = P_{0,n}\left\{\bar{Y} < -a\right\} = \frac{\mathrm{e}^{-n\left\{(\gamma_\varepsilon a^2/2)+O\left(a^4\right)\right\}}}{\sqrt{2\pi\gamma_\varepsilon n}\left(a + O\left(a^3\right)\right)}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{4.10}$$

as $n \to \infty$. Then it follows from (4.9) and (4.10) that for small $a(>0)$

$$P_{\theta,n}\left\{\left|\bar{X}-\theta\right|>a\right\}=\frac{\sqrt{2}\mathrm{e}^{-n\left\{(\gamma_\varepsilon a^2/2)+O\left(a^4\right)\right\}}}{\sqrt{\pi\gamma_\varepsilon n}\left(a+O\left(a^3\right)\right)}\left(1+O\left(\frac{1}{n}\right)\right) \tag{4.11}$$

as $n \to \infty$. By (4.4) and (4.11) we have

$$\begin{aligned}
&\lim_{a\to 0}\lim_{n\to\infty}\frac{P_{\theta,n}\left\{\left|\bar{X}-\theta\right|>a\right\}}{\sqrt{2}\mathrm{e}^{-n\left[\left\{a^2/(2(\varepsilon+1))\right\}+O\left(a^3\right)\right]}\Big/\left\{\sqrt{\pi n}\left(\left(a/\sqrt{\varepsilon+1}\right)+O\left(a^2\right)\right)\right\}}\\
&=\lim_{a\to 0}\lim_{n\to\infty}\frac{1}{\sqrt{\gamma_\varepsilon(\varepsilon+1)}}\left(1+O\left(a\right)\right)\left(1+O\left(\frac{1}{n}\right)\right)\exp\left[\frac{n}{2}\left\{\left(\frac{1}{\varepsilon+1}-\gamma_\varepsilon\right)a^2+O\left(a^3\right)\right\}\right]\\
&=\infty,
\end{aligned} \tag{4.12}$$

since $\gamma_\varepsilon < 1/(\varepsilon+1)$. Hence it is seen from (4.4) and (4.12) that $\bar{X}$ is not wLDE.

In a similar way to the previous section we consider the sample median. Let $0 < a < \varepsilon$ and $X_{(1)}, \ldots, X_{(n)}$ denote the order statistics of a random sample $(X_1, \ldots, X_n)$, i.e., $X_{(1)} \le \cdots \le X_{(n)}$. Assume that $n = 2k-1$ for $k = 1, 2, \ldots$. Then the sample median is given by $X_{(k)}$. Since

$$p := P\left\{X_1 > a\right\} = \frac{1}{2} - \frac{a}{2(\varepsilon+1)},$$

replacing $c_\varepsilon \phi(\varepsilon)$ with $1/(2(\varepsilon+1))$ in (3.20) we have from (3.27)

$$\frac{P_{\theta,n}\left\{\left|X_{(k)}-\theta\right|>a\right\}}{\sqrt{2}\left[\exp\left\{-n\left(\frac{a^2}{2(\varepsilon+1)^2}+O\left(a^3\right)\right)\right\}\right]\Big/\left\{\sqrt{\pi n}\left(\frac{a}{\varepsilon+1}+O(a^2)\right)\right\}}=1+O\left(\frac{1}{n}\right), \tag{4.13}$$

and comparing it with the lower bound by (4.4) we obtain

$$\begin{aligned}
&\lim_{a\to 0}\lim_{n\to\infty}\frac{P_{\theta,n}\left\{\left|X_{(k)}-\theta\right|>a\right\}}{\sqrt{2}\left[\exp\left\{-n\left(\frac{a^2}{2(\varepsilon+1)}+O\left(a^3\right)\right)\right\}\right]\Big/\left\{\sqrt{\pi n}\left(\frac{a}{\sqrt{\varepsilon+1}}+O(a^2)\right)\right\}}\\
&=\lim_{a\to 0}\lim_{n\to\infty}\sqrt{\varepsilon+1}(1+O(a))\left[\exp\left\{\frac{\varepsilon n}{2(\varepsilon+1)^2}\left(a^2+O\left(a^3\right)\right)\right\}\right]\left(1+O\left(\frac{1}{n}\right)\right)\\
&=\infty.
\end{aligned} \tag{4.14}$$

On the other hand, since the limiting distribution of $\sqrt{n}(X_{(k)}-\theta)$ is known to be normal with mean 0 and variance $\mathrm{e}^{2\varepsilon}/(4\gamma_\varepsilon^2)$, $X_{(k)}$ is seen to be wAMU for θ. Hence it follows from (4.14) that $X_{(k)}$ is not wLDE.

In order to compare $\bar{X}$ and $X_{(k)}$ in terms of the large deviation probability, we obtain from (4.11) and (4.13)

$$\lim_{a\to 0}\lim_{n\to\infty}\frac{P_{\theta,n}\left\{\left|\bar{X}-\theta\right|>a\right\}}{P_{\theta,n}\left\{\left|X_{(k)}-\theta\right|>a\right\}}=\begin{cases}\infty & \text{for } \varepsilon<\varepsilon_0,\\ 1 & \text{for } \varepsilon=\varepsilon_0,\\ 0 & \text{for } \varepsilon>\varepsilon_0,\end{cases} \tag{4.15}$$

where the unique solution $\varepsilon_0 (\fallingdotseq 0.4757)$ of the equation

$$\frac{2}{3}\varepsilon^3 + 2\varepsilon^2 + \varepsilon - 1 = 0$$

for $\varepsilon > 0$. It is seen from (4.15) that for $\varepsilon < \varepsilon_0$, $X_{(k)}$ has asymptotically smaller large deviation probability than $\bar{X}$, and for $\varepsilon > \varepsilon_0$, $\bar{X}$ has asymptotically smaller large deviation probability than $X_{(k)}$, and for $\varepsilon = \varepsilon_0$, $X_{(k)}$ has asymptotically same large deviation probability as $\bar{X}$.

Next we consider the case when $\varepsilon = 2a$. From (4.4) we have for any $\hat{\theta}_n \in \mathbb{A}$

$$\varliminf_{a\to 0} \varliminf_{n\to\infty} \frac{P_{\theta,n}\left\{\left|\hat{\theta}_n - \theta\right| > a\right\}}{\sqrt{2}\mathrm{e}^{-n\left((a^2/2)+O\left(a^3\right)\right)} \Big/ \left\{\sqrt{\pi n}(a + O(a^2))\right\}} \geq 1. \tag{4.16}$$

Choosing $X_{(k)}$ as $\hat{\theta}_n$ in (4.16), from (4.13) we see that the equality in (4.16) holds, i.e., $X_{(k)}$ is wLDE, which is consistent with the result in the Laplace case with a p.d.f. $f(x;\theta) = (1/2)\mathrm{e}^{-|x-\theta|}$ $(-\infty < x < \infty; \infty < \theta < \infty)$ (see Akahira 2006). Letting $\varepsilon = 2a < \varepsilon_0$, from (4.15) we see that

$$\lim_{a\to 0} \lim_{n\to\infty} \frac{P_{\theta,n}\left\{\left|\bar{X} - \theta\right| > a\right\}}{P_{\theta,n}\left\{\left|X_{(k)} - \theta\right| > a\right\}} = \infty,$$

which implies that $\bar{X}$ is not wLDE.

5 Concluding remarks

For flattened distributions in $[-\varepsilon, \varepsilon]$ which do not belong to an exponential family, the bound for the large deviation probability of wAMU estimators is obtained, and its comparisons with the sample mean and sample median are also done. Indeed, in the flattened normal case $\mathrm{fN}(\theta - \varepsilon, \theta + \varepsilon)$, the sample mean $\bar{X}$ is seen to be asymptotically better than the sample median $X_{(k)}$ for both of small and large ε, in the sense of the large deviation probability. In the case when $\varepsilon = 2a$, $\bar{X}$ is shown to be wLDE, which is the same result as in the normal case $\mathrm{N}(\theta, 1)$, but $X_{(k)}$ is not seen to be wLDE. In the flattened Laplace case $\mathrm{fL}[\theta - \varepsilon, \theta + \varepsilon]$, $X_{(k)}$ is seen to be asymptotically better than $\bar{X}$ for smaller ε, but $X_{(k)}$ is done to be asymptotically worse than $\bar{X}$ for bigger ε, in the sense of the large deviation probability. In the case when $\varepsilon = 2a$, $X_{(k)}$ is shown to be wLDE, which is the same result as in the Laplace case, but $\bar{X}$ is not seen to be wLDE.

6 Appendix

We consider the classical saddlepoint approximation. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables with a p.d.f. $f_0(x)$ w.r.t. a σ-finite measure μ. Let $\mathcal{X}$ be a sample space of X_1. Then the m.g.f. and c.g.f. of X_1 are given by

$$M_{X_1}(t) = E\left[\mathrm{e}^{tX_1}\right] = \int_{\mathcal{X}} \mathrm{e}^{tx} f_0(x)\mathrm{d}\mu$$

and $K_{X_1}(t) = \log M_{X_1}(t)$, respectively, where $t \in \mathcal{J} := \{t \mid M_{X_1}(t) < \infty\}$. Letting

$$f_t(x) = \mathrm{e}^{tx} f_0(x)/M_{X_1}(t), \quad t \in \mathcal{J},$$

we see that for each $t \in \mathcal{J}$, $f_t(x)$ is a p.d.f. (w.r.t. μ). A set of distributions with p.d.f.'s $f_t(x)$ generates an exponential family of distributions. The mean and variance of X_1 w.r.t. p.d.f. f_t are given by

$$\begin{aligned}\mu(t) :=& E_t(X_1) = \frac{\mathrm{d}}{\mathrm{d}t} K_{X_1}(t),\\ \sigma^2(t) :=& V_t(X_1) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} K_{X_1}(t),\end{aligned}$$

respectively. Here, we consider only those values of x which there exists $t = \hat{t}(x)$ such that $\mu(t) = x$, and the upper tail probability $P\left\{\bar{X} \geq x\right\}$ of $\bar{X} = (1/n)\sum_{i=1}^n X_i$, where $x \geq \mu(0)$. Then it is noted that $t \geq 0$. When $x < \mu(0)$, we can consider the lower tail probability $P\left\{\bar{X} \leq x\right\}$ by replacing X_i by $-X_i$ for each $i = 1, 2, \ldots$ and note that $t < 0$.

The approximation formula to the upper tail probability for $x \geq \mu(0)$ is given by

$$P\left\{\bar{X} \geq x\right\} = \frac{\left\{M_{X_1}(\hat{t})\right\}^n \mathrm{e}^{-n\hat{t}x}}{\sqrt{n}\hat{t}\sigma(\hat{t})} \left\{B_0(\hat{\lambda}) + O\left(\frac{1}{n}\right)\right\}, \tag{6.1}$$

where $\sigma(t) = \sqrt{\sigma^2(t)}$, $\hat{\lambda} = \sqrt{n}\hat{t}\sigma(\hat{t})$ with $\hat{t} = \hat{t}(x)$, and

$$B_0(\lambda) = \lambda \mathrm{e}^{\lambda^2/2}\{1 - \Phi(\lambda)\}$$

(see Jensen 1995, Section 2.2). The approximation formula to the lower tail probability for $x < \mu(0)$ is also given by

$$P\left\{\bar{X} \leq x\right\} = \frac{\left\{M_{X_1}(\hat{t})\right\}^n \mathrm{e}^{-n\hat{t}x}}{\sqrt{n}|\hat{t}|\sigma(\hat{t})} \left\{B_0(\hat{\lambda}) + O\left(\frac{1}{n}\right)\right\}, \tag{6.2}$$

where $\hat{\lambda} = \sqrt{n}|\hat{t}|\sigma(\hat{t})$.

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