The cardinal characteristics of the ideal generated by the F_σ measure zero subsets of the reals

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Abstract

Let $\mathcal{E}$ be the ideal generated by the F_σ measure zero subsets of the reals. The purpose of this survey paper is to study the cardinal characteristics (the additivity, covering number, uniformity, and cofinality) of $\mathcal{E}$.

1 Introduction

Let $\mathcal{M}$ and $\mathcal{N}$, as usual, denote the σ-ideal of first category subsets of $\mathbb{R}$ and the σ-ideal of Lebesgue null subsets of $\mathbb{R}$, respectively, and let $\mathcal{E}$ be the ideal generated by the F_σ measure zero subsets of $\mathbb{R}$. It is well-known that $\mathcal{E} \subseteq \mathcal{N} \cap \mathcal{M}$. Even more, it was proved that $\mathcal{E}$ is a proper subideal of $\mathcal{N} \cap \mathcal{M}$ (see [BJ95, Lemma 2.6.1]).

For $f, g \in \omega^\omega$ define

$$f \leq^* g \text{ iff } \exists m < \omega\, \forall n \geq m\colon f(n) \leq g(n).$$

Let

$$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega \text{ and } \neg\exists y \in \omega^\omega\, \forall x \in F\colon x \leq^* y\}$$

the *bounding number*, and let

$$\mathfrak{d} := \min\{|D| : D \subseteq \omega^\omega \text{ and } \forall x \in \omega^\omega\, \exists y \in D\colon x \leq^* y\}$$

the *dominating number*. As usual, $\mathfrak{c} := 2^\omega$ denotes the *size of the continuum*.

Let $\mathcal{I}$ be an ideal of subsets of X such that $\{x\} \in \mathcal{I}$ for all $x \in X$. Throughout this paper, we demand that all ideals satisfy this latter requirement. We introduce the following four

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cardinal characteristics associated with $\mathcal{I}$:

$$\begin{aligned}
\operatorname{add}(\mathcal{I}) &= \min\left\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} \notin \mathcal{I}\right\}, \\
\operatorname{cov}(\mathcal{I}) &= \min\left\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} = X\right\}, \\
\operatorname{non}(\mathcal{I}) &= \min\{|A| : A \subseteq X,\ A \notin \mathcal{I}\}, \text{ and} \\
\operatorname{cof}(\mathcal{I}) &= \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I},\ \forall A \in \mathcal{I}\ \exists B \in \mathcal{J} \colon A \subseteq B\}.
\end{aligned}$$

These cardinals are referred to as the *additivity, covering, uniformity* and *cofinality of* $\mathcal{I}$, respectively. The relationship between the cardinals defined above is illustrated in Figure 1.

$$\begin{tikzcd}
 & & \operatorname{cov}(\mathcal{I}) \arrow[r] \arrow[ddr] & \operatorname{cof}(\mathcal{I}) \arrow[dr] & \\
\aleph_0 \arrow[r] & \operatorname{add}(\mathcal{I}) \arrow[ur] \arrow[dr] & & & |\mathcal{I}| \\
 & & \operatorname{non}(\mathcal{I}) \arrow[r] \arrow[uur] & |X| \arrow[ur] &
\end{tikzcd}$$

Figure 1: Diagram of the cardinal characteristics associated with $\mathcal{I}$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$.

Over the years, research on the relationship between the cardinal characteristics associated with $\mathcal{E}$ and other classical cardinal characteristics of the continuum has been conducted by many people, e.g. [Mil81, BS92, BJ95, Bre99, Car23, CM23, GM23]. Most of them are illustrated in Figure 2, except for those that appear in Theorem 1.2:

Theorem 1.1 ([BS92], see also [BJ95, Thm. 2.6.9])**.**

(1) $\max\{\operatorname{cov}(\mathcal{M}), \operatorname{cov}(\mathcal{N})\} \leq \operatorname{cov}(\mathcal{E}) \leq \max\{\mathfrak{d}, \operatorname{cov}(\mathcal{N})\}$.

(2) $\min\{\mathfrak{b}, \operatorname{non}(\mathcal{N})\} \leq \operatorname{non}(\mathcal{E}) \leq \min\{\operatorname{non}(\mathcal{M}), \operatorname{non}(\mathcal{N})\}$.

(3) $\operatorname{add}(\mathcal{E}) = \operatorname{add}(\mathcal{M})$ *and* $\operatorname{cof}(\mathcal{E}) = \operatorname{cof}(\mathcal{M})$.

See [Bla10] for the definitions of the following cardinals. Denote by

- $\mathfrak{e}$ the *evasion number*.
- $\mathfrak{r}$ the *reaping number*.
- $\mathfrak{s}$ the *splitting number*.

Theorem 1.2.

(1) [BJ95, Lem. 7.4.3] $\mathfrak{s} \leq \operatorname{non}(\mathcal{E})$ *and* $\operatorname{cov}(\mathcal{E}) \leq \mathfrak{r}$.

(2) [Bre95] $\mathfrak{e} \leq \operatorname{non}(\mathcal{E})$.

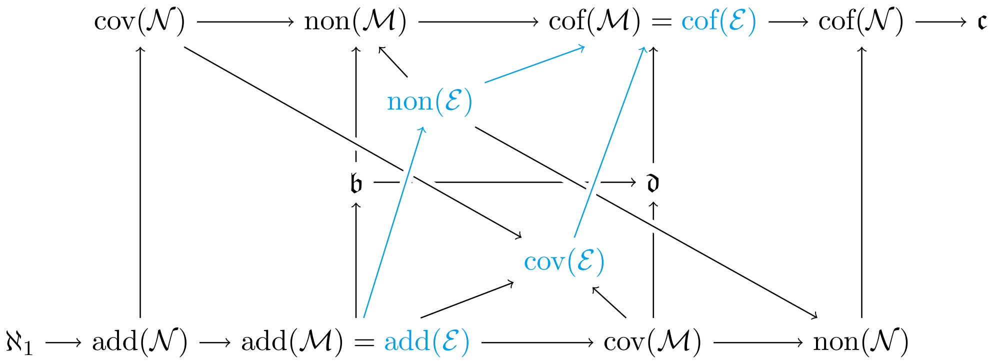
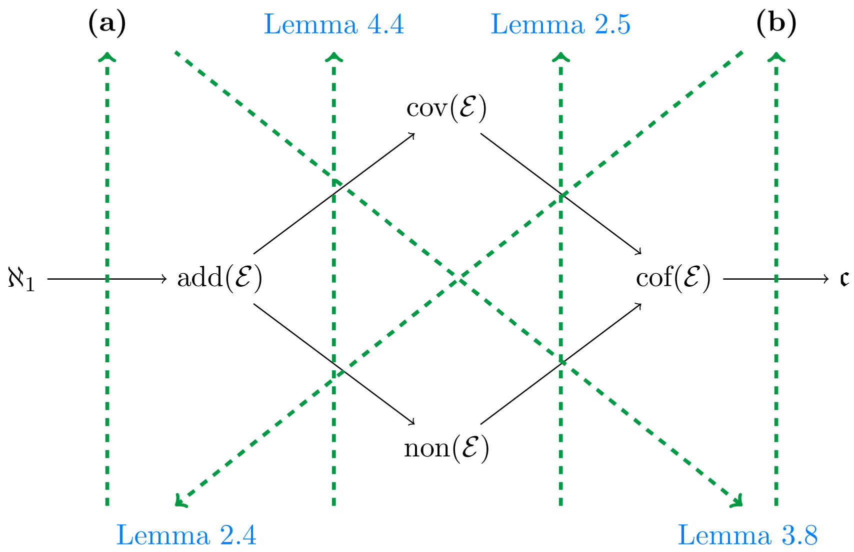


Figure 2: Cichoń's diagram and the cardinal characteristics associated with $\mathcal{E}$.

The goal of this survey paper is to prove that the next diagram of the cardinal characteristics of $\mathcal{E}$ can be distinguished at each position by constructing a model using the forcing technique. In this diagram, a dotted line means that we can obtain a model in which the cardinal characteristics of the left side are strictly smaller than the cardinal characteristics of the right side. The result attached to a dotted line is the lemma in which the consistency of the inequality is proved.



In this diagram, there are two results that we won't be proved. These are represented by (a) and (b) in the above diagram. (a) is the consistency of $\aleph_1 < \mathrm{add}(\mathcal{E})$ and (b) is the consistency of $\mathrm{cof}(\mathcal{E}) < \mathfrak{c}$. Both consistencies hold in the model $\mathbb{D}_\pi$, which is obtained by a FS iteration of length $\pi = \lambda\kappa$ of Hechler forcing where $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular (see e.g. [Mej13, Thm. 5]). There, $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \kappa$ and $\mathfrak{c} = \lambda$, so by using Theorem 1.1, $\mathbb{D}_\pi$ forces $\mathrm{add}(\mathcal{E}) = \mathrm{cof}(\mathcal{E}) = \kappa$.

Notation. A *forcing notion* is a pair $\langle \mathbb{P}, \leq \rangle$ where $\mathbb{P} \neq \emptyset$ and $\leq$ is a relation on $\mathbb{P}$ satisfying reflexivity and transitivity. We also use the expression pre-ordered set (abbreviated p.o. set or just poset) to refer to a forcing notion. The elements of $\mathbb{P}$ are called conditions and we say that a condition q is *stronger* than a condition p if $q \leq p$.

Definition 1.3. Let $\mathbb{P}$ be a forcing notion.

(1) Say that $p, q \in \mathbb{P}$ are *compatible* (in $\mathbb{P}$), denoted by $p \parallel_{\mathbb{P}} q$, if $\exists r \in \mathbb{P} \colon r \leq p$ and $r \leq q$. Say that $p, q \in \mathbb{P}$ are *incompatible* (in $\mathbb{P}$) if they are not compatible in $\mathbb{P}$, which is denoted by $p \perp_{\mathbb{P}} q$.

When $\mathbb{P}$ is clear from the context, we just write $p \parallel q$ and $p \perp q$.

(2) Say that $A \subseteq \mathbb{P}$ is an *antichain* if $p, q \in \mathbb{P} \colon p \neq q \Rightarrow p \perp q$. A is a *maximal antichain* on $\mathbb{P}$ iff A is an antichain and $\forall p \in \mathbb{P}\, \exists q \in A \colon p \parallel_{\mathbb{P}} q$.

(3) Say that $D \subseteq \mathbb{P}$ is *dense* (in $\mathbb{P}$) if $p \in \mathbb{P}\, \exists q \in D \colon q \leq p$.

(4) Say that $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-*filter* if it satisfies

  (a) $G \neq \emptyset$;

  (b) for all $p, q \in G$ there is some $r \in G$ such that $r \leq p$ and $r \leq q$; and

  (c) if $p \in \mathbb{P}$, $q \in G$ and $q \leq p$, then $p \in G$.

(5) Let $\mathcal{D}$ be a family of dense subsets of $\mathbb{P}$. Say that $G \subsetneq \mathbb{P}$ is $\mathbb{P}$-*generic over* V if G is a $\mathbb{P}$-filter and $\forall D \in \mathcal{D} \cap V \colon G \cap D \neq \emptyset$. Denote by $\dot{G}_{\mathbb{P}}$ the canonical name of the generic set. When $\mathbb{P}$ is clear from the context, we just write $\dot{G}$.

Fact 1.4 ([Gol93]). *Let $\mathbb{P}$ be a forcing notion. Let $p, q \in \mathbb{P}$.*

(1) *$p \perp q$ iff $q \Vdash p \notin \dot{G}$.*

(2) *$G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic over V iff for every maximal antichain $A \in V$, $|G \cap A| = 1$.*

Let I be a set. Denote by $\mathbb{C}_I$ be the poset that adds Cohen reals indexed by I.

Here, as usual, given a formula ϕ, $\forall^\infty n < \omega \colon \phi$ means that all but finitely many natural numbers satisfy ϕ; $\exists^\infty n < \omega \colon \phi$ means that infinitely many natural numbers satisfy ϕ

2 The consistency of $\mathrm{cov}(\mathcal{E}) < \mathrm{non}(\mathcal{E})$ and $\mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) < \mathrm{cof}(\mathcal{E})$

Throughout this section, assume that CH holds in the ground model V.

One of the fundamental properties of forcing is the Laver property:

> A forcing notion $\mathbb{P} \in V$ has the *Laver property* if for any $\mathbb{P}$-generic G over V, any function $f \in \omega^\omega \cap V$ and any $\mathbb{P}$-name $\dot{g}$ for a member in ω^ω such that $\Vdash \dot{g} \leq^* f$, there exists a function $\varphi \in ([\omega]^{<\omega})^\omega \cap V$ such that $\Vdash \dot{g}(n) \in \varphi(n)$ and $|\varphi(n)| \leq n + 1$ for every $n \in \omega$.

Example 2.1.

(1) Mathias forcing (see e.g. [BJ95, Sec. 7.4A]).

(2) Miller forcing (see e.g. [BJ95, Sec. 7.3E]).

(3) Laver forcing (see e.g. [BJ95, Sec. 7.3D]).

We now show that any poset with the Laver property preserves covering families of the ideal $\mathcal{E}$. The following is a combinatorial consequence of the Laver property.

Lemma 2.2. *Assume that $\mathbb{P}$ has the Laver property. Let $\dot{x}$ be a name for a real in 2^ω, and $p \in \mathbb{P}$. Then there are $q \leq p$ and a closed null set $C \subseteq 2^\omega$ such that $q \Vdash \dot{x} \in C$.*

Proof. Let $\langle I_n : n \in \omega\rangle$ be a partition of ω into finite intervals such that $\min(I_0) = 0$, $\max(I_n) + 1 = \min(I_{n+1})$, and $|I_n| = 2n + 1$. Define a name $\dot{h}$ for a function with domain ω such that the trivial condition forces $\dot{h}(n) = \dot{x}{\restriction}I_n$. In particular, $\dot{h}(n)$ is forced to be an element of 2^{I_n}. By the Laver property, there are $q \leq p$ in $\mathbb{P}$ and $\varphi \in \prod_{n<\omega} \mathcal{P}(2^{I_n})$ such that $|\varphi(n)| \leq 2^n$ and $q \Vdash \dot{h}(n) \in \varphi(n)$ for all $n < \omega$. Hence $q \Vdash \dot{x}{\restriction}I_n \in \varphi(n)$ for all n, so let $C := \{y \in 2^\omega : \forall n < \omega \colon y{\restriction}I_n \in \varphi(n)\}$ (coded in the ground model). It is not hard to prove that C is a closed set, so it remains to see that C has measure zero set. Indeed,

$$\begin{aligned}
\mathrm{Lb}(C) &= \prod_{n\in\omega} \mathrm{Lb}\big(\{y \in 2^\omega : y{\restriction}I_n \in \varphi(n)\}\big)\\
&= \prod_{n\in\omega} \frac{|\varphi(n)|}{2^{|I_n|}}\\
&= \prod_{n\in\omega} \frac{2^n}{2^{|I_n|}} \leq \prod_{n\in\omega} \frac{2^n}{2^{2n+1}} = \prod_{n\in\omega} \frac{1}{2^{n+1}} = 0.
\end{aligned}$$

By Lb we denote Lebesgue measure zero on 2^ω. Since $q \Vdash \dot{x} \in C$, we are done. □

Employing the earlier result:

Corollary 2.3. *Assume that $\mathbb{P}$ has the Laver property. Then $\mathbb{P}$ preserves covering families of the ideal $\mathcal{E}$.*

As a direct consequence, we obtain:

Lemma 2.4. *Let $\mathbb{M}_{\omega_2}$ be a CS (countable support) iteration of length ω_2 of Mathias forcing. Then in $V^{\mathbb{M}_{\omega_2}}$, $\mathrm{add}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \aleph_1$, and $\mathrm{non}(\mathcal{E}) = \mathrm{cof}(\mathcal{E}) = \aleph_2$.*

Proof. It is well-known that $\mathbb{M}_{\omega_2}$ forces $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{s} = \mathfrak{b} = \mathfrak{d} = \mathrm{non}(\mathcal{E}) = \aleph_2 = \mathfrak{c}$. (see e.g. [BJ95, Sec. 7.4A]). In addition to this, since $\mathfrak{s} \leq \mathrm{non}(\mathcal{E})$ by Theorem 1.2, we have that $\mathbb{M}_{\omega_2}$ forces $\mathrm{non}(\mathcal{E}) = \aleph_2$ and $\mathbb{M}_{\omega_2}$ forces $\mathrm{cov}(\mathcal{E}) = \aleph_1$ thanks to Example 2.1 (1) and Corollary 2.3. □

Lemma 2.5. *Let $\mathbb{MI}_{\omega_2}$ be a CS iteration of length ω_2 of Miller forcing. Then, in $V^{\mathbb{MI}_{\omega_2}}$, $\mathrm{add}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \mathrm{non}(\mathcal{E}) = \aleph_1$, and $\mathrm{cof}(\mathcal{E}) = \aleph_2$.*

Proof. It is well-known that $\mathbb{MI}_{\omega_2}$ forces $\mathfrak{b} = \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$ and $\mathfrak{d} = \aleph_2$ (see e.g. [BJ95, Sec. 7.3E]). Additionally, since $\mathfrak{b} = \mathrm{non}(\mathcal{M}) = \aleph_1$, by using Theorem 1.1, we get that $\mathrm{non}(\mathcal{E}) = \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$, so $\mathbb{MI}_{\omega_2}$ forces $\mathrm{non}(\mathcal{E}) = \aleph_1$. On the other hand, by using Example 2.1 (2) and Corollary 2.3, $\mathbb{MI}_{\omega_2}$ forces $\mathrm{cov}(\mathcal{E}) = \aleph_1$. Lastly, since $\mathrm{cof}(\mathcal{E}) = \mathrm{cof}(\mathcal{M})$ by Theorem 1.1, $\mathbb{MI}_{\omega_2}$ forces $\mathrm{cof}(\mathcal{E}) = \aleph_2$. □

In comparison to Lemma 2.4, the upcoming result is stronger, since it can be forced $\mathrm{non}(\mathcal{E}) > \mathfrak{d}$.

Lemma 2.6. *Let $\mathbb{P}$ be a CS iteration of length ω_2 of the tree forcing from [Bre99, Lem. 2]. Then, in $V^{\mathbb{P}}$, $\mathfrak{d} = \mathrm{cov}(\mathcal{E}) = \aleph_1$ and $\mathrm{non}(\mathcal{E}) = \aleph_2$.*

Proof. We just are going to give only a brief outline of the proof. Details can be found in the references. The forcing from [Bre99, Lem. 2] we iterate belongs to a class of forcing notions introduced by Shelah [She92] (see also [BJ95, Sec. 7.3B]). This forcing is ω^ω-bounding and does not add random reals. Furthermore, Brendle proved that this forcing increases $\mathrm{non}(\mathcal{E})$ (see [Bre99, Lem. 2]). We therefore have that, in $V^{\mathbb{P}}$, $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ and $\mathrm{non}(\mathcal{E}) = \aleph_2$. Since $\mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \aleph_1$, by applying Theorem 1.1 $\mathrm{cov}(\mathcal{E}) = \aleph_1$. ☐

3 The consistency of $\mathrm{non}(\mathcal{E}) < \mathrm{cov}(\mathcal{E})$

Before delving into the specifics, we provide all of the necessary to prove the consistency of $\mathrm{non}(\mathcal{E}) < \mathrm{cov}(\mathcal{E})$. This section is based on [CM23, Sec. 7] and [Car23, Sec. 5].

Definition 3.1. For an increasing function $b \in \omega^\omega$, define the following sets of slaloms:

(1) $\mathcal{S}_b := \{\varphi \in \prod_{n<\omega} \mathcal{P}(b(n)) : \forall\, n \in \omega \colon \frac{|\varphi(n)|}{2^{b(n+1)-b(n)}} \leq \frac{1}{2^n}\}$.

(2) $\mathcal{S}^{\mathrm{w}}_b := \{\varphi \in \prod_{n<\omega} \mathcal{P}(b(n)) : \exists^\infty\, n \in \omega \colon \frac{|\varphi(n)|}{2^{b(n+1)-b(n)}} \leq \frac{1}{2^n}\}$.

Note that $\mathcal{S}_b \subseteq \mathcal{S}^{\mathrm{w}}_b$ and that

$$\varphi \in \mathcal{S}^{\mathrm{w}}_b \text{ implies } \liminf_{n\to\infty} \frac{|\varphi(n)|}{|b(n)|} < 1,$$
$$\varphi \in \mathcal{S}_b \text{ implies } \limsup_{n\to\infty} \frac{|\varphi(n)|}{|b(n)|} < 1.$$

Also observe that, if b is a function with the domain ω such that $b(i) \neq \varnothing$ for all $i < \omega$, and $h \in \omega^\omega$, then

$$(\otimes) \qquad \liminf_{n\to\infty} \frac{h(n)}{|b(n)|} < 1 \text{ iff } \forall\, m < \omega \colon \prod_{n\geq m} \frac{h(n)}{|b(n)|} = 0,$$
$$\limsup_{n\to\infty} \frac{h(n)}{|b(n)|} < 1 \text{ iff } \forall\, A \in [\omega]^{\aleph_0} \colon \prod_{n\in A} \frac{h(n)}{|b(n)|} = 0.$$

We can construct sets in $\mathcal{E}$ using slaloms in the following way.

Lemma 3.2. *Let $\tilde{b} \in \omega^\omega$ be increasing and $b(n) := 2^{\tilde{b}(n+1)-\tilde{b}(n)}$ for all $n \in \omega$. If $\varphi \in \prod_{n<\omega} \mathcal{P}(b(n))$ and $\liminf_{n\to\infty} \frac{|\varphi(n)|}{|b(n)|} < 1$, then the set*

$$H_{\tilde{b},\varphi} := \{x \in 2^\omega : \forall^\infty\, n \in \omega \colon x{\restriction}[\tilde{b}(n), \tilde{b}(n+1)) \in \varphi(n)\}$$

belongs to $\mathcal{E}$.

Proof. Notice that $H_{\tilde{b},\varphi}$ is a countable union of the closed null sets

$$B^m_{\tilde{b},\varphi} := \{x \in 2^\omega : \forall\, n \geq m \colon x{\restriction}[\tilde{b}(n), \tilde{b}(n+1)) \in \varphi(n)\} \text{ for } m \in \omega.$$

Indeed,

$$\begin{aligned}\mathrm{Lb}(B^m_{\tilde{b},\varphi}) &= \prod_{n \geq m} \mathrm{Lb}\left(\{x \in 2^\omega : x{\restriction}[\tilde{b}(n), \tilde{b}(n+1)) \in \varphi(n)\}\right)\\ &= \prod_{n\geq m} \frac{|\varphi(n)|}{2^{\tilde{b}(n+1)-\tilde{b}(n)}} = 0,\end{aligned}$$

where the latter equality holds by ($\otimes$). Hence $\mathrm{Lb}(B^m_{\tilde{b},\varphi}) = 0$ for all $m < \omega$, so

$$H_{\tilde{b},\varphi} = \bigcup_{m<\omega} B^m_{\tilde{b},\varphi}$$

is an F_σ null set and thus belongs to $\mathcal{E}$. $\square$

Corollary 3.3. *Let $b \in \omega^\omega$ be increasing. If $\varphi \in \mathcal{S}^{\mathrm{w}}_b$ then $H_{b,\varphi} \in \mathcal{E}$.*

Thanks to the foregoing lemma, we obtain a basis of $\mathcal{E}$.

Lemma 3.4 ([BS92, Thm. 4.3])**.** *Suppose that $C \in \mathcal{E}$. Then there is some increasing $\tilde{b} \in \omega^\omega$ and some $\varphi \in \mathcal{S}_{\tilde{b}}$ such that $C \subseteq H_{\tilde{b},\varphi}$.*

Proof. Let us assume wlog that $C \subseteq 2^\omega$ is a null set of type F_σ. Then C can be written as $\bigcup_{n\in\omega} C_n$ where $\langle C_n : n \in \omega\rangle$ is an increasing family of closed sets of measure zero. Note that each C_n is a compact set. It is easy to see that, if $K \subseteq 2^\omega$ is a compact null set, then

$$\forall\, \varepsilon > 0\ \forall^\infty n\ \exists\, T \subseteq 2^n \colon K \subseteq [T] \text{ and } \frac{|T|}{2^n} < \varepsilon$$

where $[T] := \bigcup_{t\in T}[t]$. Hence, we can define an increasing $\tilde{b} \in \omega^\omega$ by $\tilde{b}(0) := 0$ and

$$\tilde{b}(n+1) := \min\{m > \tilde{b}(n) : \exists\, T_n \subseteq 2^m \colon C_n \subseteq [T_n] \text{ and } \frac{|T_n|}{2^m} < \frac{1}{4^{\tilde{b}(n)}}\} \text{ for } n > 1.$$

Next, choose $T_n \subseteq 2^{\tilde{b}(n+1)}$ such that $C_n \subseteq [T_n]$ and $\frac{|T_n|}{2^{\tilde{b}(n+1)}} < \frac{1}{4^{\tilde{b}(n)}}$. Now define, for $n \in \omega$,

$$\varphi(n) := \{s{\restriction}[\tilde{b}(n), \tilde{b}(n+1)) : s \in T_n\}.$$

It is clear that $|\varphi(n)| \leq |T_n|$ for every $n < \omega$, hence

$$\frac{|\varphi(n)|}{2^{\tilde{b}(n+1)-\tilde{b}(n)}} < \frac{1}{2^{\tilde{b}(n)}} \leq \frac{1}{2^n}.$$

Thus, $\varphi \in \mathcal{S}_{\tilde{b}}$ and, by Corollary 3.3, $H_{\tilde{b},\varphi} \in \mathcal{E}$. We also have $C \subseteq H_{\tilde{b},\varphi}$. $\square$

The following is a technical lemma that connects the structure of ω^ω with $\mathcal{E}$. Before stating it, for each increasing $f \in \omega^\omega$ define the increasing function $f^* \colon \omega \to \omega$ such that $f^*(0) = 0$ and $f^*(n+1) = f(f^*(n)+1)$ for $n > 0$.

Lemma 3.5 ([CM23, Lem. 7.5])**.** *Let $b_0, b_1 \in \omega^\omega$ be increasing functions, let $b_1^* \in \omega^\omega$ be as above, and let $\varphi \in \mathcal{S}_{b_0}$.*

(1) *If $b_1 \not\leq^* b_0$ then there is some $\varphi^* \in \mathcal{S}^{\mathrm{w}}_{b_1^*}$ such that $H_{b_0,\varphi} \subseteq H_{b_1^*,\varphi^*}$.*

(2) *If $b_0 \leq^* b_1$ then there is some $\varphi_* \in \mathcal{S}_{b_1^*}$ such that $H_{b_0,\varphi} \subseteq H_{b_1^*,\varphi_*}$.*

As a result of the previous, we infer:

Lemma 3.6 ([BS92, Lem. 5.1])**.** *Suppose that $\mathbb{P}$ is a ccc forcing notion. Let $\dot{C}$ be a $\mathbb{P}$-name for a member of $\mathcal{E}$.*

(1) *If $\mathbb{P}$ does not add dominating reals, then there exists $b \in \omega^\omega \cap V$ and a $\mathbb{P}$-name $\dot{\varphi}$ such that $\dot{\varphi} \in \mathcal{S}^{\mathrm{w}}_b$ and $\Vdash$ "$\dot{C} \subseteq H_{b,\dot{\varphi}}$".*

(2) *If $\mathbb{P}$ is ω^ω-bounding, then there exists $b \in \omega^\omega \cap V$ and a $\mathbb{P}$-name $\dot{\varphi}$ such that $\dot{\varphi} \in \mathcal{S}_b$ and $\Vdash$ "$\dot{C} \subseteq H_{b,\dot{\varphi}}$".*

The previous lemma is essential for proving the consistency of $\mathrm{non}(\mathcal{E}) < \mathrm{cov}(\mathcal{E})$.

Lemma 3.7 ([BS92, Thm. 5.3], see also [Car23, Thm. 5.2])**.** *Assume that $\mathbb{B}$ is a complete Boolean algebra with a strictly positive σ-additive measure μ and let $\dot{C}$ be a $\mathbb{P}$-name of a closed measure zero subset of 2^ω. Then there is a closed measure zero subset C^* of 2^ω (in the ground model) such that*

$$\forall x \in 2^\omega \smallsetminus C^* \colon \ \Vdash x \notin \dot{C}.$$

Proof. By employing Lemma 3.6, choose $b \in \omega^\omega$ in V and a $\mathbb{B}$-name $\dot{\varphi}$ of a member of $\mathcal{S}_b$ such that $\Vdash_{\mathbb{B}} \dot{C} \subseteq H_{b,\dot{\varphi}}$. For $s \in 2^{[b(n),b(n+1))}$ set $B_{n,s} := [\![s \in \dot{\varphi}(n)]\!] \in \mathbb{B}$.

Now, for $n < \omega$, define $\psi(n)$ by

$$\psi(n) := \left\{ s \in 2^{[b(n),b(n+1))} : \mu(B_{n,s}) \geq 2^{-\lfloor \frac{n}{2} \rfloor} \right\}.$$

We claim that

$$\frac{|\psi(n)|}{2^{b(n+1)-b(n)}} \leq 2^{-\lfloor \frac{n}{2} \rfloor}.$$

Suppose that for some $n_0 \in \omega$,

$$\frac{|\psi(n_0)|}{2^{b(n_0+1)-b(n_0)}} > 2^{-\lfloor \frac{n_0}{2} \rfloor}.$$

For $S \subseteq \psi(n)$ set

$$i(S) := \max\left\{ |X| : X \subseteq S,\ \mu\Big(\bigwedge_{s \in X} B_{n,s} \Big) > 0 \right\}.$$

By [Kel59, Prop. 1],

$$2^{-\lfloor \frac{n}{2} \rfloor} \leq \inf\{\mu(B_{n,s}) : s \in \psi(n)\} \leq \inf\left\{ \frac{i(S)}{|S|} : \emptyset \subsetneq S \subseteq \psi(n) \right\},$$

in particular,

$$2^{b(n_0+1)-b(n_0)-n_0} < \frac{|\psi(n_0)|}{2^{\lfloor \frac{n_0}{2} \rfloor}} \leq i(\psi(n_0)).$$

Choose $X \subseteq \psi(n_0)$ such that $|X| > 2^{b(n_0+1)-b(n_0)-n_0}$ and $\mu\left(\bigwedge_{s\in X} B_{n_0,s}\right) > 0$. Hence, $\bigwedge_{s\in X} B_{n_0,s} \Vdash X \subseteq \dot{\varphi}(n_0)$, so $\bigwedge_{s\in X} B_{n_0,s} \Vdash |\dot{\varphi}(n_0)| > 2^{b(n_0+1)-b(n_0)-n_0}$, which is a contradiction.

Thus

$$C^* := \{x \in 2^\omega : \forall^\infty n \in \omega \colon x{\restriction}[b(n), b(n+1)) \in \psi(n)\}$$

is a member of $\mathcal{E}$ by Lemma 3.2.

To end the proof, let us argue that if $x \notin C^*$, then $\Vdash_{\mathbb{B}} x \notin \dot{C}$. Suppose that $x \notin C^*$. Towards a contradiction assume that $p \Vdash_{\mathbb{B}} x \in \dot{C}$ for some $p \in \mathbb{B}$. Since $\Vdash_{\mathbb{B}} \dot{C} \subseteq H_{b,\dot{\varphi}}$, we can assume wlog that there is some $m \in \omega$ such that $p \Vdash \forall n \geq m \colon x{\restriction}[b(n), b(n+1)) \in \dot{\varphi}(n))$ and $\mu(p) > 2^{-m}$.

On the other hand, since $x \notin C^*$, we can find an $n \geq 2m$ such that $x{\restriction}[b(n), b(n+1)) \notin \psi(n)$. In particular,

$$\mu(B_{n, x{\restriction}[b(n),b(n+1)}) < \frac{1}{2^m}.$$

Now define $q := p \smallsetminus B_{n,x{\restriction}[b(n),b(n+1))}$. Then $\mu(q) > 0$ and $q \Vdash x{\restriction}[b(n), b(n+1)) \notin \dot{\varphi}(n)$, which is a contradiction. □

We are ready to prove the consistency of $\mathrm{non}(\mathcal{E}) < \mathrm{cov}(\mathcal{E})$.

Lemma 3.8. *Assume $\aleph_1 \leq \nu \leq \lambda = \lambda^{\aleph_0}$ with ν regular. Let $\mathbb{B}_\pi$ be a FS iteration of random forcing of length $\pi = \lambda\nu$. Then, in $V^{\mathbb{B}_\pi}$, $\mathrm{non}(\mathcal{E}) = \mathfrak{b} = \aleph_1$, $\mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \nu$, and $\mathrm{cov}(\mathcal{E}) = \mathfrak{d} = \lambda$.*

Proof. $\mathbb{B}_\pi$ forces $\mathfrak{b} = \aleph_1$, $\mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \nu$, and $\mathfrak{d} = \lambda$ is a well-known fact (see e.g [Car23, Thm. 5.4]). Furthermore, by employing Lemma 3.7, we can ensure that $\mathbb{B}_\pi$ forces $\mathrm{non}(\mathcal{E}) = \aleph_1$ and $\mathrm{cov}(\mathcal{E}) = \lambda$. □

As an immediate consequence of the foregoing, we obtain as well:

Corollary 3.9 ([BS92, Thm. 5.5 and 5.6])**.** *It is consistent with ZFC that $\mathrm{non}(\mathcal{E}) < \min\{\mathrm{non}(\mathcal{N}), \mathrm{non}(\mathcal{M})\}$ and $\mathrm{cov}(\mathcal{E}) > \max\{\mathrm{cov}(\mathcal{N}), \mathrm{cov}(\mathcal{M})\}$.*

We close this section by displaying another model where the consistency of $\mathrm{non}(\mathcal{E}) < \mathrm{cov}(\mathcal{E})$ holds.

Theorem 3.10 (see e.g [Bre09])**.** *Let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$. Then $\mathbb{C}_\lambda$ forces $\mathrm{non}(\mathcal{M}) = \aleph_1$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$ (Figure 3). In particular, $\mathbb{C}_\lambda$ forces $\mathrm{non}(\mathcal{E}) = \aleph_1$ and $\mathrm{cov}(\mathcal{E}) = \lambda$.*

$$
\begin{tikzcd}
 & \operatorname{cov}(\mathcal{N}) \arrow[r] & \operatorname{non}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & {} \arrow[r, phantom, "\aleph_1"] & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] \arrow[r, phantom, "\lambda"] & {} & \\
\aleph_1 \arrow[r] & \operatorname{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \operatorname{add}(\mathcal{M}) \arrow[r] \arrow[u] & \operatorname{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \operatorname{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

Figure 3: The constellation of Cichoń's diagram after adding $\lambda = \lambda^{\aleph_0}$ many Cohen reals.

4 The consistency of $\operatorname{add}(\mathcal{E}) < \operatorname{non}(\mathcal{E})$

In this section, we shall use a FS iteration of (b, h)-localization forcing (denoted by $\mathbb{LOC}_{b,h}$) proposed by Brendle and Mejía [BM14] to prove the consistency of $\operatorname{add}(\mathcal{E}) < \operatorname{non}(\mathcal{E}) = \operatorname{cov}(\mathcal{E})$. We first start by introducing $\mathbb{LOC}_{b,h}$ that we want to iterate.

Definition 4.1. Given a sequence of non-empty sets $b = \langle b(n) : n \in \omega \rangle$ and $h \colon \omega \to \omega$, define

$$
\begin{aligned}
\prod b &:= \prod_{n \in \omega} b(n), \\
\mathcal{S}(b, h) &:= \prod_{n \in \omega} [b(n)]^{\leq h(n)}.
\end{aligned}
$$

For two functions $x \in \prod b$ and $\varphi \in \mathcal{S}(b, h)$, we define

$$x \in^* \varphi \text{ iff } \forall^\infty n \in \omega \colon x(n) \in \varphi(n).$$

We define the *localization cardinals*

$$
\begin{aligned}
\mathfrak{b}^{\mathrm{Lc}}_{b,h} &= \min\{|F| : F \subseteq \prod b \;\&\; \neg\exists \varphi \in \mathcal{S}(b, h)\, \forall x \in F \colon x \in^* \varphi\}, \\
\mathfrak{d}^{\mathrm{Lc}}_{b,h} &= \min\{|G| : G \subseteq \mathcal{S}(b, h) \;\&\; \forall x \in \prod b\, \exists \varphi \in G \colon x \in^* \varphi\}.
\end{aligned}
$$

Using the notion of Definition 4.1, we define the following poset meant to increase $\operatorname{non}(\mathcal{E})$.

Definition 4.2. Fix b and h as in Definition 4.1. Define the localization forcing $\mathbb{LOC}_{b,h}$ by

$$\mathbb{LOC}_{b,h} := \{(p, n) : p \in \mathcal{S}(b, h),\ n < \omega \text{ and } \exists m < \omega\, \forall i < \omega \colon |p(i)| \leq m\},$$

ordered by $(p', n') \leq (p, n)$ iff $n \leq n'$, $p' {\restriction} n = p$, and $\forall i < \omega \colon p(i) \subseteq p'(i)$.

Let $G \subseteq \mathbb{LOC}_{b,h}$ be a $\mathbb{LOC}_{b,h}$-generic filter over V. In $V[G]$, define the generic slalom by

$$\dot{\varphi}_{\mathrm{gn}}(i) = \bigcup\{p(i) : (p, n) \in G\},$$

and for any $x \in \prod b \cap V$, there is some $n \in \omega$ such that for any $m \geq n$, $x(m) \in \dot{\varphi}_{\mathrm{gn}}(m)$. Consequently, $\mathbb{LOC}_{b,h}$ increases $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$.

$\mathbb{LOC}_{b,h}$ is σ-m-linked for all $2 \leq m < \omega$ when h diverges to infinity and $b(n)$ is countable for all $n < \omega$.

Why does $\mathbb{LOC}_{b,h}$ increase $\mathrm{non}(\mathcal{E})$? Because there exists a connection between $\mathrm{non}(\mathcal{E})$ and $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$. Indeed, we have:

Lemma 4.3 ([CM23, Fact. 7.7 and Lem. 7.8])**.** *If $b \in \omega^\omega$ and $\limsup_{n\to\infty} \frac{h(n)}{b(n)} < 1$, then $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{non}(\mathcal{E})$.*

We now prove the consistency of $\mathrm{add}(\mathcal{E}) < \mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E})$.

Lemma 4.4. *Assume $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular. Let $\mathbb{P}_\pi$ be a FS iteration of the (b, h)-localization forcing of length $\pi = \lambda\kappa$. Then, in $V^{\mathbb{P}_\pi}$, $\mathfrak{b} = \aleph_1$, $\mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \kappa$, and $\mathfrak{c} = \lambda$.*

Proof. In V, find increasing functions h, b in ω^ω such that $\limsup_{n\to\infty} \frac{h(n)}{b(n)} < 1$, so by Lemma 4.3 $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{d}^{\mathrm{Lc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{Lc}}_{b,h} \leq \mathrm{non}(\mathcal{E})$. In view of [CM22, Lem. 2.9], $\mathbb{P}_\pi$ forces $\mathrm{non}(\mathcal{E}) = \mathfrak{b}^{\mathrm{Lc}}_{b,h} = \mathrm{cov}(\mathcal{E}) = \mathfrak{d}^{\mathrm{Lc}}_{b,h} = \kappa$. Moreover, $\mathbb{P}_\pi$ forces $\mathfrak{c} = \lambda$. It remains to prove that $\mathbb{P}_\pi$ forces $\mathfrak{b} = \aleph_1$. For this, it suffices to show that $\mathbb{LOC}_{b,h}$ is σ-Fr-linked. □

Before attempting to prove that $\mathbb{LOC}_{b,h}$ is σ-Fr-linked, we begin with some notation:

- Denote by $\mathrm{Fr} := \{\omega \smallsetminus a : a \in [\omega]^{<\aleph_0}\}$ the *Fréchet filter*.
- A filter F on ω is *free* if $\mathrm{Fr} \subseteq F$. A set $x \subseteq \omega$ is *F-positive* if it intersects every member of F. Denote by F^+ the family of all F-positive sets. Note that $x \in \mathrm{Fr}^+$ iff x is an infinite subset of ω.

For not adding dominating reals, we have the following notion.

Definition 4.5 ([Mej19, BCM21])**.** Let $\mathbb{P}$ be a poset and F be a filter on ω. A set $Q \subseteq \mathbb{P}$ is *F-linked* if, for any $\bar{p} = \langle p_n : n < \omega\rangle \in Q^\omega$, there is some $q \in \mathbb{P}$ forcing that

$$q \Vdash \{n \in \omega : p_n \in \dot{G}\} \in F^+.$$

Observe that, in the case $F = \mathrm{Fr}$, the above equation is "$\{n \in \omega : p_n \in \dot{G}\}$ is infinite".

We say that Q is *uf-linked (ultrafilter-linked)* if it is F-linked for any filter F on ω containing the *Fréchet filter* Fr.

For an infinite cardinal μ, $\mathbb{P}$ is *μ-F-linked* if $\mathbb{P} = \bigcup_{\alpha<\mu} Q_\alpha$ for some F-linked Q_α ($\alpha < \mu$). When these Q_α are uf-linked, we say that $\mathbb{P}$ is *μ-uf-linked*.

For ccc posets we have:

Lemma 4.6 ([Mej19, Lem 5.5])**.** *If $\mathbb{P}$ is ccc then any subset of $\mathbb{P}$ is uf-linked iff it is Fr-linked.*

Our objective is to demonstrate that $\mathbb{LOC}_{b,h}$ is σ-Fr-linked, witnessed by

$$L_{b,h}(s,m) := \{(p,n) \in \mathbb{LOC}_{b,h} : s \subseteq p,\ n = |s|,\ \text{and } \forall i < \omega\colon |p(i)| \leq m\}$$

for $s \in \mathcal{S}_{<\omega}(b,h) = \bigcup_{k\in\omega} \prod_{n<k} [b(n)]^{\leq h(n)}$, and $m < \omega$.

In order to see that $L_{b,h}(s,m)$ is Fr-linked, it suffices to prove:

Lemma 4.7 ([Mej19], see also [Car23]). *Let $b, h \in \omega^\omega$ be such that $\forall i < \omega\colon b(i) > 0$ and h goes to infinity, let D be a non-principal ultrafilter on ω and $(s,m) \in \mathcal{S}_{<\omega}(b,h) \times \omega$. If $\bar{p} = \langle (p_n, |s|) : n < \omega \rangle$ is a sequence in $L_{b,h}(s,m)$ then there is a $(q,n) \in L_{b,h}(s,m)$ such that if $a \in D$, then (q,n) forces that $\{n \in \omega : p_n \in \dot{G}\} \cap a \neq \emptyset$ is infinite.*

Proof. Assume that $\bar{p} = \langle (p_n, |s|) : n < \omega \rangle$ is a sequence in $L_{b,h}(s,m)$. For each $i \geq |s|$, since $[b(i)]^{\leq m}$ is finite, we can find $q(i) \in [b(i)]^{\leq m}$ and an $a_i \in D$ such that $p_n(i) = q(i)$ for any $n \in a_i$. To get $(q,n) \in L_{b,h}(s,m)$ we define $n := |s|$ and $q(i) := s(i)$ for all $i < n$. It remains to show that (q,n) forces $\{n \in \omega : p_n \in \dot{G}\} \cap a \neq \emptyset$ whenever $a \in D$. To see this, assume that $(r, n^*) \leq (q,n)$ and $a \in D$. Since $(r,n^*) \in \mathbb{LOC}_{b,h}$, we can find $k, m_0 < \omega$ such that $k \geq n^* \geq n = |s|$, $|r(i)| \leq m_0$ for any $i < \omega$, and for any $i \geq k$, $m_0 + m \leq h(i)$. Choose some $n_0 \in \bigcap_{i<k} a_i \cap a$ (put $a_i := \omega$ for $i < |s|$). Note that $p_{n_0}(i) = q(i)$ for all $i < k$.

Now we define $q'(i)$ by

$$q'(i) = \begin{cases} r(i) & \text{if } i < k, \\ r(i) \cup p_{n_0}(i) & \text{if } i \geq k. \end{cases}$$

Since $|q'(i)| \leq m_0 + m \leq h(i)$ for all $i \geq k$, (q',k) is a condition in $\mathbb{LOC}_{b,h}$. Moreover, (q',k) is a condition stronger than r and p_{n_0}, so it forces $n_0 \in \{n \in \omega : p_n \in \dot{G}\} \cap a$. $\square$

The next step now is to indicate that Fr-linked behaves well regarding preserving unbounded families on ω^ω, which helps to keep $\mathfrak{b}$ small in generic extension. This will be proved in a sequence of lemmas.

Lemma 4.8 (The author and Mejía). *Let $\mathbb{P}$ be a poset and $Q \subseteq \mathbb{P}$. Then Q is Fr-linked iff, for each $\mathbb{P}$-name $\dot{n}$ of a natural number there is some $m < \omega$ such that $\forall p \in Q\,(p \not\Vdash m < \dot{n})$.*

Proof. The direction from left into right is due to Mejía [Mej19, Lem 3.26]. Assume that, for any $n \in \omega$, there is some $p_n \in Q$ such that $p_n \Vdash m \leq \dot{n}$ Then, if G is $\mathbb{P}$-generic over V, then $\{n \in \omega : p_n \in \dot{G}\}$ must be finite because $p_n \in \dot{G} \Rightarrow n \leq \dot{m}[G] < \omega$. Therefore, in V, Q cannot be Fr-linked.

On the other hand, assume that Q is not Fr-linked, accordingly pick some sequence $\langle p_n : n < \omega \rangle$ in Q such that $\Vdash$ "$\{n \in \omega : p_n \in \dot{G}\}$ finite", so there is a $\mathbb{P}$-name $\dot{n}$ of a natural number such that $\Vdash \{n \in \omega : p_n \in \dot{G}\} \subseteq \dot{n}$. Towards a contradiction suppose that $\exists m \forall q \in Q\colon q \not\Vdash m < \dot{n}$. Choose $m \in \omega$ such that $\forall q \in Q\colon q \not\Vdash m < \dot{n}$. For any $n \in \omega$, $p_n \not\Vdash m < \dot{n}$. Next, find $r \leq p_m$ such that $r \Vdash m \geq \dot{n}$, on the other hand, since $r \Vdash p_m \in G$, $r \Vdash m < \dot{n}$. This is a contradiction. $\square$

Lemma 4.9 ([Mej24])**.** *Let $\mathbb{P}$ be a poset and Q be an* Fr*-linked subset of $\mathbb{P}$. If $\dot{y}$ is a $\mathbb{P}$-name of a member of ω^ω, then there is $y' \in \omega^\omega$ (in the ground model) such that, for any $x \in \omega^\omega$*

$$x \not\leq^* y' \Rightarrow \forall n \in \omega\, \forall p \in Q \colon p \not\Vdash \forall m \geq n \colon x(m) \leq \dot{y}(m).$$

Proof. By applying Lemma 4.8, for each $m \in \omega$ find $y' \in \omega^\omega$ such that, for each $m < \omega$ no member of Q forces $y'(m) < \dot{y}(m)$.

Now suppose that $x \in \omega^\omega$ and $x \not\leq^* y'$. Towards a contradiction assume that there is a $n \in \omega$ and $p \in Q$ so that

$$p \Vdash \forall m \geq n \colon x(m) \leq \dot{y}(m).$$

Choose $m \geq n$ so that $x(m) > y'(m)$. On the other hand, $p \not\Vdash y'(m) < \dot{y}(m)$, so there is some $q \leq p$ such that $q \Vdash y'(m) \geq \dot{y}(m)$. Then

$$q \Vdash \dot{y}(m) \geq x(m) > y'(m) \geq \dot{y}(m),$$

which is a contradiction. $\square$

Using the earlier results about Fr-linkedness, we infer that $\mathbb{P}_\pi$ forces $\mathfrak{b} = \aleph_1$. So we are done with the proof of Lemma 4.4.

We now provide another proof for the consistency of $\mathrm{add}(\mathcal{E}) < \mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E})$.

Theorem 4.10 (The author and Mejía)**.** *Assume $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular. Let $\mathbb{E}_\pi$ be a FS iteration of eventually different real forcing $\mathbb{E}$ of length $\pi = \lambda\kappa$. Then, in $V^{\mathbb{E}_\pi}$, $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_1$, $\mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \kappa$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c} = \lambda$ (see Figure 4).*

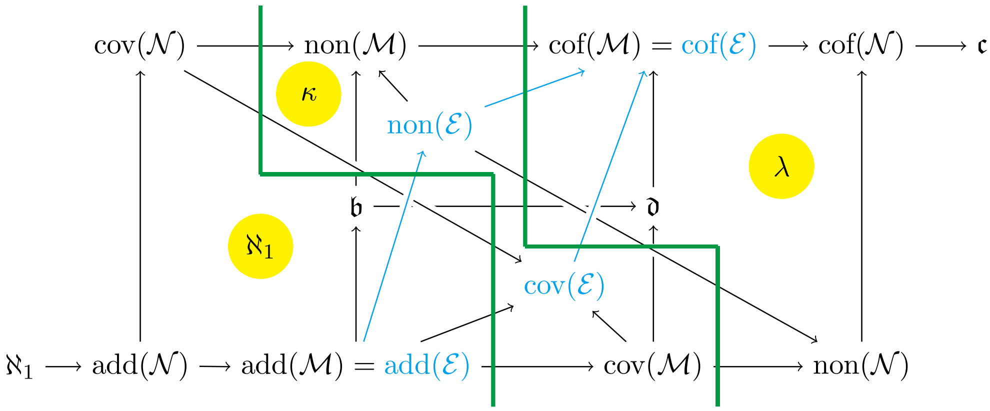


Figure 4: Cichon's diagram after adding κ-many eventually different reals with $\mathbb{E}$.

Proof. It is well-known that $\mathbb{E}_\pi$ forces $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_1$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \kappa$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c} = \lambda$. (see e.g. [CM22, Sec. 3.1]). Moreover, we know that $\mathbb{E}$ is Fr-linked by [Mej19, Lem. 3.29]. It remains to prove that $\mathbb{E}_\pi$ forces $\mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \kappa$. For this, it suffices to see that $\mathbb{E}$ forces all ground model reals belongs to $\mathcal{E}$.

Think of $\mathbb{E}$ as

$$\mathbb{E} = \left\{ (s, \varphi) : \ s \in \omega^{<\omega} \text{ and } \varphi \in \bigcup_{m \in \omega} \mathcal{S}(\omega, m) \right\}$$

Ordered by $(s', \varphi') \leq (s, \varphi)$ iff $s \subseteq s'$ and $\varphi(n) \subseteq \varphi'(n)$ for all $n \in \omega$, and $s'(n) \notin \varphi(n)$ for any $n \in |s'| \smallsetminus |s|$. Let G be a $\mathbb{E}$-generic filter over V. In $V[G]$, define

$$e_{\mathrm{gn}} = \bigcup \{s : \exists \varphi \colon (s, \varphi) \in G\}.$$

Notice that for any $x \in \omega^\omega \cap V$, $\forall^\infty n \in \omega \colon x(n) \neq e_{\mathrm{gn}}(n)$.

Claim 4.11. $\Vdash_{\mathbb{E}} \omega^\omega \cap V \subseteq \{x \in \omega^\omega : \forall^\infty n \in \omega \colon x(n) \neq \dot{e}_{\mathrm{gn}}(n)\} \in \mathcal{E}$.

Proof. It is clear that $\{x \in \omega^\omega : \forall^\infty n \in \omega \colon x(n) \neq \dot{e}_{\mathrm{gn}}(n)\} = \bigcup_{m \in \omega} \bigcap_{n \geq m} \{x \in \omega^\omega : x(n) \neq \dot{e}_{\mathrm{gn}}(n)\}$ is an F_σ set and $\Vdash_{\mathbb{E}} \omega^\omega \cap V \subseteq \{x \in \omega^\omega : \forall^\infty n \in \omega \colon x(n) \neq \dot{e}_{\mathrm{gn}}(n)\}$, so it remains to prove that in $V[e]$, $\mathrm{Lb}_\omega(\{x \in \omega^\omega : \forall^\infty n \in \omega \colon x(n) \neq e_{\mathrm{gn}}(n)\}) = 0$ (by Lb_ω we denote Lebesgue measure zero on ω^ω). Indeed, note that

$\mathrm{Lb}_\omega(\{x \in \omega^\omega : \forall^\infty n \in \omega \colon x(n) \neq e_{\mathrm{gn}}(n)\}) = \lim_{n \to \infty} \prod_{m \geq n} \left(1 - \frac{1}{2^{e_{\mathrm{gn}}(m)+1}}\right)$ and

$$\prod_{m \geq n} \left(1 - \frac{1}{2^{e_{\mathrm{gn}}(m)+1}}\right) \leq \prod_{m \geq n} \mathbf{e}^{-\frac{1}{2^{e_{\mathrm{gn}}(m)+1}}} \leq \mathbf{e}^{-\Sigma_{m \geq n} \frac{1}{2^{e_{\mathrm{gn}}(m)+1}}}.$$

To conclude the proof, let us argue

$$(🚇) \qquad \Vdash_{\mathbb{E}} \Sigma_{n \in \omega} \frac{1}{2^{\dot{e}_{\mathrm{gn}}(n)+1}} = \infty.$$

For this, it suffices to see that

$$\forall n \, \forall p \in \mathbb{E} \, \exists q \leq p \, \exists m \geq n \, \exists m' > m \colon q \Vdash \Sigma_{i=m}^{m'+1} \frac{1}{2^{\dot{e}_{\mathrm{gn}}(i)+1}} > 1.$$

Let n and $p = (s, \varphi) \in \mathbb{E}$, with $M := \sup\{|\varphi(i)| : i < \omega\}$. Wlog assume that $|s| \geq n$. Extend p to $q = (t, \varphi)$ such that $t \supseteq s$ and $\forall i \in |t| \smallsetminus |s| \colon t(i) \leq M$ and $|t| - |s| > 2^{M+1}$. Put $m := |s|$ and $m' := |t|$. Then it can be proved that q is as desired.

Utilizing (🚇), we conclude that in $V[e]$, $\mathrm{Lb}_\omega(\{x \in \omega^\omega : \forall^\infty n \in \omega \colon x(n) \neq e_{\mathrm{gn}}(n)\}) = 0$. $\square$

By employing Claim 4.11, it can be proved that $\mathbb{E}_\pi$ forces $\mathrm{non}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}) = \kappa$. $\square$

We will conclude this section by presenting an additional concept related to not adding dominating reals, which indeed matches Fr-linked.

Brendle and Judah [BJ93] presented the property that helps us not add dominating reals. Given a partial order $\mathbb{P}$, a function $h \colon \mathbb{P} \to \omega$ is a *height function* iff $q \leq p$ implies $h(q) \geq h(p)$. A pair $\langle \mathbb{P}, h \rangle$ fulfills the property (⚽) iff:

(⚽) given a maximal antichain $\{p_n : n \in \omega\} \subseteq \mathbb{P}$ and $m \in \omega$, there is an $n \in \omega$ such that: whenever p is incompatible with $\{p_j : j \in n\}$ then $h(p) > m$.

They proved that FS iterations of ccc posets with the property (⚽) do not add dominating reals. Inspired by (⚽), in [Car24], the author introduced the following linkedness property:

Definition 4.12 ([Car24, Def. 3.1])**.** Let $\mathbb{P}$ be a poset. A set $Q \subseteq \mathbb{P}$ is leaf*-linked* if, for any maximal antichain $\{p_n : n \in \omega\} \subseteq \mathbb{P}$, there is some $n \in \omega$ such that $\forall p \in Q\, \exists j < n \colon p \parallel p_j$. For an infinite cardinal θ. Say that $\mathbb{P}$ is θ-leaf*-linked* if $\mathbb{P} = \bigcup_{\alpha<\theta} Q_\alpha$ where Q_α is leaf-linked ($\alpha < \mu$).

Indeed, Fr-linked and leaf-linked coincide for ccc posets.

Lemma 4.13 ([Car24, Lem. 3.6])**.** *Let $\mathbb{P}$ be a ccc poset and a set $Q \subseteq \mathbb{P}$. The following statements are equivalent:*

(1) *Q is* Fr*-linked.*

(2) *Q is* leaf*-linked.*

(3) *for each $\mathbb{P}$-name $\dot{n}$ of a natural number there is some $m < \omega$ such that $\forall p \in Q \colon p \not\Vdash m < \dot{n}$.*

Proof. (1) $\Rightarrow$ (2). Let $\{p_n : n \in \omega\}$ be a maximal antichain in $\mathbb{P}$. Towards a contradiction assume that $\forall m\, \exists q_m \in Q\, \forall n < m \colon q_m \perp p_n$. By Fr-linkedness, choose $q \in \mathbb{P}$ such that $q \Vdash$ "$\exists^\infty m \in \omega \colon q_m \in \dot{G}$". Since each q_m fulfills $\forall n < m \colon q_m \perp p_n$, $q \Vdash \exists^\infty m\, \forall n < m \colon p_n \notin \dot{G}$. Hence, $q \Vdash \forall n < \omega \colon p_n \notin \dot{G}$. On the other hand, since $\{p_n : n \in \omega\}$ is a maximal antichain, by using Fact 1.4, $\Vdash \exists n \in \omega \colon p_n \in \dot{G}$, which reaches a contradiction.

(2) $\Rightarrow$ (3). Assume that $\dot{n}$ is a $\mathbb{P}$-name of a natural number and assume towards a contradiction that for each $m \in \omega$, there is some $q_m \in Q$ that forces $m < \dot{n}$. Let $\{p_k : k \in \omega\}$ be a maximal antichain deciding the value $\dot{n}$ and let n_k be such that $p_k \Vdash$ "$\dot{n} = n_k$". By soft-linkedness, choose $m \in \omega$ such that $\forall p \in Q\, \exists k < m \colon p \parallel p_k$. Next, let $k^\star$ be larger than $\max\{n_k : k < m\}$. Then $p_{k^\star} \Vdash k^\star < \dot{n}$ and since $q^\star \in Q$, there is $k < m$ such that $p_{k^\star} \parallel p_k$. Let $r \leq q_{k^\star}, p_k$. Then $r \Vdash \dot{n} = k < k^\star < \dot{n}$, which is a contradiction.

(3) $\Rightarrow$ (1). This implication follows by Lemma 4.8. □

As an immediate consequence, we infer:

Corollary 4.14. *If $\mathbb{P}$ is ccc then any subset of $\mathbb{P}$ is leaf-linked iff it is Fr-linked.*

Next, we establish some consequences of our leaf-linked property:

Lemma 4.15 ([Car24, Lem. 3.9])**.** *Let $\mathbb{P}$ be a poset and θ be a cardinal number. Consider the following properties of $\mathbb{P}$:*

(1) *$\mathbb{P}$ is θ-leaf-linked.*

(2) *There is a dense set $Q \subseteq \mathbb{P}$ and a function $h \colon Q \to \theta$ such that, for every $\alpha < \theta$, the set $\{p \in Q : h(p) = \alpha\}$ is leaf-linked.*

(3) *There is a dense set $Q \subseteq \mathbb{P}$ and a function $h \colon Q \to \theta$ such that, for every $\alpha < \theta$, the set $\{p \in Q : h(p) \leq \alpha\}$ is leaf-linked.*

Then (3) $\Rightarrow$ (2); (1) $\Rightarrow$ (2); (3) $\Rightarrow$ (1); *and* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) *for $\theta = \omega$.*

Proof. The implication (3) $\Rightarrow$ (2) is trivial.

(1) $\Rightarrow$ (2). Let $\langle Q_\alpha \mid \alpha < \theta \rangle$ be a sequence of leaf-linked subsets of $\mathbb{P}$ witnessing that $\mathbb{P}$ is θ-leaf-linked. The set $Q := \bigcup_{\alpha<\theta} Q_\alpha$ is a dense subset of $\mathbb{P}$. For $p \in Q$ define $h(p) = \min\{\alpha < \theta : p \in Q_\alpha\}$. It is clear that every set $\{p \in Q : h(p) = \alpha\}$ is a subset of the leaf-linked set Q_α and is therefore leaf-linked.

(3) $\Rightarrow$ (1) for regular θ. For every $\alpha < \theta$ the set $Q_\alpha = \{p \in Q : h(p) \leq \alpha\}$ is leaf-linked and $Q = \bigcup_{\alpha<\theta} Q_\alpha$ is a dense subset of $\mathbb{P}$.

(2) $\Rightarrow$ (3) for $\theta = \omega$. Note that $\{p \in Q : h(p) \leq n\}$ is a finite union of leaf-linked sets $\{p \in Q : h(p) = k\}$ for $k \leq n$ and is therefore leaf-linked. $\square$

We now give a useful sufficient condition under which the pairs $\langle \mathbb{P}, h \rangle$ are σ-leaf-linked. We begin by considering the following property of pairs $\langle \mathbb{P}, h \rangle$ where $\mathbb{P}$ is a poset and h is a height function on $\mathbb{P}$: Say that $\langle \mathbb{P}, h \rangle$ has property $(\clubsuit)$ if:

$(\clubsuit)$

(I) if $\{p_n : n < \omega\}$ is decreasing and $\exists m \in \omega\, \forall n \in \omega\, (h(p_n) \leq m)$, then $\exists p \in \mathbb{P}\, \forall n \in \omega\, (p \leq p_n)$;

(II) $\forall m \in \omega\, \forall P \in [\mathbb{P}]^{<\omega} \smallsetminus \{\emptyset\}\, \exists R \in [\mathbb{P}]^{<\omega}\, (R \perp P)$ and $\forall p \in \mathbb{P}\, (h(p) \leq m)\, [p \perp P \Rightarrow \exists r \in R\, (p \leq r)]$;

The following lemma justifies the introduction of the property $(\clubsuit)$:

Lemma 4.16 ([BJ93, Lem. 1.2], see also [Car24, Lem. 3.13])**.** *A ccc poset $\mathbb{P}$ with a height function $h\colon \mathbb{P} \to \omega$ satisfying $(\clubsuit)$ is σ-leaf-linked.*

Proof. For $m \in \omega$ let

$$Q_m := \{p \in \mathbb{P} : h(p) \leq m\} \text{ and } Q := \bigcup_{m\in\omega} Q_m.$$

We show that each Q_m is leaf-linked, so assume that $\{p_n : n \in \omega\} \subseteq \mathbb{P}$ is a maximal antichain in $\mathbb{P}$ and show that there is some $n \in \omega$ such that $\forall q \in Q_m\, \exists j < n\, (q \parallel p_j)$. Suppose not. So for each $n \in \omega$ let R_n be a finite subset of $\mathbb{P}$ obtained by employing (II) of $(\clubsuit)$ to the family $P = \{p_i : i < n\}$. For each $n \in \omega$, enumerate R_n as $\{q^n_j : j < k_n\}$ for $k_n < \omega$.

By assumption, none of these sets can be empty and we can assume that $q^n_j \in Q_m$ for all j, n. By applying (II) of $(\clubsuit)$ they form an ω-tree with finite levels concerning "$\leq$". Then by König's lemma, there is a function $f \in \omega^\omega$ such that $q^0_{f(0)} \leq q^1_{f(1)} \leq q^3_{f(3)} \leq \cdots$. By (I) of $(\clubsuit)$ there is a condition $q \leq q^n_{f(n)}$ for all n, contradicting the fact that $\{p_n : n \in \omega\}$ is a maximal antichain.

Finally, $\mathbb{P}$ is σ-leaf-linked by (3) of Lemma 4.15 applied to $\theta = \omega$ and for the height function h. $\square$

We also consider the following property of pairs $\langle \mathbb{P}, h \rangle$ where $\mathbb{P}$ is a poset and h is a height function on $\mathbb{P}$: Say that $\langle \mathbb{P}, h \rangle$ has property (🐞) if:

(🐞) given $m \in \omega$ and a finite non-maximal non-empty antichain $P \subseteq \mathbb{P}$, there is some finite subset R of $\mathbb{P}$ such that $R \perp P$ and $\forall p \in \mathbb{P}\, [(p \perp P$ and $h(p) \leq m) \Rightarrow \exists r \in R\, (p \leq r)]$.

Lemma 4.17 ([Car24, Lem. 3.11])**.** *A ccc poset $\mathbb{P}$ with a height function $h\colon \mathbb{P} \to \omega$ satisfying (🐞) is σ-leaf-linked.*

Proof. For $m \in \omega$ let

$$Q_m := \{p \in \mathbb{P} : h(p) \leq m\} \text{ and } Q := \bigcup_{m\in\omega} Q_m.$$

We prove that every set Q_m is leaf-linked. Let $P = \{p_k : k \in \omega\}$ be arbitrary maximal antichain in $\mathbb{P}$. Denote $P_n = \{p_k : k \leq n\}$. By using (🐞), for every set P_n there is some finite subset R_n of $\mathbb{P}$ such that $R_n \perp P_n$ and $\forall p \in Q_m\, [p \perp P_n \Rightarrow \exists r \in R_n\, (p \leq r)$. We can assume that $R_n \in [Q_m]^{<\omega}$ because $(p \in Q_m \text{ and } p \leq r) \Rightarrow r \in Q_m$. Therefore,

($\boxplus$) $\forall n \in \omega\, \exists R_n \in [Q_m]^{<\omega} \smallsetminus \{\varnothing\}\, [R_n \perp P_n \text{ and } \forall p \in Q_m\, (p \perp P_n \Rightarrow \exists r \in R_n\, (r \leq p))]$.

Employing ($\boxplus$) for every $p \in R_{n+1}$ there is $r \in R_n$ such that $r \leq p$ (because $p \in Q_m$ and $p \perp P_n$). In this way, by induction, we prove that $\forall n \in \omega\, \forall p \in R_n\, \exists r \in R_0)\, (r \leq p)$. Denote by R'_n the set $\{r \in R_0 : \exists p \in R_n\, (r \leq p)\}$. Then ($\boxplus$) holds with the sets R'_n instead of R_n and therefore without loss of generality, we can assume that $R_n \subseteq R_0$ for all $n \in \omega$. Since P is a maximal antichain, there is $m_0 \in \omega$ such that $\forall p \in R_0\, (p \not\perp P_{m_0})$. Then it follows that $R_{m_0} = \varnothing$ because $R_{m_0} \perp P_{m_0}$ and $R_{m_0} \subseteq R_0$. Then by ($\boxplus$) for $n = m_0$ we get $\forall p \in Q_m\, (p \not\perp P_{m_0})$. This finishes the proof that Q_m is leaf-linked. Then, $\mathbb{P}$ is σ-leaf-linked by (3) of Lemma 4.15 applied to $\theta = \omega$ and for the height function h. □

Example 4.18 ([BJ93])**.**

(1) Consider random forcing $\mathbb{B}$ as

$$\mathbb{B} = \{B \subseteq 2^\omega : \forall s \in 2^{<\omega} \colon [t] \cap B \neq \emptyset \Rightarrow \mathrm{Lb}([t] \cap B) > 0\}.$$

We define $h\colon \mathbb{B} \to \omega$ by $h(B) = \min\{n \in \omega : \mathrm{Lb}(B) \geq \frac{1}{n}\}$. Brendle and Judah established that $\langle \mathbb{B}, h\rangle$ satisfies the property ($\clubsuit$). Hence, by using Lemma 4.16, $\mathbb{B}$ is σ-leaf-linked.

(2) For $m \in \omega$ put

$$B \cap 2^m := \{t \in 2^m : [t] \cap B \neq \emptyset\}.$$

We define the poset $\mathbb{B}^*$ as $\{(B, n) : B \in \mathbb{B} \text{ and } n \in \omega\}$. This forcing is ccc and it is used to add a perfect set of random reals. Now, define $\mathbb{B}^*_0$ by

$$\mathbb{B}^*_0 = \{(B, n) : \forall s \in 2^n \cap B \colon \mathrm{Lb}([s] \cap B) \geq 2^{\mathrm{dom}(s)+1}\}.$$

It is clear that $\mathbb{B}^*_0 \subseteq \mathbb{B}^*$ and $\mathbb{B}^*_0$ is dense in $\mathbb{B}^*$ (recall the Lebesgue density). We define a height on $\mathbb{B}^*_0$ as follows: For (B, n) let $h_1((B, n)) = n$. In [BJ93, Lem. 1.5], it was demonstrated that $\langle \mathbb{B}^*_0, h_1\rangle$ fulfills the property ($\clubsuit$), so by using Lemma 4.16 again, we obtain that $\mathbb{B}^*_0$ is σ-leaf-linked. Since $\mathbb{B}^*_0$ is dense in $\mathbb{B}^*$, $\mathbb{B}_0$ is still σ-leaf-linked.

Example 4.19 ([Bre95])**.** Given a $b \in \omega^\omega$ define the forcing $\mathbb{PR}_b$ as the poset whose conditions are of the form $p = (A, \langle \pi_n : n \in A\rangle, F)$ such that

(i) $A \in [\omega]^{<\aleph_0}$,

(ii) $\forall n \in A\colon \pi_n\colon \prod_{m<n} b(m) \to b(n)$, and

(iii) F is a finite set of functions and $\forall f \in F\, \exists k \leq \omega\colon f \in \prod_{n<k} b(n)$, i.e., F is a finite subset of $\mathrm{seq}_{<\omega}(b) \cup \mathrm{seq}(b)$.

The order is defined by $(A', \pi', F') \leq (A, \pi, F)$ iff

(a) $A \subseteq A'$, $\pi \subseteq \pi'$,

(b) if $A' \neq A$ and $A \neq \emptyset$, then $\max A < \min(A' \smallsetminus A)$,

(c) $\forall f \in F\, \exists g \in F'\colon f \subseteq g$, and

(d) $\forall f \in F\, \forall n \in (A' \smallsetminus A) \cap \mathrm{dom}(f)\colon \pi'_n(f{\restriction}n) = f(n)$.

Furthermore, $\mathbb{PR}_b$ is σ-centered (and thus ccc). Let $h\colon \mathbb{PR}_b \to \omega$ be defined by $h(p) = \max(A_p \cup \{|F_p|\})$, which can be easily seen that is a height function where $p = (A_p, \pi_p, F_p)$. Brendle proved that the pair $\langle \mathbb{PR}_b, h\rangle$ satisfies the conditions of the property ($\clubsuit$). Therefoe it is σ-leaf-linked (σ-Fr-linked).

Motivated for all the discussions in the preceding sections, one could ask the following:

Problem 4.20. *Are each one of the following constellations consistent with ZFC?*

(1) *The constellation on the left side in Figure 5.*

(2) *The constellation on the right side in Figure 5.*

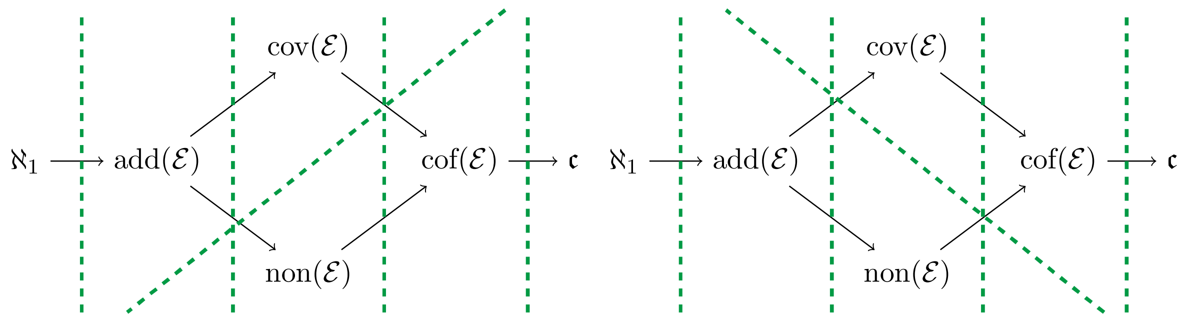


Figure 5: Possible models where the four cardinal characteristics associated with $\mathcal{E}$ can be pairwise different.

We have solved Problem 4.20 (2), concretely, we proved:

Theorem 4.21 ([Car23, Thm. 5.6])**.** *Assuming $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ are uncounatble cardinals satisfying certain conditions. Then there is a ccc poset forcing the constellation of Figure 6.*

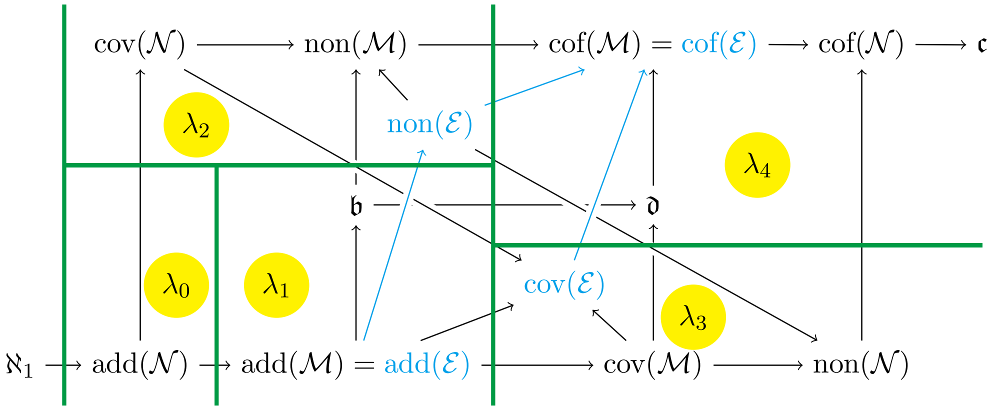


Figure 6: Constellation forced in Theorem 4.21

We conclude this section by presenting several problems related to Theorem 4.21, which we are interested in:

Problem 4.22. *Are each one of the following constellations consistent with ZFC?*

(1) *Constellation of Figure 7.*

(2) *Constellation of Figure 8.*

(3) *Constellation of Figure 9.*

(4) *Constellation of Figure 10.*

(5) *Constellation of Figure 11.*

(6) *Constellation of Figure 12.*

(7) *Constellation of Figure 13.*

(8) *Constellation of Figure 14.*

(9) *Constellation of Figure 15.*

(10) *Constellation of Figure 16.*

(11) *Constellation of Figure 17.*

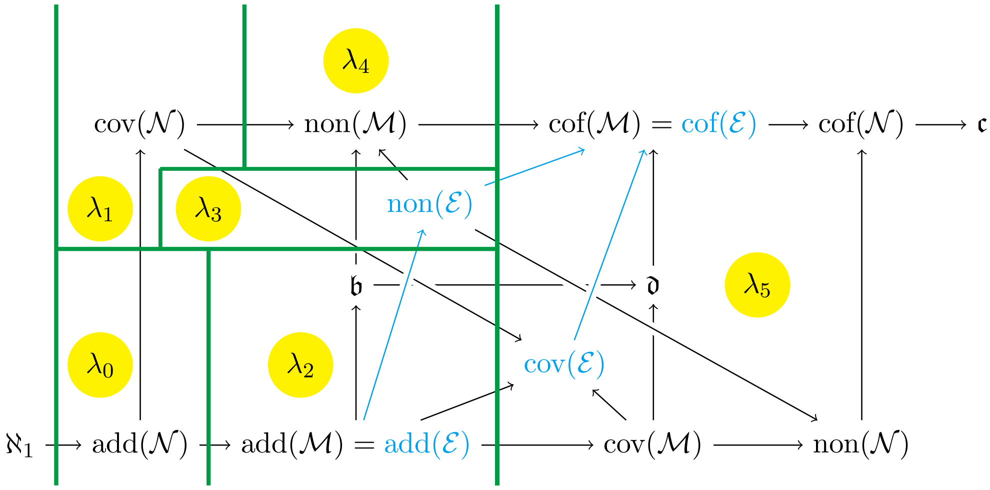


Figure 7: Constellation forced in Problem 4.22 (1)

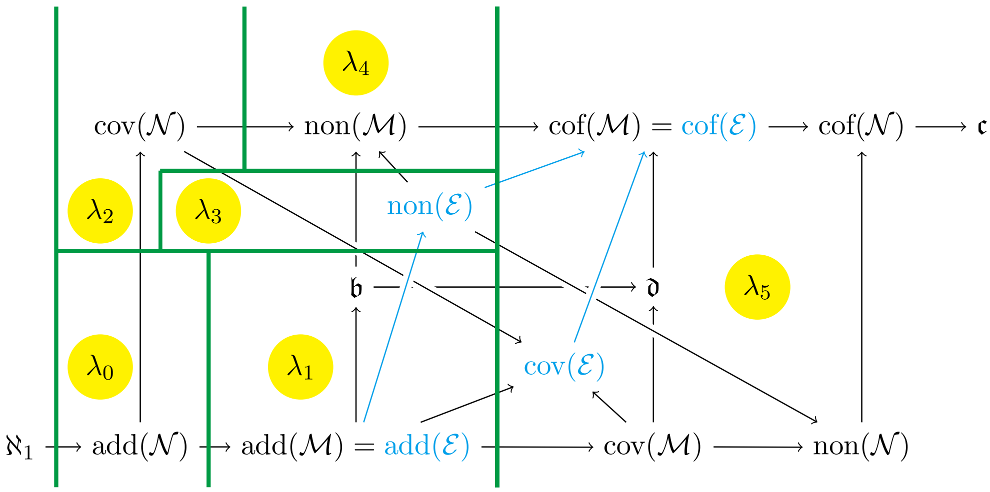


Figure 8: Constellation of Problem 4.22 (2)

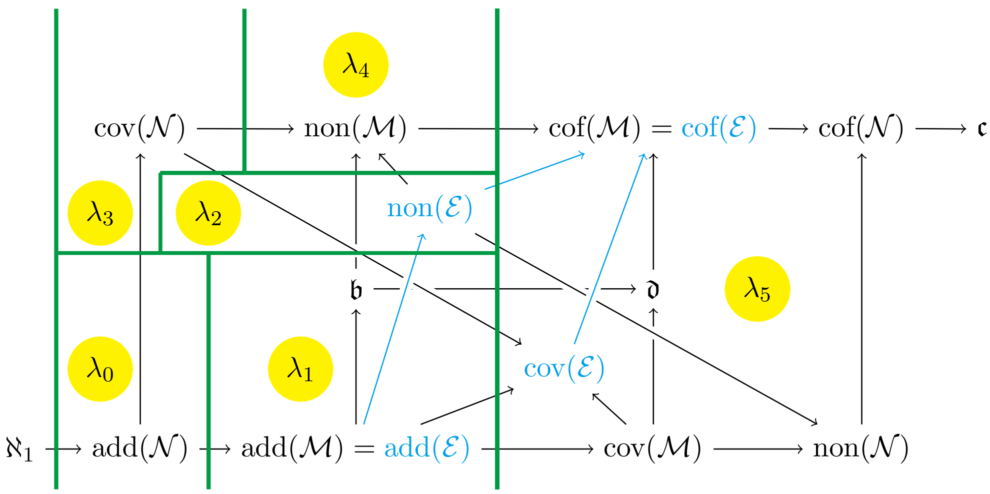


Figure 9: Constellation of Problem 4.22 (3)

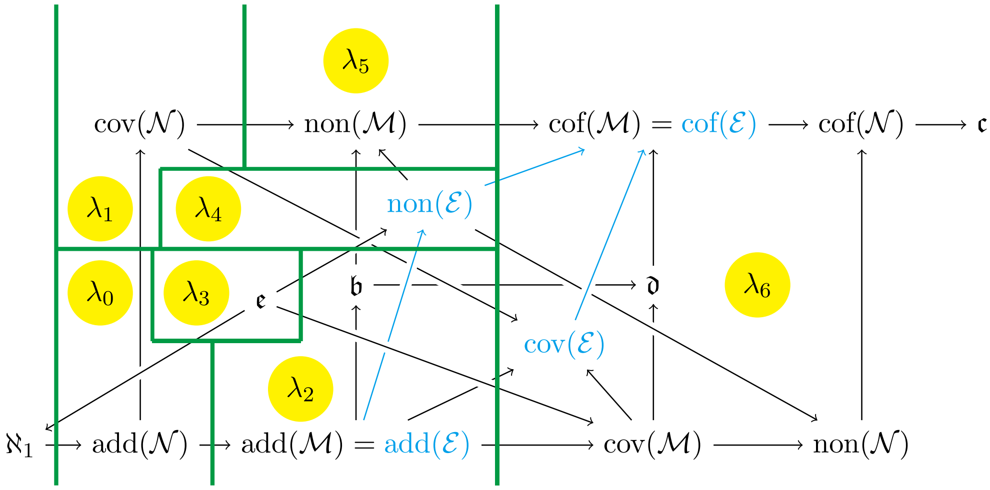


Figure 10: Constellation of Problem 4.22 (4)

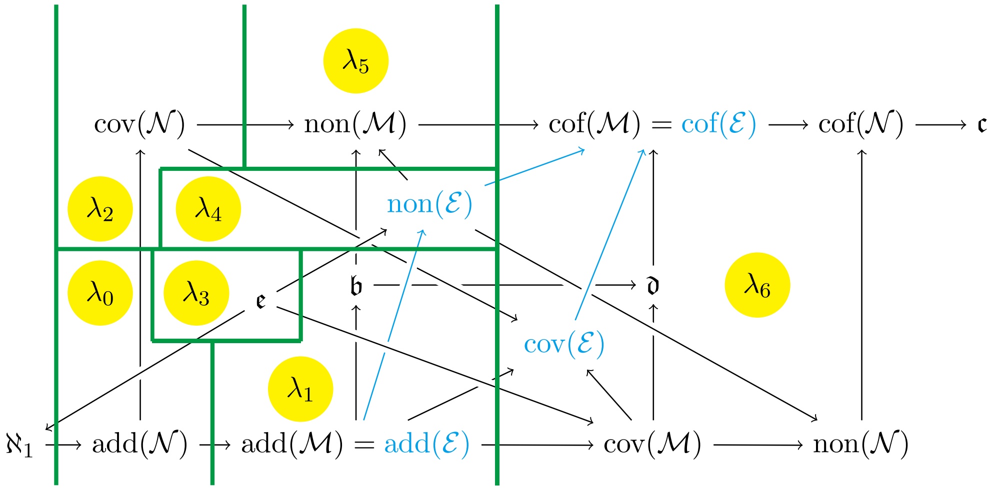


Figure 11: Constellation of Problem 4.22 (5)

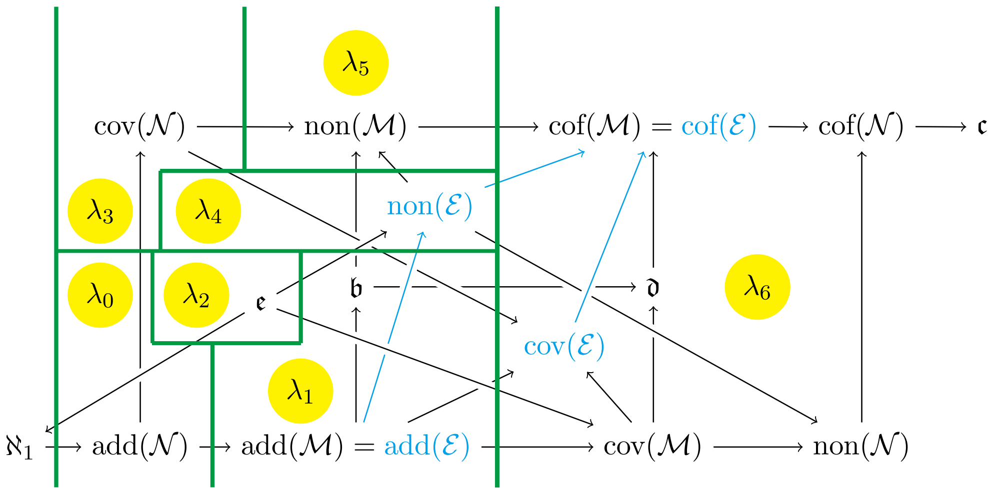


Figure 12: Constellation of Problem 4.22 (6)

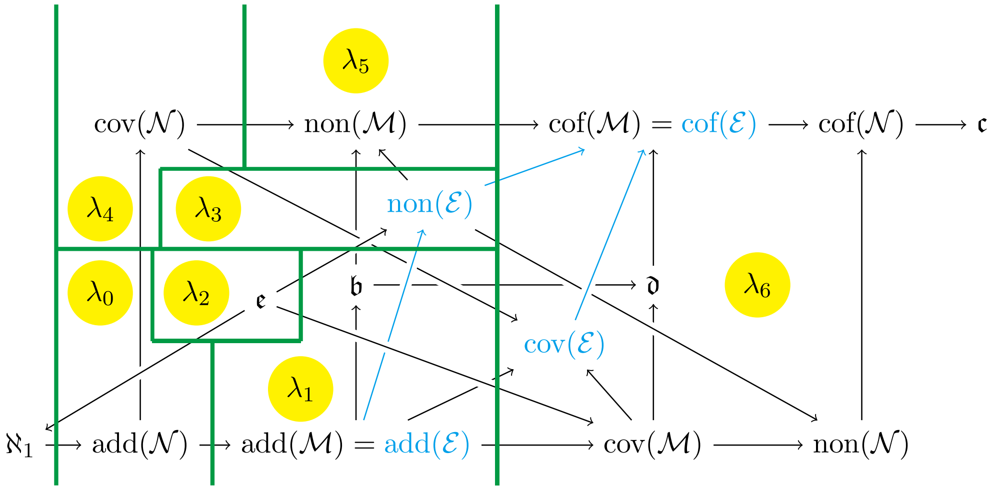


Figure 13: Constellation of Problem 4.22 (7)

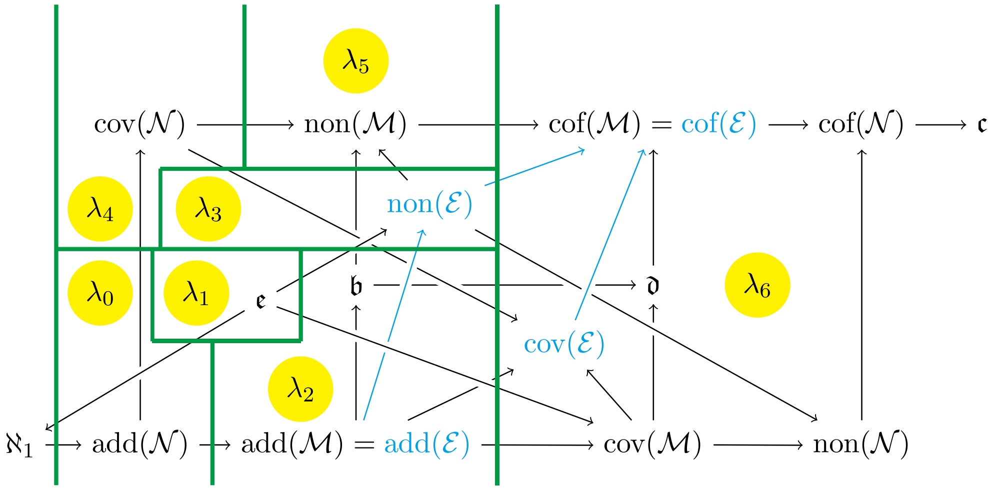


Figure 14: Constellation of Problem 4.22 (8)

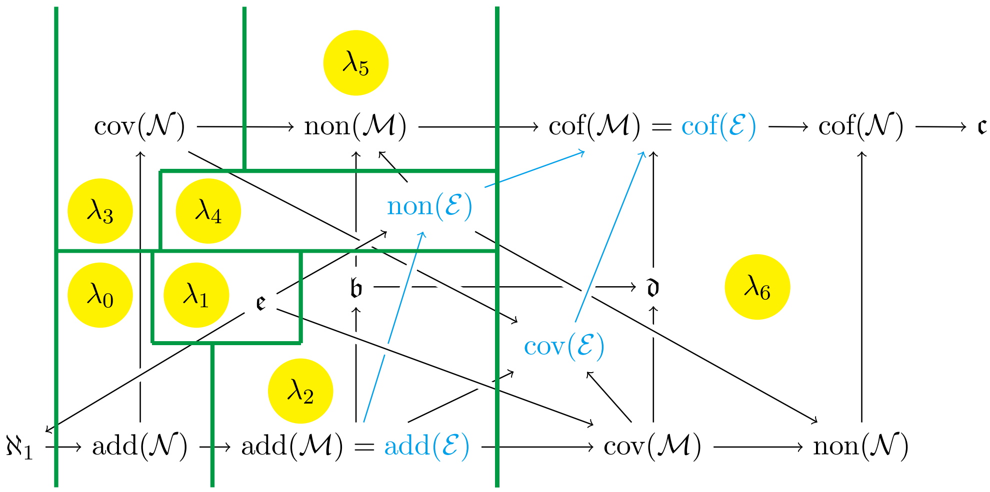


Figure 15: Constellation of Problem 4.22 (9)

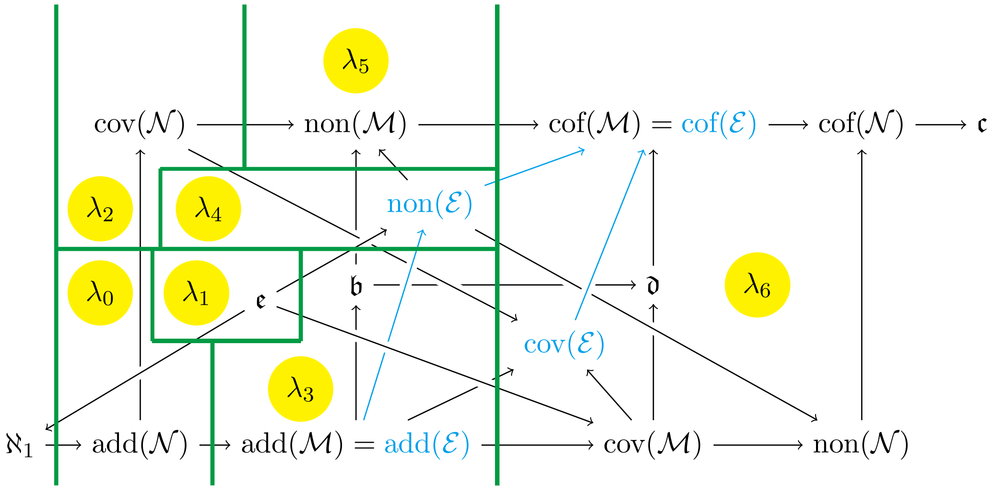


Figure 16: Constellation of Problem 4.22 (10)

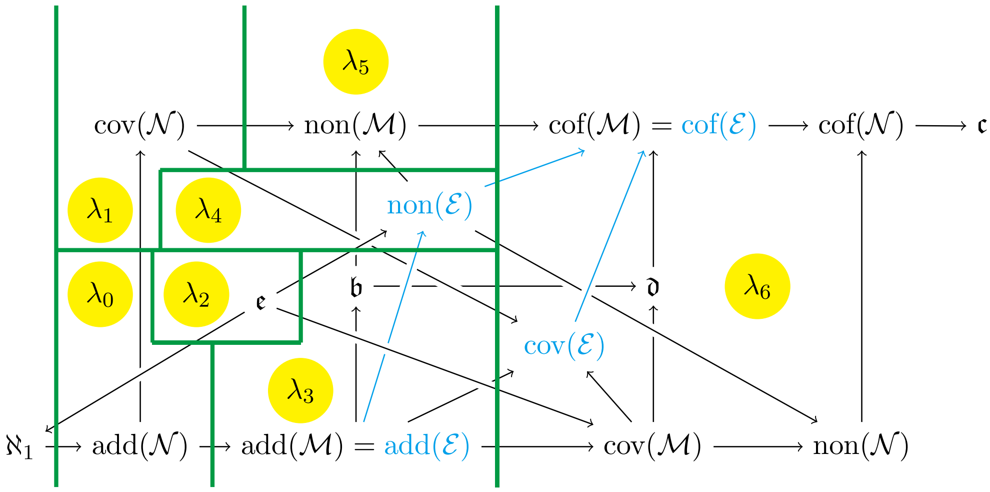


Figure 17: Constellation of Problem 4.22 (11)

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