The first homology of IA_n with coefficients in spaces of Jacobi diagrams

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1 Introduction

The IA-automorphism group IA_n of the free group F_n of rank n is a normal subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of F_n. If $n = 2$, then IA_2 is equal to the inner automorphism group $\mathrm{Inn}(F_2)$ of F_2 and if $n \geq 3$, then $\mathrm{Inn}(F_n)$ is a proper subgroup of IA_n. The group IA_n is defined in a way similar to the Torelli group of a closed surface or a surface with one boundary component. That is, IA_n is the kernel of the canonical map from $\mathrm{Aut}(F_n)$ to the general linear group $\mathrm{GL}(n, \mathbb{Z})$ induced by the abelianization map of F_n. Therefore, we have a short exact sequence of groups

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z}) \to 1.$$

It follows from this short exact sequence that the rational (co)homology of IA_n admits an action of $\mathrm{GL}(n, \mathbb{Z})$. Since IA_n and the Torelli groups are defined in a similar way, some strategies of studying the (co)homology of the Torelli groups can be used to study the (co)homology of IA_n and vice versa.

Cohen–Pakianathan [3], Farb (unpublished) and Kawazumi [11] independently determined the first homology of IA_n by using the Johnson homomorphism for $\mathrm{Aut}(F_n)$, and we have

$$H_1(\mathrm{IA}_n, \mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_{\mathbb{Z}}, \bigwedge^2 H_{\mathbb{Z}}), \quad H_{\mathbb{Z}} = H_1(F_n, \mathbb{Z}).$$

In higher degrees, the rational (co)homology of IA_n has been studied by many authors [14, 15, 2, 4, 16, 7] and we have a conjectural structure of the whole rational cohomology of IA_n for sufficiently large n [5].

For any (right) $\mathrm{Aut}(F_n)$-module M, the homology of IA_n with coefficients in M also admits a $\mathrm{GL}(n, \mathbb{Z})$-module structure. To the best of our knowledge, however, the homology of IA_n with any non-trivial coefficients has not been computed.

The *Jacobi diagrams* are uni-trivalent graphs encoding the algebraic structures of Lie algebras and their representations. The space of Jacobi diagrams appears as the target space of the *Kontsevich invariant*, which is a universal finite type invariant for links and unifies all quantum invariants of links [13, 1]. Habiro and Massuyeau [6] extended the Kontsevich invariant to construct a functor from the category of bottom tangles in handlebodies to the category of Jacobi diagrams in handlebodies. By using the restriction of this functor to the degree 0 part, the author constructed a functor A_d from the opposite category $\mathbf{F}^{\mathrm{op}}$ of the category $\mathbf{F}$ of finitely generated free groups to the category of filtered vector spaces [8, 10]. By restricting the functor A_d to the automorphism group $\mathrm{Aut}_{\mathbf{F}^{\mathrm{op}}}(n) \cong \mathrm{Aut}(F_n)^{\mathrm{op}}$, we obtain a (right) action of $\mathrm{Aut}(F_n)$ on the space $A_d(n)$.

In this report, we will exhibit our recent computation of the first homology of IA_n with non-trivial coefficients in the space $A_2(n)$ of Jacobi diagrams of degree 2.

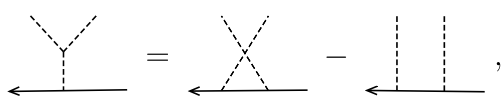
2 Preliminaries

2.1 The space $A_d(n)$ of Jacobi diagrams

A *Jacobi diagram* is a uni-trivalent graph such that each trivalent vertex is equipped with a cyclic order.

Let $n \geq 0$. A *Jacobi diagram* on n-component oriented arcs X_n is a Jacobi diagram such that each connected component has at least one univalent vertex and univalent vertices are embedded into the interior of X_n. Two Jacobi diagrams on X_n are regarded as the same if there is a homeomorphism between them whose restriction to the arc components is isotopic to the identity map. The *degree* of a Jacobi diagram is defined as half the number of its vertices.

Let $A_d(n)$ denote the vector space over $\mathbb{Q}$ spanned by Jacobi diagrams of degree d on X_n modulo the STU relation:



where the dotted lines represent a part of a Jacobi diagram and the solid lines represent a part of oriented arcs and where the trivalent vertex has the counter-clockwise order.

The vector space $A_d(n)$ admits a descending filtration of finite length

$$A_d(n) = A_{d,0}(n) \supset A_{d,1}(n) \supset A_{d,2}(n) \supset \cdots \supset A_{d,2d-2}(n) \supset A_{d,2d-1}(n) = 0,$$

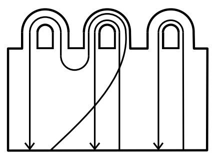
where $A_{d,k}(n)$ is the subspace of $A_d(n)$ spanned by Jacobi diagrams of degree d with at least k trivalent vertices.

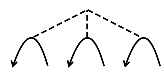
2.2 The action of $\mathrm{Aut}(F_n)$ on $A_d(n)$

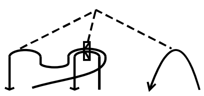
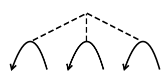
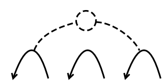
The author introduced an action of $\mathrm{Aut}(F_n)$ on $A_d(n)$ in [8] and studied the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$ in [8, 10]. The action is defined by using the composition of morphisms of the category $\mathbf{A}$ of Jacobi diagrams in handlebodies, which was introduced in [6]. The objects of $\mathbf{A}$ are non-negative integers and the hom-space from m to n is the vector space spanned by (m, n)-Jacobi diagrams, which are Jacobi diagrams on X_n mapped into a handlebody of genus m in such a way that the endpoints of the arcs are uniformly distributed on the bottom line and the i-th component goes from the $2i$-th point to the $(2i-1)$-st point, where we count the endpoints from left to right. An automorphism $f \in \mathrm{Aut}(F_n)$ is identified with an (n, n)-Jacobi diagram and the action of f on a Jacobi diagram $u \in A_d(n)$ is defined as the composition of u and the corresponding diagram f in the category $\mathbf{A}$. For example, the automorphism

$$f : F_3 \to F_3, \quad x_1 \mapsto x_1x_2,\ x_2 \mapsto x_2,\ x_3 \mapsto x_3$$

is identified with

$f =$  $\in \mathbf{A}(3,3)$,

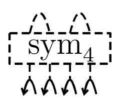
and the action of f on the Jacobi diagram $u =$  is given by

$u \cdot f =$  $=$  $- \dfrac{1}{2}$  .

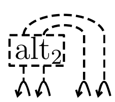
2.3 The $\mathrm{Aut}(F_n)$-module structure of $A_2(n)$

Here we recall the result in [8] on the $\mathrm{Aut}(F_n)$-module structure of $A_2(n)$.

Let $A_2'(n)$ and $A_2''(n)$ be the $\mathrm{Aut}(F_n)$-submodules of $A_2(n)$ generated by the elements P' and P'', respectively, where

$P' =$ 

$\in A_2(4), \quad \mathrm{sym}_4 = \sum_{\sigma \in \mathfrak{S}_4} \sigma,$

$P'' =$ 

$\in A_2(4), \quad \mathrm{alt}_2 = \sum_{\tau \in \mathfrak{S}_2} \mathrm{sgn}(\tau)\tau.$

For a pair (λ, μ) of partitions of non-negative integers, let $V_{\lambda,\mu}$ denote the irreducible algebraic $\mathrm{GL}(n, \mathbb{Z})$-representation corresponding to (λ, μ).

Theorem 2.1 ([8, Proposition 7.5 and Theorem 7.9])**.** *For $n \geq 3$, we have*

$$A_2(n) = A_2'(n) \oplus A_2''(n),$$

where $A_2'(n) \cong V_{0,4}$ is a simple $\mathrm{Aut}(F_n)$*-module, and where $A_2''(n)$ is an indecomposable* $\mathrm{Aut}(F_n)$*-module with the unique composition series*

$$A_2''(n) \supsetneq A_{2,1}(n) \supsetneq A_{2,2}(n) \supsetneq 0,$$

whose composition factors are

$$A_2''(n)/A_{2,1}(n) \cong V_{0,2^2}, \quad A_{2,1}(n)/A_{2,2}(n) \cong V_{0,1^3}, \quad A_{2,2}(n) \cong V_{0,2}.$$

3 Main theorem

Here we will observe the computation of $H_1(\mathrm{IA}_n, A_2(n))$.

By Theorem 2.1, we have

$$H_1(\mathrm{IA}_n, A_2(n)) \cong H_1(\mathrm{IA}_n, A_2'(n)) \oplus H_1(\mathrm{IA}_n, A_2''(n)).$$

Since the action of $\mathrm{Aut}(F_n)$ on $A_2'(n)$ factors through $\mathrm{GL}(n,\mathbb{Z})$, which means that IA_n acts trivially on $A_2'(n)$, we have

$$H_1(\mathrm{IA}_n, A_2'(n)) \cong H_1(\mathrm{IA}_n, \mathbb{Q}) \otimes A_2'(n).$$

We can compute the irreducible decomposition of the tensor product of two algebraic $\mathrm{GL}(n,\mathbb{Z})$-representations by using the formula in [12], which is given by combining the Littlewood–Richardson rule. Then we have

$$\begin{aligned} H_1(\mathrm{IA}_n, \mathbb{Q}) \otimes A_2'(n) &\cong (V_{0,1} \otimes V_{1^2,0}) \otimes V_{0,4} \\ &\cong \begin{cases} V_{1^2,41} \oplus V_{1^2,5} \oplus V_{1,4}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,3} & n \geq 4 \\ V_{1^2,5} \oplus V_{1,4}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,3} & n = 3. \end{cases} \end{aligned}$$

We also determined the irreducible decomposition of the first homology of IA_n with coefficients in $A_2''(n)$.

Theorem 3.1 ([9, Theorem 1.1])**.** *We have*

$$H_1(\mathrm{IA}_n, A_2''(n)) \cong \begin{cases} V_{1^2,2^21} \oplus V_{1^2,32} \oplus V_{1,21^2} \oplus V_{1,2^2}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,21}^{\oplus 2} & n \geq 5 \\ V_{1^2,32} \oplus V_{1,21^2} \oplus V_{1,2^2}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,21}^{\oplus 2} & n = 4 \\ V_{1,2^2} \oplus V_{1,31} \oplus V_{0,21}^{\oplus 2} & n = 3. \end{cases}$$

Outline of the proof. Use the long exact sequences of homology associated to the short exact sequences

$$0 \to A_{2,2}(n) \to A_{2,1}(n) \to A_{2,1}(n)/A_{2,2}(n) \to 0$$

and

$$0 \to A_{2,1}(n) \to A_2''(n) \to A_2''(n)/A_{2,1}(n) \to 0$$

and compute the image of the boundary homomorphisms. The case of $n = 3$ needs more computation than the other cases. □

We also computed the twisted first homology of IO_n, which is the analogue of IA_n to the outer automorphism group of F_n.

We have

$$H_1(\mathrm{IO}_n, A_2(n)) \cong H_1(\mathrm{IO}_n, A_2'(n)) \oplus H_1(\mathrm{IO}_n, A_2''(n)),$$

where

$$H_1(\mathrm{IO}_n, A_2'(n)) \cong \begin{cases} V_{1^2,41} \oplus V_{1^2,5} \oplus V_{1,4} \oplus V_{1,31} & n \geq 4 \\ V_{1^2,5} \oplus V_{1,4} \oplus V_{1,31} & n = 3. \end{cases}$$

Theorem 3.2 ([9, Theorem 10.4]). *We have*

$$H_1(\mathrm{IO}_n, A_2''(n)) \cong \begin{cases} V_{1^2,2^21} \oplus V_{1^2,32} \oplus V_{1,21^2} \oplus V_{1,2^2} \oplus V_{1,31} \oplus V_{0,21} & n \geq 5 \\ V_{1^2,32} \oplus V_{1,21^2} \oplus V_{1,2^2} \oplus V_{1,31} \oplus V_{0,21} & n = 4 \\ V_{1,31} \oplus V_{0,21} & n = 3. \end{cases}$$

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References

[1] Dror Bar-Natan. Vassiliev homotopy string link invariants. *J. Knot Theory Ramifications*, 4(1):13–32, 1995.

[2] Mladen Bestvina, Kai-Uwe Bux, and Dan Margalit. Dimension of the Torelli group for $\mathrm{Out}(F_n)$. *Invent. Math.*, 170(1):1–32, 2007.

[3] Fred Cohen and Jonathan Pakianathan. On automorphism groups of free groups, and their nilpotent quotients. in preparation.

[4] Matthew Day and Andrew Putman. On the second homology group of the Torelli subgroup of $\mathrm{Aut}(F_n)$. *Geom. Topol.*, 21(5):2851–2896, 2017.

[5] Kazuo Habiro and Mai Katada. On the stable cohomology of the (IA-)automorphism groups of free groups. *arXiv:2211.13458v4*, 2022.

[6] Kazuo Habiro and Gwénaël Massuyeau. The Kontsevich integral for bottom tangles in handlebodies. *Quantum Topol.*, 12(4):593–703, 2021.

[7] Mai Katada. Stable rational homology of the IA-automorphism groups of free groups. *arXiv:2207.00920v2*, 2022.

[8] Mai Katada. Actions of automorphism groups of free groups on spaces of Jacobi diagrams. I. *Ann. Inst. Fourier (Grenoble)*, 73(4):1489–1532, 2023.

[9] Mai Katada. The first homology of IA_n with coefficients in spaces of jacobi diagrams. *arXiv:2312.10428*, 2023.

[10] Mai Katada. ACTIONS OF AUTOMORPHISM GROUPS OF FREE GROUPS ON SPACES OF JACOBI DIAGRAMS. II. *J. Inst. Math. Jussieu*, 23(1):1–69, 2024.

[11] Nariya Kawazumi. Cohomological aspects of Magnus expansions. *arXiv:math/0505497*, 2005.

[12] Kazuhiko Koike. On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters. *Adv. Math.*, 74(1):57–86, 1989.

[13] Maxim Kontsevich. Vassiliev's knot invariants. In *I. M. Gel′fand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 137–150. Amer. Math. Soc., Providence, RI, 1993.

[14] Sava Krstić and James McCool. The non-finite presentability of $\mathrm{IA}(F_3)$ and $\mathrm{GL}_2(\mathbf{Z}[t, t^{-1}])$. *Invent. Math.*, 129(3):595–606, 1997.

[15] Alexandra Pettet. The Johnson homomorphism and the second cohomology of IA_n. *Algebr. Geom. Topol.*, 5:725–740, 2005.

[16] Takao Satoh. On the low-dimensional cohomology groups of the IA-automorphism group of the free group of rank three. *Proc. Edinb. Math. Soc. (2)*, 64(2):338–363, 2021.

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