ON THE CHEN-YANG VOLUME CONJECTURE

RENAUD DETCHERRY

Abstract. In this survey, we will introduce the Chen-Yang volume conjecture, which predicts a relationship between the growth rate of Turaev-Viro invariants of a 3-manifold and its hyperbolic volume. We will give a summary of partial results on this subject.

1. A volume conjecture for Turaev-Viro invariants

For M a hyperbolic 3-manifold, let $\mathrm{Vol}(M)$ be its hyperbolic volume. The function Vol can be extended to any compact oriented 3-manifold: thanks to Kneser's decomposition theorem one can write M as a connected sum of prime 3-manifolds, and JSJ decomposition allows us to cut the resulting pieces along tori into hyperbolic or Seifert pieces. Then one can set $\mathrm{Vol}(M)=\sum \mathrm{Vol}(M_i)$, where the M_i are the hyperbolic pieces in the decomposition of M. The latter extension of the hyperbolic volume is often called the simplicial volume of M.

For M a compact oriented 3-manifold, a partially ideal triangulation of M is a decomposition of M into tetrahedra with some vertices truncated, where the truncation faces give a triangulation of ∂M. Some celebrated topological invariants of M can be computed from a partially ideal triangulation of M : the Turaev-Viro invariants. They depend on a choice of an odd integer $r \geq 3$ and a complex number q such that q^2 is a primitive r-th root of unity. In the following, we will always assume $q=\exp(\frac{2i\pi}{r})$.

Let E be the set of long edges of the triangulation (that is, discarding edges that come from the truncations), V the set of interior vertices, and τ the set of tetrahedron. We say that a map

$$c : E \longrightarrow \{0,2,\ldots,r-3\}$$

is an admissible coloring of E, if for any e,f,g edges that bound a face of a tetrahedron of τ, we have the inequalities

$$c(e)+c(f)+c(g) \leq 2r-4$$

$$c(e) \leq c(f)+c(g).$$

We will denote the set of admissible colorings as $\mathrm{Adm}(\tau)$. We will also set, for n an integer, $[n]=\frac{q^n-q^{-n}}{q-q^{-1}}$, and $[n]!=[1]\ldots[n]$. Finally, we set $\eta_r=\frac{q-q^{-1}}{\sqrt{-r}}$.

For e an integer, we set $w(e)=(-1)^e[e+1]$. For (i,j,k,l,m,n) integers in $\{0,2,\ldots,r-3\}$ such that $(i,j,k),(j,l,n),(i,m,n)$ and (k,l,m) are admissible, we say that (i,j,k,l,m,n) is an admissible 6-uple. For (i,j,k,l,m,n) an admissible 6-uple, let

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_q$$

be the quantum $6j$-symbol at q, a formula of which is given in [DKY18, Section 2.2]. For τ a partially ideal triangulation of M, and $\Delta \in \tau$ a tetrahedron, and $c \in \mathrm{Adm}(\tau)$, we let $|\Delta|_c$ be the quantum $6j$-symbol corresponding to the tetrahedron Δ whose edges are colored according to c.

The Turaev-Viro invariants are defined and can be computed using the following state sum:

Theorem 1.1. [TV92] *Given τ a partially ideal triangulation of M a compact oriented 3-manifold, the quantity*

$$TV_r(M,q) = \eta_r^{2|V|} \sum_{c \in \mathrm{Adm}(\tau)} \prod_{e \in E} w(e) \prod_{\Delta \in \tau} |\Delta|_c$$

is a topological invariant of M.

For closed 3-manifolds, the Turaev-Viro invariants $TV_r(M)$ can be related to the Reshetikhin-Turaev $RT_r(M)$ invariants of 3-manifolds by a theorem of Roberts. The Reshetikhin-Turaev invariants of M are defined using a surgery presentation of M. We will not recall this definition here, but we will need some of the properties of the RT_r invariants. Their main feature is that they are part of a $2+1$-TQFT (with anomaly), which means that they can be extended to a (projective) functor from the category of $2+1$-cobordism to the category of vector spaces. In particular, and in simple terms, we have the following:

- For any compact oriented surface $\Sigma_{g,n}$, a finite dimensional vector space $RT_r(\Sigma_{g,n})$, together with a Hermitian form $\langle , \rangle$ on $RT_r(\Sigma_{g,n})$ and a representation
$$\rho_r : \mathrm{Mod}(\Sigma_g) \longrightarrow PGL(RT_r(\Sigma_{g,n})).$$
- For a compact oriented 3-manifold M with boundary Σ, a vector $RT_r(M) \in RT_r(\Sigma)$, well defined up to a $4r$-th root of unity.
- If M_1 and M_2 are 3-manifolds with boundary Σ and $M = M_1 \bigcup_{\Sigma} M_2$ then
$$RT_r(M) = \langle RT_r(M_1), RT_r(M_2) \rangle,$$
up to a $4r$-th root of unity.

A theorem of Roberts for closed 3-manifolds, generalized by Benedetti and Petronio to all compact oriented 3-manifolds, explains the relationship between the TV_r and RT_r invariants:

Theorem 1.2. [BP96] *For M a compact oriented 3-manifold, we have*

$$TV_r(M) = \eta_r^{-\chi(M)} ||RT_r(M)||^2,$$

where $\chi(M)$ is the Euler characteristic of M.

Turaev-Viro invariants are one of the most well-known quantum invariants of 3-manifolds. As such, they are quite efficient at distinguishing 3-manifolds, but their geometric meaning is mysterious. However, the geometry of the manifold can be recovered from their asymptotics, in particular according to the following striking conjecture of Chen and Yang:

Conjecture 1.1. [CY18] *(Chen-Yang's volume conjecture) Let M be a compact oriented 3-manifold. Then*

$$\lim_{r\to\infty,\ r\ odd} \frac{2\pi}{r} \log TV_r(M, e^{\frac{2i\pi}{r}}) = \mathrm{Vol}(M).$$

This conjecture is reminiscent of Kashaev's volume conjecture [Kas97][MM01] which states the following: Let K be a hyperbolic knot, and $J_{K,n}(t)$ be the n-th normalized colored Jones polynomial of K (here, we use the convention that $J_{K,1}(t)$ is trivial for any knot K, that $J_{K,2}(t)$ is the Jones polynomial of K, and $J_{U,n}(t) = 1$ for any n where U is the unknot.) Then Kashaev's volume conjecture is that:

$$\lim_{n\to\infty} \frac{2\pi}{n} \log |J_{K,n}(\exp(\frac{2i\pi}{n})) = \mathrm{Vol}(S^3 \setminus K).$$

However, compared to Kashaev's volume conjecture, it is remarkable that Chen-Yang's conjecture is applicable to any 3-manifolds, while already adapting Kashaev's conjecture to the case of links in S^3 is subtle. The two conjectures nonetheless look very similar, and it is natural to wonder whether they are related. The relationship between the conjectures is elucidated by the following theorem:

Theorem 1.3. [DKY18]*Let L be a link in S^3 with n components and E_L be its exterior. Let $r = 2m+1 \geq 3$, then we have*

$$TV_r(E_L, e^{\frac{2i\pi}{r}}) = 2^{n-1}\eta_r' \sum_{1\leq \underline{i} \leq m} [i]^2 |J_{L,\underline{i}}(e^{\frac{2i\pi}{m+1/2}})|^2,$$

where $J_{L,\underline{i}}$ is the $\underline{i}$-th normalized colored Jones polynomial of L.

This formula is a consequence of Benedetti-Petronio's theorem [BP96] which relates the Turaev-Viro invariants TV_r to the Reshetikhin-Turaev invariants RT_r, and the TQFT properties of the invariants RT_r. Notice that the last term of the sum is very similar to the Kashaev invariant $\tilde{J}_{K,m}(e^{\frac{2i\pi}{m}})$ that appears in Kashaev's volume conjecture. However, it seems that the shift of the order of the root of unity by $1/2$ makes the asymptotic properties more robust: while it is well-known that the unnormalized n-th colored Jones vanishes at $e^{\frac{2i\pi}{m}}$, and that $J_{K,m}(e^{\frac{2i\pi}{m-k}})$ grows at most polynomially for any positive integer k, one observes numerically that:

Conjecture 1.2. *For any link L in S^3, and any integer k, one has*

$$\lim_{m\to+\infty}\frac{2\pi}{m}\log\left|J_{L,m}(e^{\frac{2i\pi}{m+1/2+k}})\right|=\mathrm{Vol}(M).$$

2. A few examples of the conjecture

The formula given in Theorem 1.3 allowed us to verify the first exmaples of Chen-Yang's volume conjecture:

Theorem 2.1. [DKY18][DK20] *The Chen-Yang volume conjecture is true for the complements of the figure eight knot and the borromean rings.*

The above theorem uses also the compact formulae, given by Habiro [Hab00], for the colored Joens polynomials of the figure-eight knot and the Borromean link. For the figure-eight knot $K=4_1$, the formula is:

$$J_{4_1,i}(q)=1+\sum_{j=1}^{i-1}\frac{\{i+j\}!}{\{i\}\{i-j-1\}!},$$

where we set $\{n\}!=\{n\}\{n-1\}\ldots\{1\}$ and

$$\{n\}=q^n-q^{-n}=e^{\frac{2in\pi}{r}}-e^{-\frac{2in\pi}{r}}.$$

These formulae express the colored Jones polynomials in terms of sums of products of quantum factorials. One finds in particular that in the sum given by Theorem 1.3, the last term dominates. Moreover, an important property is that the asymptotics of quantum factorial is connected to the Lobachevski function:

$$\Lambda(x)=-\int_0^x\log|\sin(t)|dt,$$

which appears in the expression of the hyperbolic volume of an ideal hyperbolic tetrahedron. An ideal hyperbolic tetrahedron whose faces have dihedral angles α,β,γ has volume

$$\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma),$$

and for example the figure-eight complement 4_1 whose hyperbolic structure can be obtained by gluing two ideal regular hyperbolic tetrahodra has volume:

$$\mathrm{Vol}(E_{4_1})=2v_3=6\Lambda(\frac{\pi}{3}).$$

The following lemma and Habiro's formulae then allows one to easily compute the asymptotics of the colored Joens polynomials of the figure-eight knot complement and of the Borromean link:

Lemma 2.2. *For any sequence of integers $0\leq n_r\leq\frac{r-1}{2}$, we have*

$$\frac{2\pi}{r}\log|\{n_r\}!|=\Lambda(\frac{2n_r\pi}{r})+O(\frac{\log r}{r}).$$

3. A coarse approach to Chen-Yang's volume conjecture

More examples of the conjecture have been proved, with the most remarkable case being the hyperbolic Dehn-filling of the figure-eight knot by Ohtuski [Oht18] (for integral slopes) and by Wong and Yang [WY22] (general slopes).

Theorem 3.1. [Oht18][WY22] *The Chen-Yang volume conjecture is true for all hyperbolic Dehn-fillings of the figure-eight knot.*

In another direction, one may wonder if the conjecture is stable under some natural geometric operations. One such operation is cabling. Let $C_{p,q}$ be the complement of a torus knot $T_{p,q}$ inside a solid torus. This 3-manifold $C_{p,q}$ can also be considered as a cobordism from T^2 to T^2. The complement of the (p,q)-cabling of a knot K is the 3-manifold $E_K \underset{T^2}{\cup} C_{p,q}$. More generally, let us call M' a (p,q)-cabling of M if M' is obtained from M by gluing $C_{p,q}$ on a toroidal boundary component of M.

Theorem 3.2. [Det19] *Let $n \in \mathbb{Z}$, and let M, M' be two compact oriented* 3*-manifolds with empty or toroidal boundary. If M' is a* $(2, 2n+1)$*-cabling of M, then the Chen-Yang conjecture is true for M if and only if it is true for M'.*

The proof of this theorem uses the TQFT properties of the invariants RT_r. By Benedetti-Petronio's theorem, $TV_r(M) = ||RT_r(M)||^2$, $TV_r(M') = ||RT_r(M')||^2$, and one also has $RT_r(M') = RT_r(C_{2,2n+1})(RT_r(M))$. It is then sufficient to analyze the properties of the cobordism map $RT_r(C_{2,2n+1})$ associated to the cobordism $C_{2,2n+1}$. One can in particular show that $RT_r(C_{2,2n+1})$ is invertible, and that the operators norms of $RT_r(C_{2,2n+1})$ and of $RT_r(C_{2,2n+1})^{-1}$ both grow polynomially in r. Let us also note that if M' is a cabling of M, then $\mathrm{Vol}(M) = \mathrm{Vol}(M')$.

Finally, we remark that Theorem 3.2 was recently generalized to all (p,q)-cabling Kumar and Melby [KM23], with a closely related argument.

Verifying Chen-Yang's volume becomes vastly more difficult and technical as one considers more complicated 3-manifold. However, it is still possible to give general results, and a good motivation for the conjecture, by considering a coarse version of the conjecture. Let us define the upper and lower exponential growth rates of Turaev-Viro invariants by

$$LTV(M) := \limsup_{r\to\infty,\ r \text{ odd}} \frac{2\pi}{r} \log TV_r(M, e^{\frac{2i\pi}{r}})$$

and

$$lTV(M) := \liminf_{r\to\infty,\ r \text{ odd}} \frac{2\pi}{r} \log TV_r(M, e^{\frac{2i\pi}{r}}).$$

One finds that there is a general inequality relating $LTV(M)$ and $\mathrm{Vol}(M)$:

Theorem 3.3. [DK20] *There exists a universal constant $C>0$ such that for any compact oriented 3-manifold M with empty or toroidal boundary, one has*

$$LTV(M) := \limsup_{r\to\infty,\ r \ impair} \frac{2\pi}{r}\log TV_r(M, e^{\frac{2i\pi}{r}}) \leq C\mathrm{Vol}(M)$$

Let us point out that in particular this theorem essentially settles the conjecture for all 3-manifolds with simplicial volume 0 (also called graph manifolds). The proof of the above relies on the observation that the growth rate $LTV(M)$ has many geometric properties reminiscent of the simplicial volume. Indeed we have:

Proposition 3.4. [DK20]

(1) *Let M be a compact oriented 3-manifold with toroidal boundary, and let M' be a 3-manifold obtained from M by Dehn-filling of some of the boudnary components of M. Then:*

$$LTV(M') \leq LTV(M).$$

(2) *Let M be a compact oriented 3-manifold with toroidal boundary, and let T be a 2-torus embedded in M. Then*

$$LTV(M) \leq LTV(M\setminus T).$$

Finally, the same inequalities hold replacing LTV by lTV.

The latter proposition is a consequence of Benedetti-Petronio [BP96] about the connection between Turaev-Viro and Reshetikhin-Turaev invariants, and the TQFT properties of Reshetikhin-Turaev.

Proof sketch of proposition 2.1: Let us prove point (2) in the case where M is closed and T is a separating torus. Denote by M_1 and M_2 the two connected components of $M\setminus T$. Then, by Benedetti-Petronio's theorem, for any $r\geq 3$ odd:

$$TV_r(M, e^{\frac{2i\pi}{r}}) = |RT_r(M, e^{\frac{i\pi}{r}})|^2$$

but furthermore, since RT_r is a $2+1$ TQFT:

$$RT_r(M) = \langle RT_r(M_1), RT_r(M_2)\rangle.$$

Let us note that M_1 and M_2 are 3-manifolds with boundary T^2. Since the natural Hermitian form $\langle,\rangle$ on $RT_r(\Sigma)$ is positive definite when $\Sigma=T^2$, no matter which $2r$-th root of unity is chosen, the required inequality is a consequence of Cauchy-Schwartz's inequality. □

The next ingrédient of the proof of Theorem 3.3 is a majoration of the exponential growth rate of the quantum $6j$-symbols. Indeed one has:

Proposition 3.5. [BDKY22] *Let $(a_1^r, a_2^r, a_3^r, a_4^r, a_5^r, a_6^r)$ be a sequence of admissible 6-uplets, then*

$$\limsup_{r\to\infty}\frac{2\pi}{r}\log\begin{vmatrix} a_1^r & a_2^r & a_3^r \\ a_4^r & a_5^r & a_6^r\end{vmatrix}_{q=e^{\frac{2i\pi}{r}}} \leq v_8,$$

where v_8 is the hyperbolic volume of the regular ideal hyperbolic octahedron.

This inequality is optimal, in the sense that v_8 is indeed the exponential growth rate of the $6j$ symbols where all the a_i^r are chosen so that $\lim\limits_{r\to\infty} \frac{a_i^r}{r} = \frac{1}{2}$. This asymptotic formula for quantum $6j$-symbols also allows to prove the Chen-Yang volume conjecture for an interesting family of hyperbolic knots: the *fundamental shadow links*, which were first introduced by Costantino [Cos07].These are hyperbolic links in a connected sum $\overset{k}{\underset{i=1}{\#}}(S^2 \times S^1)$, with hyperbolic volumes $(k+1)v_8$, and have the following two remarkable properties:

- The family $\mathcal{L}$ of fundamental shadow links is a *universal* family of links, meaning that any 3-manifold with toroidal or empty boundary is a Dehn-filling (with integral slopes) of a link in $\mathcal{L}$.
- The colored Jones polynomials of links in $\mathcal{L}$ are products of quantum $6j$-symbols.

Thanks to the second property and Proposition 3.5, one gets:

Theorem 3.6. [BDKY22] *The Chen-Yang volume conjecture is true for any fundamental shadow link.*

On the other hand, the first property of fundamental shadow links gives rise to a stronger version of Theorem 3.3:

Corollary 3.7. [BDKY22] *Let $\varepsilon > 0$. For a generic 3-manifold M with empty or toroidal boundary, one has:*

$$LTV(M) \leq (1+\varepsilon)\mathrm{Vol}(M).$$

The above corollary, the word *generic* must be understood in the following sense. Any 3-manifold with empty or toroidal boundary is a Dehn surgery on a fundamental shadow link L with k components for some integer k. For any $\varepsilon > 0$, by Thurston's work, there exists an integer n such that the surgery $L(n_1, \dots, n_k)$ has volume $> \frac{1}{1+\varepsilon}\mathrm{Vol}(L)$ if $|n_i| \geq n$. Corollary 3.7 then follows from Proposition 3.4 and Theorem 3.6, where generic 3-manifold means a surgery on a link in $\mathcal{L}$ with sufficiently large slopes. The fact that $\mathcal{L}$ is a universal family of links justifies to call such 3-manifolds "generic".

4. Connection to the AMU conjecture

For M a compact oriented 3-manifold, we will say that M is *q-hyperbolic* if $lTV(M) > 0$. A weak form of Chen-Yang's volume conjecture is the following:

Conjecture 4.1. *(Weak Chen-Yang volume conjecture) A compact oriented 3-manifold is q-hyperbolic if and only if it admits a hyperbolic piece.*

One strength of Proposition 3.4 is that it allows one to construct many q-hyperbolic 3-manifold. Indeed, whenever a 3-manifold M is q-hyperbolic, any link complement in M is also q-hyperbolic. Here M can be taken to be a hyperbolic 3-manifold for which we know the Chen-Yang volume conjecture is true for example.

Using this idea and the theorem of [Oht18] and [WY22] that hyperbolic Dehn-fillings of the figure eight knot verify the Chen-Yang volume conjecture, Kalfagianni and Melby recently verified the weak Chen-Yang volume conjecture for many hyperbolic knots with less than 10 crossings [KM24].

One reason to be interested in the weak form of the Chen-Yang volume conjecture is that it implies another famous conjecture in quantum topology, the AMU conjecture about quantum representations of mapping class groups of surfaces. Recall that a mapping class f of the closed compact oriented surface $\Sigma_{g,n}$ of genus $g \geq 2$ and with n boundary components is either of finite order, reducible (if some power of f fixes a simple closed curves), or a pseudo-Anosov map (else). If a mapping class is reducible, it can be restricted to smaller subsurfaces, and its piece are either finite order or pseudo-Anosov.

Let us write ρ_r for the quantum representation of $\mathrm{Mod}(\Sigma_{g,n})$, coming from the TQFT RT_r, for some odd level $r \geq 3$.

Conjecture 4.2. [AMU06] *A mapping class has a pseudo-Anosov part if and only if $\rho_r(f)$ has infinite order for any large enough r.*

We claim that:

Proposition 4.1. *Let $\Sigma_{g,n}$ be the compact oriented surface of genus g and with n boundary components. For $f \in \mathrm{Mod}(\Sigma_g)$, if M_f, the mapping torus of f satisfies the weak Chen-Yang conjecture, then f satisfies the AMU conjecture.*

Sketch of proof. We will consider the case where $n = 0$ for simplicity. By Benedetti-Petronio's theorem and TQFT properties of RT_r, one has that, if M_f is the mapping torus of a mapping class f, then $TV_r(M_f) = ||\mathrm{Tr}(\rho_r(f))||^2$. If f has a pseudo-Anosov part, then the simplicial volume of M_f is strictly positive and the weak Chen-Yang conjecture implies that $\mathrm{Tr}(\rho_r(f))$ grows exponentially. However, the dimensions of the TQFT spaces $RT_r(\Sigma_g)$ grow only polynomially, so for large enough r the map $\rho_r(f)$ must have an eigenvalue of modulus > 1, and therefore be of infinite order. □

Thanks to Proposition 3.4, one can construct many hyperbolic mapping tori M_f, such that M_f satisfies the weak Chen-Yang conjecture. One approach for this is to construct hyperbolic fibered links in S^3 that have a figure-eight components. In [DK19], we showed that that for any hyperbolic link L in S^3, one can add one component to L to make it hyperbolic and fibered.

Another approach, which is needed to produce examples of the AMU conjecture in surfaces with arbitrary genus and at most one boundary component, was used in [DK22]. The idea is to construct hyperbolic fibered knots in a hyperbolic Dehn-filling of the figure-eight knot, which are q-hyperbolic by [Oht18] and [WY22]. Using the theory of open book decompositions, we showed that in any closed hyperbolic 3-manifold there is a hyperbolic fibered knot. We get the following:

Theorem 4.2. [DK19][DK22] *Let* $\Sigma_{g,n}$ *be a compact oriented surface with* $n \geq 1$ *boundary components and genus* $g \geq 9$ *if* $n = 1$ *or* $g \geq \max(3, n)$ *else. Then there exists a pseudo-Anosov map in* $\mathrm{Mod}(\Sigma_{g,n})$ *that satisfies the AMU conjecture.*

It should be noted DK:AMU that the constructions in [DK19] and [DK22] are fairly flexible and in fact often produce infinite families of independent examples of the AMU conjecture. In particular, the presence of a Stallings twist on the fiber surface of such a hyperbolic fibered link in S^3 allows us to change the monodromy f by a power of a Dehn twist, while still keeping that f satisfies the AMU conjecture.

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Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université Bourgogne Franche-Comté, F-2100 Dijon, France

Email address: renaud.detcherry@u-bourgogne.fr