Anatomy of $\tilde{\mathbb{E}}$

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Abstract

We present a detailed general framework to describe the forcing $\tilde{\mathbb{E}}$, defined by Kellner, Shelah and Tanăsie to prove the consistency with ZFC of an alternative order of Cichoń's maximum. Our presentation is close to the framework of tree-creature forcing notions from Horowitz and Shelah. We show that the posets in this class have strong FAM limits for intervals (in recent terminology, they are σ-FAM-linked) and, furthermore, that they also have strong ultrafilter limits for intervals.

1 Introduction

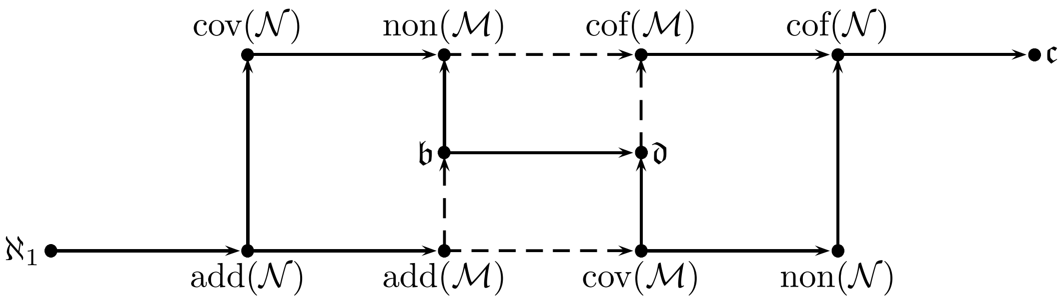

Figure 1: Cichoń's diagram. The arrows mean $\leq$ and dotted arrows represent $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$, which we call the *dependent cardinals*.

Cichoń's maximum refers to the situation when all non-dependent cardinals in Cichoń's diagram (see Figure 1) are pairwise different. The first proof of the consistency of Cichoń's maximum with ZFC is due to Goldstern, Kellner and Shelah [GKS19]. They first refined a ccc poset from [GMS16], constructed via a FS (finite support) iteration, to force the non-dependent cardinals on the left side of the diagram pairwise different (i.e. with $\mathrm{non}(\mathcal{M}) <$

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$\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$), and applied *Boolean ultrapowers* to this poset to force, in addition, that the non-dependent cardinals on the right side can also be separated. To force $\mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ with the first ccc poset, we introduced in [GMS16] the notion of *ultrafilter limits for posets* to show that some restrictions of the forcing $\mathbb{E}$, the standard ccc poset that adds an eventually different real, do not add dominating reals along the iteration.

Although there are many possible instances of Cichoń's maximum, only four are possible to be forced using FS iterations of ccc posets. The reason is that such iterations add Cohen reals at limits steps, which make them force $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$. Under this restriction, Cichoń's diagram gets reduced as illustrated in Figure 2, which can only be extended to four different linear orders. The constellation proved consistent in [GMS16, GKS19] satisfies $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$ and $\mathfrak{d} < \mathrm{non}(\mathcal{N})$.

$$\begin{tikzcd}
 & & \mathrm{cov}(\mathcal{N}) \arrow[dr] & & & \mathfrak{d} \arrow[dr] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[ur] \arrow[dr] & & \mathrm{non}(\mathcal{M}) \arrow[r, red] & \mathrm{cov}(\mathcal{M}) \arrow[ur] \arrow[dr] & & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[ur] & & & \mathrm{non}(\mathcal{N}) \arrow[ur] & &
\end{tikzcd}$$

Figure 2: Cichoń's diagram after a FS iteration of (non-trivial) ccc posets with length of uncountable cofinality.

The other possible constellation for the left side of Cichoń's diagram in the context of FS iterations is the one where $\mathfrak{b} < \mathrm{cov}(\mathcal{N})$. The challenge to force this constellation is to avoid adding dominating reals when forcing with restrictions of random forcing and $\mathbb{E}$ (to increase $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{M})$, respectively). From [GMS16] we know that we can use the method of ultrafilter limits for $\mathbb{E}$, but it is unknown (and very unlikely) whether random forcing has ultrafilter limits. On the other hand, the method of fam-limits (fam stands for *finitely additive measure*) from Shelah [She00] (which preceeds ultrafilter limits) works for random forcing, but not for $\mathbb{E}$ (see Example 3.7 (3)). So, one way to solve this issue is to either find a forcing with fam-limits increasing $\mathrm{non}(\mathcal{M})$, or a forcing with ultrafilter limits adding random reals[1]

Kellner, Shelah and Tanăsie [KST19] found a forcing with fam-limits that adds eventually different reals (i.e. increasing $\mathrm{non}(\mathcal{M})$), which they denoted by $\tilde{\mathbb{E}}$ ([KST19, Def. 1.14]). This is a modification of a tree-creature ccc forcing introduced by Horowitz and Shelah [HS16]. In this way, they succeeded in forcing, with a ccc FS iteration, the separation of the cardinals of the left side of Cichoń's diagram with $\mathfrak{b} < \mathrm{cov}(\mathcal{N})$, which can be used to force Cichoń's maximum with $\mathfrak{b} < \mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N}) < \mathfrak{d}$ after Boolean ultrapowers. We remark that the method of Boolean ultrapowers uses large cardinals. On the other hand, we found another method to force the constellations of Cichoń's maximum

[1] Currently, forcings with ultrafilter-limits can only be iterated when they are μ-centered for small μ, but there is no way a forcing adding random reals can have this property [Bre91].

from [GKS19, KST19] without using large cardinals [GKMS22]. The consistency of the two remaining constellations of Cichoń's maximum compatible with Figure 2 are still unknown.

In this way, the forcing $\tilde{\mathbb{E}}$ plays an important role in forcing different values in Cichoń's diagram (and also simultaneously with other cardinal characteristics). However, $\tilde{\mathbb{E}}$ is a tree-forcing defined by declaring many parameters ([KST19, Def. 1.12]) and using creature-type norms on the successors of the nodes, which makes it a little bit difficult to digest. Considering this, the proof that $\tilde{\mathbb{E}}$ has *strong FAM limits for intervals* (in the terminology from [KST19]) and some other properties become quite involved.

In this note, we make an effort to describe the elements of $\tilde{\mathbb{E}}$ in more detail, to give a clearer picture of how this forcing behaves. To achieve this, we do not rely on the many parameters fixed in [KST19], but we reduce them to a more general framework, closer to [HS16], containing much less information. Since this framework depends on a *tree with creatures*, we start by developing in detail the type of creatures we use to define the forcing. So our framework results in a class of forcings of the form $\tilde{\mathbb{E}}_{\mathbf{t}}$ where $\mathbf{t}$ is what we call a *tree-creature frame*.

Afterwards, we focus on two main results about $\tilde{\mathbb{E}}_{\mathbf{t}}$: we show that, when $\mathbf{t}$ satisfies certain hypothesis (Assumption 3.1), the forcing is σ-$\mathcal{Y}_*$-linked and σ-$\mathcal{D}_*$-linked. The first property corresponds to a generalization of strong FAM limits for intervals introduced in [Uri23, CMU], which can be used in FS iterations to produce large posets with fam-limits. The second property is the version of ultrafilter limits for intervals. The proof of these properties is based on the original proof of [KST19, Lem. 1.20].

2 Tree-creature frames and the forcing

This section is based on [HS16, KST19]. We first look at the atomic structures that we use to define our forcing.

Definition 2.1. Let C be a set. A *norm of subsets of* C is a function $\|\cdot\|\colon \mathcal{P}(C) \to [0,\infty]$ satisfying:

(i) $\|A\| \leq \|B\|$ whenever $A \subseteq B \subseteq C$.

(ii) $\|\emptyset\| = 0$.

Example 2.2 (cf. [KST19, Def. 1.12])**.** Given a non-empty set C and a natural number $m > 1$, the following is a norm of subsets of C:

$$\|A\|^C_m := \log_m \left(\frac{|C|}{|C| - |A|} \right).$$

Equivalently, $\|A\|^C_m$ is the solution $x \in \mathbb{R}$ to the equation

$$|A| = |C| \left(1 - m^{-x}\right). \tag{2.2.1}$$

Notice that $\|C\|^C_m = \infty$ and, whenever $|A| = |C| - 1$, $\|A\|^C_m = \log_m |C|$.

Typically, $m \ll |C|$. For example, when $|C| > \frac{m}{m-1}$, we obtain that all singletons have norm <1. Moreover, a set A with norm ≥ 1 must have size close to C as long as m is very large.

Lemma 2.3 (cf. [KST19, Lem. 1.16]). *Let C be a non-empty set and $m > 1$ a natural number. Then the norm $\|\cdot\| := \|\cdot\|_m^C$ satisfies the following properties.*

(a) *For $A \subseteq C$, $\|A\| \geq 1$ iff $|A| \geq |C|\left(1 - \dfrac{1}{m}\right)$.*

(b) *Let I be a finite set, $\bar{A} = \langle A_i \colon i \in I\rangle$ a sequence of subsets of C, and $g \colon I \to [0,1]$ such that $\sum_{i\in I} g(i) = 1$. For a real $\varepsilon > 0$, consider the set*

$$B_{\bar{A},g}(\varepsilon) = \left\{k \in C \colon \sum\{g(i) \colon i \in I,\ k \in A_i\} > 1 - \varepsilon\right\}.$$

Then $\|B_{\bar{A},g}(\varepsilon)\| \geq \min_{i\in I} \|A_i\| - \log_m \frac{1}{\varepsilon}$.

Proof. (a): By Equation 2.2.1, $\|A\| \geq 1$ iff $|A| = |C|(1 - m^{-\|A\|}) \geq |C|(1 - m^{-1})$.
(b) Let $x := \min_{i\in I} \|A_i\|$ and $B := B_{\bar{A},g}(\varepsilon)$. Then

$$\begin{aligned}
|C|(1 - m^{-x}) &= \sum_{i\in I} g(i)|C|(1 - m^{-x}) \leq \sum_{i\in I} g(i)|A_i| = \sum_{i\in I}\sum_{k\in A_i} g(i)\\
&= \sum_{k\in C}\sum\{g(i) \colon i \in I,\ k \in A_i\}\\
&= \sum_{k\in B}\sum\{g(i) \colon i \in I,\ k \in A_i\} + \sum_{k\in C\smallsetminus B}\sum\{g(i) \colon i \in I,\ k \in A_i\}\\
&\leq |B| + (|C| - |B|)(1 - \varepsilon) = |B|\varepsilon + |C|(1 - \varepsilon).
\end{aligned}$$

Then $|C|(\varepsilon - m^{-x}) \leq |B|\varepsilon$, so

$$|B| \geq |C|\left(1 - \frac{m^{-x}}{\varepsilon}\right) = |C|\left(1 - m^{-\left(x - \log_m \frac{1}{\varepsilon}\right)}\right).$$

Therefore, $\|B\| \geq x - \log_m \dfrac{1}{\varepsilon}$. □

Property (b) in Lemma 2.3 is essential for the forcing arguments. We isolate it below as a notion of *co-bigness* as in [HS16].

Definition 2.4. A norm $\|\cdot\|$ of subsets of C is *m-co-big* if it satisfies (b) of Lemma 2.3.

Co-bigness is a property that allows "homogenizing" many sets without losing much of the norm. A way to homogenize is taking intersection, for which we may not lose much norm, either.

Lemma 2.5 (cf. [KST19, Lem. 1.16 (c)]). *If $\|\cdot\|$ is an m-co-big norm of subsets of C then, whenever I is a finite set and $\langle A_i \colon i \in I\rangle$ is a sequence of subsets of C,*

$$\left\|\bigcap_{i\in I} A_i\right\| \geq \min_{i\in I} \|A_i\| - \log_m |I|.$$

Proof. Apply the m-co-bigness to $g(i) = \varepsilon = \frac{1}{|I|}$. In this case, $B_{\bar{A},g} = \bigcap_{i\in I} A_i$. $\square$

Our forcing is composed by trees. When T is a tree and $t \in T$, we denote by $\operatorname{succ}_T(t)$ the *set of immediate successors of t in T*. When T has a single root (i.e. only one element of height 0), the *stem of T* is the node s of smallest height such that $|\operatorname{succ}_T(s)| \geq 2$ (if it exists). For any ordinal α, $\operatorname{Lv}_\alpha(T)$ denotes the set of nodes of T at level α.

Definition 2.6. A tuple $\mathbf{t} = \langle T_*, b_*, \langle \|\cdot\|_t \colon t \in T_* \rangle\rangle$ is a *tree-creature frame* when:

(T1) T_* is a finitely-branching tree of height ω with a single root and without maximal nodes. Without loss of generality, we can assume that T_* is a subtree of $\omega^{<\omega}$.

(T2) $b_* \colon T_* \to \omega$ such that $\lim\limits_{n\to\infty} \min\limits_{t\in \operatorname{Lv}_n(T_*)} b_*(t) = \infty$.

(T3) For each $t \in T_*$, $\|\cdot\|_t$ is a $b_*(t)$-co-big norm of subsets of $\operatorname{succ}_{T_*}(t)$ such that all singletons have norm <1 and $\|\operatorname{succ}_{T_*}(t)\|_t = \infty$.

Tree-creature frames are easy to construct. In the practice, T_* and b_* are constructed simultaneously by induction on the height n such that $b^*(t)$ for $t \in \operatorname{Lv}_n(T_*)$ is much larger than $|\operatorname{Lv}_n(T_*)|$ and the $b^*(s)$ defined so far, and $|\operatorname{succ}_{T_*}(t)| \gg b_*(t)$. Co-big norms are easily obtained from Example 2.2. In [KST19, Def. 1.12], $b^*(t)$ and $\operatorname{succ}_{T_*}(t)$ only depend on the height of t. On the other hand, [HS16] proceeds by induction on t using the lexicographic order $\lhd$ of $\omega^{<\omega}$ satisfying $|s| < |t| \Rightarrow s \lhd t$, and defines $b_*(t)$ much larger than everything defined for $s \lhd t$ in T_* (and $|\operatorname{succ}_{T_*}(t)| \gg b_*(t)$).

The following is the central definition of this work.

Definition 2.7 ([KST19, Def. 1.14])**.** Let $\mathbf{t}$ be a tree-creature frame as in Definition 2.6. Define the poset $\tilde{\mathbb{E}} := \tilde{\mathbb{E}}_{\mathbf{t}}$ whose conditions are subtrees $p \subseteq T_*$ such that, for each $t \in p$ above $\operatorname{st}(p)$, $\|p\|_t := \|\operatorname{succ}_p(t)\|_t \geq 1 + \frac{1}{|\operatorname{st}(p)|}$ (we interpret $\frac{1}{0} = \infty$). We order $\tilde{\mathbb{E}}$ by $\subseteq$.

Note that $T_* \in \tilde{\mathbb{E}}$ and that it is the maximum condition of $\tilde{\mathbb{E}}$. Also, whenever $p \in \tilde{\mathbb{E}}$ and $t \in p$ is above the stem, $p|^t \in \tilde{\mathbb{E}}$ (the subtree of p whose nodes are precisely the nodes in p compatible with t) and it is stronger than p.

Conditions with stem $\langle\,\rangle$ are somewhat uninteresting.

Fact 2.8. *Whenever $p \in \tilde{\mathbb{E}}$ and $\operatorname{st}(p) = \langle\,\rangle$, $\|p\|_t = \infty$ for all $t \in p$. As a consequence, the set of conditions with stem $\langle\,\rangle$ is centered.*

Proof. The first part follows by Definition 2.7 and the convention $\frac{1}{0} = \infty$. The second part follows by Lemma 2.5: when I is finite and $\langle r_i : i \in I\rangle$ is a sequence of conditions with empty stem, $r := \bigcap_{i\in I} r_i$ is in $\tilde{\mathbb{E}}$ (even with empty set) because, for any $t \in r$,

$$\left\| \bigcap_{i\in I} \operatorname{succ}_{r_i}(t) \right\|_t \geq \infty - \log_{b_*(t)} |I| = \infty. \qquad \square$$

It is not hard to show that $\tilde{\mathbb{E}}$ has the ccc. Moreover, it is σ-k-linked for any $2 \leq k < \omega$. Recall that, given a poset $\mathbb{P}$, $Q \subseteq \mathbb{P}$ is *k-linked* if any subset of Q of size $\leq k$ has a lower bound in $\mathbb{P}$; and the poset $\mathbb{P}$ is *σ-k-linked* if it can be covered by countably many k-linked subsets.

Lemma 2.9. *Let $\mathbf{t}$ be a tree-creature frame. Then, for any $2 \leq k < \omega$:*

(a) *For each $s \in T_*$, the set* $E_{s,k} := \left\{ p \in \tilde{\mathbb{E}} \colon \operatorname{st}(p) = s \text{ and } \|p\|_t \geq 1 + \frac{1}{|s|} + \log_{b_*(t)} k \right\}$ *is k-linked in $\tilde{\mathbb{E}} := \tilde{\mathbb{E}}_{\mathbf{t}}$.*

(b) $\bigcup\{E_{s,k} \colon s \in T_*,\ k < \omega\}$ *is dense in $\tilde{\mathbb{E}}$. In particular, $\tilde{\mathbb{E}}$ is σ-k-linked.*

Proof.
(a): If $\{p_i \colon i < k\} \subseteq E_{s,k}$, let $p := \bigcap_{i<k} p_i$. This is a subtree of T_* with $s \in p$ and, for $t \in p$ above s, by Lemma 2.5,

$$\|p\|_t \geq 1 + \frac{1}{|s|} + \log_{b_*(t)} k - \log_{b_*(t)} k = 1 + \frac{1}{|s|},$$

so $p \in \tilde{\mathbb{E}}$. This shows that $E_{s,k}$ is k-linked.

(b): Let $p \in \tilde{\mathbb{E}}$ and $s_0 := \operatorname{st}(p)$. By (T2), find $s \in p$ (above s_0) such that $\frac{1}{|s|} + \log_{b_*(t)} k < \frac{1}{|s_0|}$ for all $t \in T_*$ above s. Then, $p|^s \in E_{s,k}$ and it is stronger than p. $\square$

3 Fam-limits

From now on, we fix a tree-creature frame $\mathbf{t}$ as in Definition 2.6, and let $\tilde{\mathbb{E}} := \tilde{\mathbb{E}}_{\mathbf{t}}$. We also fix the following assumption (stronger than (T2)) until the end of this paper.

Assumption 3.1. $\lim_{n\to\infty} \min_{t \in \mathrm{Lv}_n(T_*)} \log_n b_*(t) = \infty$. Equivalently, the function

$$n \mapsto \min_{t \in \mathrm{Lv}_n(T_*)} b_*(t)$$

dominates the set $\{m^{\mathrm{id}} \colon m < \omega\}$, where m^{id} is the function sending $n \mapsto m^n$.

Based on [KST19, Lem. 1.20], we plan to show that $\tilde{\mathbb{E}} = \tilde{\mathbb{E}}_{\mathbf{t}}$ has (strong) fam-limits under the previous assumption. We first review the formalization of the notion of *strong fam-limits* from [Uri23, CMU].

Definition 3.2. Let $\mathbb{B}$ be a Boolean algebra. A *finitely additive measure (fam) on* $\mathbb{B}$ is a map $\Xi \colon \mathbb{B} \to [0, \infty]$ satisfying

(i) $\Xi(0_{\mathbb{B}}) = 0$,

(ii) $\Xi(a \vee b) = \Xi(a) + \Xi(b)$ whenever $a, b \in \mathbb{B}$ and $a \wedge b = 0_{\mathbb{B}}$.

If in addition $\Xi(1_{\mathbb{B}}) = 1$, we say that Ξ is a *probability fam*. From now on, we will assume that all our fams are of probability.

Denote by $\mathbf{P}^{\mathbb{B}}$ the collection of *finite partitions of* $1_{\mathbb{B}}$, i.e. $P \in \mathbf{P}^{\mathbb{B}}$ iff $P \subseteq \mathbb{B}$ is finite, $a \wedge b = o_{\mathbb{B}}$ for $a \neq b$ in P, and $\bigvee P = 1_{\mathbb{B}}$. When Ξ is a fam on $\mathbb{B}$, we also write $\mathbf{P}^{\Xi}$ for $\mathbf{P}^{\mathbb{B}}$.

Let K be a non-empty set. Recall that a *field of sets over* K is a subalgebra of $\mathcal{P}(K)$ (under the set-theoretic operations). When $\mathbb{B}$ is a field of sets over K and Ξ is a fam on $\mathbb{B}$, we say that Ξ is *free* if, for all $k \in K$, $\{k\} \in \mathbb{B}$ and $\Xi(\{k\}) = 0$ (this implies that all finite subsets of K have measure zero).

We use the following particular type of fams.

Definition 3.3 ([CMPU]). Let $\mathbb{B}$ be a field of sets over K and Ξ a (probability) fam on $\mathbb{B}$. We say that Ξ has the *uniform approximation property (uap)* if, for any $\varepsilon > 0$ and any $P \in \mathbf{P}^{\Xi}$, there is some non-empty finite $u \subseteq K$ such that, for all $B \in P$,

$$\left| \frac{|u \cap B|}{|u|} - \Xi(B) \right| < \varepsilon.$$

Lemma 3.4 ([CMPU]). *Any fam over a field of sets $\mathbb{B}$ over K satisfying that all finite sets in $\mathbb{B}$ have measure zero has the uap. In particular, any free fam has the uap.*

Surprisingly, fams with the uap having a finite set of positive measure are easily characterized. Concretely, for any such fam Ξ on $\mathbb{B}$ there is some natural number $d > 0$ such that, for any $B \in \mathbb{B}$, $\Xi(B)$ has the form $\frac{\ell}{d}$ for some $0 \leq \ell \leq d$, and $\ell \leq |B|$ when B is finite. Details will be available in [CMPU].

For fams with the uap, the size of the set u has a bound that depends only on ε and $|P|$.

Lemma 3.5 ([KST19, Lem. 1.2], [CMPU]). *Let Ξ be a fam with the uap on a field of sets $\mathbb{B}$ over K. Then, for any $\varepsilon > 0$ and $m \in \omega$, there is some $M := M_{\varepsilon,m} \in \omega$ such that, for any $P \in \mathbf{P}^{\Xi}$, if $|P| \leq m$ then the u in Definition 3.3 can be found of size $\leq M$.*

We are ready to introduce our formalization of forcings with fam-limits.

Definition 3.6 ([Uri23, CMU]). Let $\mathbb{P}$ be a poset.

(1) Let $\Xi \colon \mathcal{P}(K) \to [0,1]$ be a fam with the uap, $\bar{I} = \langle I_k \colon k \in K\rangle$ a partition of a set W into finite sets, and $\varepsilon > 0$.

A set $Q \subseteq \mathbb{P}$ is $(\Xi, \bar{I}, \varepsilon)$*-linked* if there is a function $\lim \colon Q^W \to \mathbb{P}$ and a $\mathbb{P}$-name $\dot{\Xi}'$ of a fam with the uap on $\mathcal{P}(K)$ extending Ξ such that, for any $\bar{p} = \langle p_\ell \colon \ell \in W\rangle \in Q^W$,

$$\lim \bar{p} \Vdash \int_K \frac{|\{\ell \in I_k \colon p_\ell \in \dot{G}\}|}{|I_k|} d\dot{\Xi}' \geq 1 - \varepsilon. \tag{3.6.1}$$

(2) Let μ be an infinite cardinal, and let $\mathcal{Y} \subseteq \mathcal{Y}_*$, where $\mathcal{Y}_*$ is the class of all pairs $(\Xi, \bar{I})$ such that Ξ is a fam with the uap on some $\mathcal{P}(K)$ and $\bar{I} = \langle I_k \colon k \in K\rangle$ is a pairwise disjoint family of finite non-empty sets.[2]

The poset $\mathbb{P}$ is μ-$\mathcal{Y}$*-linked*, witnessed by $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$, if:

(i) Each $Q_{\alpha,\varepsilon}$ is $(\Xi, \bar{I}, \varepsilon)$-linked for any $(\Xi, \bar{I}) \in \mathcal{Y}$.

(ii) For $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\bigcup_{\alpha<\omega} Q_{\alpha,\varepsilon}$ is dense in $\mathbb{P}$.

(3) The poset $\mathbb{P}$ is *uniformly* μ-$\mathcal{Y}$*-linked* if there is some $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$ as above, such that in (1) the name $\dot{\Xi}'$ only depends on $(\Xi, \bar{I})$ (and not on any $Q_{\alpha,\varepsilon}$, although we may have different limits on each $Q_{\alpha,\varepsilon}$).

We write σ-$\mathcal{Y}$*-linked* when $\mu = \aleph_0$.

[2]Notice that K is not a fixed set.

Example 3.7.

(1) Any singleton is $(\Xi, \bar{I}, \varepsilon)$-linked: If Q is a singleton, Q^W only contains one constant sequence, so $\lim \bar{p}$ can be defined as this constant value. Notice that $\lim \bar{p}$ forces that the integral of Equation 3.6.1 is 1 for any $\dot{\Xi}'$ extending Ξ. Hence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$-$\mathcal{Y}_*$-linked. In particular, Cohen forcing is uniformly σ-$\mathcal{Y}_*$-linked.

(2) Shelah [She00] proved, implicitly, that random forcing is uniformly σ-$\mathcal{Y}_{**}$-linked, where $\mathcal{Y}_{**}$ is the class of all $(\Xi, \bar{I}) \in \mathcal{Y}_*$ such that Ξ is free. More generally, we proved that any measure algebra with Maharam type μ is uniformly μ-$\mathcal{Y}_{**}$-linked [MU23].

(3) With Cardona and Uribe-Zapata we have proved that σ-$\mathcal{Y}_*$-linked posets do not increase $\mathrm{non}(\mathcal{E})$, where $\mathcal{E}$ denotes the ideal generated by the F_σ measure zero subsets of 2^ω, and that $\mathbb{E}$ (the standard ccc poset adding an eventually different real) and localization posets increase $\mathrm{non}(\mathcal{E})$ (see [Car23, CM23]). As a consequence, $\mathbb{E}$ and the localization posets cannot be σ-$\mathcal{Y}_*$-linked.

The following is the main result of this section.

Theorem 3.8. *Under Assumption 3.1, the poset $\tilde{\mathbb{E}}$ is uniformly σ-$\mathcal{Y}_*$-linked.*

Before proving the theorem, we review some results about fams. Given a fam Ξ on a field of sets $\mathbb{B}$ over K, and a bounded function $\Xi\colon \mathbb{B} \to \mathbb{R}$, we can define the Ξ-*integral of* f, denoted $\int_K f d\Xi$, similarly as the Riemann integral (using $\mathbf{P}^\Xi$, lower sums and upper sums), and we say that f is Ξ-*integrable* when its Ξ-integral is defined. When $\mathbb{B} = \mathcal{P}(K)$, any bounded real-valued function on K is Ξ-integrable. For details, see [Uri23, CMPU].

Theorem 3.9 ([CMPU])**.** *Let Ξ_0 be a fam with the uap on a field of sets $\mathbb{B}$ over K, and let I be a set. For each $i \in I$, let C_i be a closed subset of $\mathbb{R}$ and $f_i\colon K \to \mathbb{R}$ bounded. Then, the following statements are equivalent.*

(I) *For any $P \in \mathbf{P}^{\Xi_0}$, $\varepsilon > 0$, any finite set $J \subseteq I$, and any open $G_i \subseteq \mathbb{R}$ containing C_i for $i \in J$, there is some non-empty finite $u \subseteq K$ such that:*

(i) $\left| \dfrac{|B \cap u|}{|u|} - \Xi_0(B) \right| < \varepsilon$ *for any $B \in P$, and*

(ii) $\dfrac{1}{|u|} \displaystyle\sum_{k \in u} f_i(k) \in G_i$ *for any $i \in J$.*

(II) *There is some fam Ξ on $\mathcal{P}(K)$ with the uap extending Ξ_0 such that, for any $i \in I$, $\displaystyle\int_K f_i d\Xi \in C_i$.*

The previous result allows the following characterization.

Theorem 3.10 (cf. [Uri23, CMU])**.** *Let μ be a cardinal, $\mathbb{P}$ a poset, $\langle Q_{\alpha,\varepsilon}\colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$ a sequence of subsets of $\mathbb{P}$, $\mathcal{Y} \subseteq \mathcal{Y}_*$ and, for each $(\Xi, \bar{I}) \in \mathcal{Y}$, $\alpha < \mu$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\lim^{\Xi,\bar{I}}\colon Q^W_{\alpha,\varepsilon} \to \mathbb{P}$ where $W := \bigcup_{k \in K} I_k$. Then, the following statements are equivalent.*

(I) *$\mathbb{P}$ is uniformly μ-$\mathcal{Y}$-linked witnessed by $\langle Q_{\alpha,\varepsilon}\colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$.*

(II) *Definition 3.6 (2) (ii) holds and, for any*

- $(\Xi, \bar{I}) \in \mathcal{Y}$,
- $i^* < \omega$,
- $(\alpha_i, \varepsilon_i) \in \mu \times ((0,1) \cap \mathbb{Q})$,
- $\bar{r}^i = \langle r^i_\ell \colon \ell \in W \rangle \in Q^W_{\alpha_i, \varepsilon_i}$ *for* $i < i^*$,
- $P \in \mathbf{P}^\Xi$,
- $\varepsilon' > 0$, *and*
- $q \in \mathbb{P}$ *stronger than* $\lim^{\Xi, \bar{I}} \bar{r}^i$ *for all* $i < i^*$,

there is some $q' \leq q$ *in* $\mathbb{P}$ *and* $u \subseteq K$ *finite non-empty such that*

(1) $\left| \dfrac{|u \cap B|}{|u|} - \Xi(B) \right| < \varepsilon'$ *for all* $B \in P$, *and*

(2) $\dfrac{1}{|u|} \displaystyle\sum_{k \in u} \dfrac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} > 1 - \varepsilon_i - \varepsilon'$ *for all* $i < i^*$.

Proof. (I) $\Rightarrow$ (II): Assume (I) and the assumptions of (II). Let $\dot{\Xi}'$ be as in Definition 3.6 (3), and let G be $\mathbb{P}$-generic over V such that $q \in G$, and $\Xi' := \dot{\Xi}'[G]$. In $V[G]$, since $q \leq \lim^{\Xi, \bar{I}} \bar{r}^i$ for all $i < i^*$, by Equation 3.6.1

$$\int_K f_i d\Xi' \geq 1 - \varepsilon_i, \text{ where } f_i(k) := \frac{|\{\ell \in I_k \colon r^i_\ell \in \dot{G}\}|}{|I_k|}.$$

Then, by Theorem 3.9, there is some non-empty finite $u \subseteq K$ such that

(i) $\left| \dfrac{|B \cap u|}{|u|} - \Xi'(B) \right| < \varepsilon'$ for any $B \in P$, and

(ii) $\dfrac{1}{|u|} \displaystyle\sum_{k \in u} f_i(k) > 1 - \varepsilon_i - \varepsilon'$ for any $i < i^*$.

Since $\Xi'(B) = \Xi(B)$ for all $B \in \mathcal{P}(K) \cap V$, we obtain (1).

Back in V, find $q' \leq q$ forcing the above and such that either $q' \leq r^i_\ell$ or $q' \perp r^i_\ell$ for all $i < i^*$ and $\ell \in \bigcup_{k \in u} I_k$. Then, q' decides the value of $f_i(k)$ for all $i < i^*$ and $k \in u$, even more, q' forces

$$f_i(k) = \frac{\{\ell \in I_k \colon q' \leq r^i_\ell\}}{|I_k|}.$$

Then, (2) follows.

(II) $\Rightarrow$ (I): Assume (II) and $(\Xi, \bar{I}) \in \mathcal{Y}$. Let G be $\mathbb{P}$-generic over V and work in $V[G]$. Consider the set

$$\Omega := \left\{ (\alpha, \varepsilon, \bar{r}) \colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q},\ \bar{r} \in Q^W_{\alpha, \varepsilon} \cap V,\ \lim^{\Xi, \bar{I}} \bar{r} \in G \right\}.$$

For each $\alpha < \mu$, $\varepsilon \in (0,1) \cap \mathbb{Q}$ and $\bar{r} \in Q^W_{\alpha, \varepsilon} \cap V$, define

$$f_{\alpha, \varepsilon, \bar{r}}(k) := \frac{\{\ell \in I_k \colon r_\ell \in \dot{G}\}}{|I_k|}, \qquad K_{\alpha, \varepsilon, \bar{r}} := [1 - \varepsilon, \infty).$$

To prove (I), it is enough to check Theorem 3.9 (I) for Ω. Indeed, assume

- $i^* < \omega$,
- $(\alpha_i, \varepsilon_i, \bar{r}^i) \in \Omega$ for $i < i^*$,
- $P \in \mathbf{P}^{\Xi}$, and
- $\varepsilon' > 0$.

Back in V, let $q \in \mathbb{P}$ be a condition forcing the above such that, wlog, $q \leq \lim^{\Xi,\bar{I}} \bar{r}^i$ for all $i < i^*$. Then, by (II), there exists some $q' \leq q$ in $\mathbb{P}$ and a finite non-empty $u \subseteq K$ satisfying (1) and (2). This density argument allows to find such a q' in G. Therefore, in $V[G]$, for any $i < i^*$,

$$\frac{1}{|u|} \sum_{k \in u} f_{\alpha_i, \varepsilon_i, \bar{r}^i}(k) \geq \frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} > 1 - \varepsilon_i - \varepsilon'. \qquad \square$$

We now proceed to prove Theorem 3.8. Some preparation is needed to define the correct limit function and the witness for the uniform σ-$\mathcal{Y}_*$-linkedness.

Definition 3.11. For $s \in T_* \smallsetminus \{\langle\ \rangle\}$, define

$$E'_s := \left\{ p \in \tilde{\mathbb{E}} \colon \operatorname{st}(p) = s \text{ and } \|p\|_t \geq 1 + \frac{1}{|s|} + 4 \log_{b_*(t)} |t| \text{ for all } t \in p \text{ above } s \right\},$$
$$E' := \bigcup_{s \in T_* \smallsetminus \{\langle\ \rangle\}} E'_s.$$

Note that, by Assumption 3.1, E' is dense in $\tilde{\mathbb{E}}$ (which can be proved similarly as Lemma 2.9).

In the following lemma, we show how to homogenize finitely many conditions in $\tilde{\mathbb{E}}$.

Lemma 3.12 (cf. [KST19, Lem.1.20, Step 1])**.** *Let $s \in T_* \smallsetminus \{\langle\ \rangle\}$, I a finite set and $\{r_\ell \colon \ell \in I\} \subseteq E'_s$. Then there is a condition $r^* \in \tilde{\mathbb{E}}$ with stem s such that*

(i) $\|\operatorname{succ}_{r^*}(t)\| \geq 1 + \frac{1}{|s|} + 2 \log_{b_*(t)} |t|$ *for any $t \in r^*$ above s, and*

(ii) *The set* $\left\{ r \in \tilde{\mathbb{E}} \colon \frac{\{\ell \in I \colon r \leq r_\ell\}|}{|I|} > 1 - \frac{1}{|s|} \right\}$ *is dense below r^*.*

The condition r^* constructed in the proof is called the *pseudo-fusion of* $\{r_\ell \colon \ell \in I\}$.

Proof. For $n \geq |s|$, denote

$$\delta_n := \begin{cases} \left(1 - \frac{1}{|s|}\right)\left(1 + \frac{1}{n-1}\right) & \text{if } n > |s|, \\ 1 & \text{if } n = |s|. \end{cases}$$

For $t \in T_*$ above s and $n \geq |s|$, define

$$J_t := \{\ell \in I \colon t \in r_\ell\},$$
$$L_n := \left\{ t \in T_* \colon |t| = n,\ \frac{|J_t|}{|I|} \geq \delta_n \right\}.$$

We define r^* by recursion on the height such that its n-th level is contained in L_n for $n \geq |s|$: up to level $|s|$, r^* is determined by s (i.e. just the stem); and, when $n \geq |s|$ and r^* is defined up to height n, for each $t \in r^*$ at level n we set $\operatorname{succ}_{r^*}(t) := \operatorname{succ}_{T_*}(t) \cap L_{n+1}$. Apply $b_*(t)$-co-bigness to $\langle \operatorname{succ}_{r_\ell}(t) \colon \ell \in J_t \rangle$ and $\ell \in J_t \mapsto \frac{1}{|J_t|}$ to obtain

$$\begin{aligned}\left\| \left\{ t' \in \operatorname{succ}_{T_*}(t) \colon |J_{t'}| \geq |J_t| \left(1 - \frac{1}{n^2}\right) \right\} \right\|_t &\geq 1 + \frac{1}{|s|} + 4 \log_{b_*(t)} |t| - \log_{b_*(t)} n^2 \\ &= 1 + \frac{1}{|s|} + 2 \log_{b_*(t)} |t|.\end{aligned}$$

Since $t \in L_n$, any t' in this set satisfies

$$|J_{t'}| \geq |J_t| \left(1 - \frac{1}{n^2}\right) \geq |I|\delta_n \left(1 - \frac{1}{n^2}\right) = |I|\delta_{n+1},$$

so $t' \in L_{n+1}$, i.e. $t' \in \operatorname{succ}_{r^*}(t)$. Therefore, $\|\operatorname{succ}_{r^*}(t)\|_t \geq 1 + \frac{1}{|s|} + 2 \log_{b_*(t)} |t|$ which establishes (i).

We know show (ii). Let $r \leq r^*$. By strengthening r if necessary, by Lemma 2.9 we may assume that $r \in E_{t,|I|+1}$ for some $t \in r_*$ above s of length $\geq |I| + 1$. For $\ell \in J_t$, $t \in r_\ell$ and $r_\ell \in E'_s$, so $r_\ell|^t \in E_{t,|I|+1}$. Then there is some $r' \in \tilde{\mathbb{E}}$ stronger than r and r_ℓ for all $\ell \in J_t$ because $E_{t,|I|+1}$ is $(|I| + 1)$-linked. On the other hand, $t \in L_{|t|}$, so $\frac{|J_t|}{|I|} \geq \delta_{|t|}$. Therefore,

$$\frac{|\{\ell \in I \colon r' \leq r_\ell\}|}{|I|} \geq \frac{|J_t|}{|I|} \geq \delta_{|t|} > 1 - \frac{1}{|s|}. \qquad \square$$

Using the previous result, we show how to define fam-limits on $\tilde{\mathbb{E}}$.

Theorem 3.13 (cf. [KST19, Lem. 1.20])**.** *Let $\Xi \colon \mathcal{P}(K) \to [0, 1]$ be a fam with the uap and $\bar{I} = \langle I_k \colon k \in K \rangle$ a partition of a set W into finite sets. Then, for any $s \in T_* \smallsetminus \{\langle\ \rangle\}$ there is a function $\lim^{\Xi,\bar{I}} \colon (E'_s)^W \to \tilde{\mathbb{E}}$ such that $\lim^{\Xi,\bar{I}} \bar{r}$ has stem s for any $\bar{r} \in (E'_s)^W$, and satisfying: For any*

- $i^* < \omega$,
- $s_i \in T_* \smallsetminus \{\langle\ \rangle\}$,
- $\bar{r}^i = \langle r^i_\ell \colon \ell \in W \rangle \in (E'_{s_i})^W$ *for* $i < i^*$,
- $P \in \mathbf{P}^\Xi$,
- $\varepsilon' > 0$, *and*
- $q \in \tilde{\mathbb{E}}$ *stronger than* $\lim^{\Xi,\bar{I}} \bar{r}^i$ *for all* $i < i^*$,

there is some $q' \leq q$ in $\tilde{\mathbb{E}}$ and $u \subseteq K$ finite non-empty such that

(1) $\left| \frac{|u \cap B|}{|u|} - \Xi(B) \right| < \varepsilon'$ *for all* $B \in P$, *and*

(2) $\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} > 1 - \frac{2}{|s_i|} - \varepsilon'$ *for all* $i < i^*$.

Proof. Let $s \in T_*$. We first show how to define $\lim^{\Xi,\bar{I}} \bar{r}$ for $\bar{r} = \langle r_\ell \colon \ell \in W\rangle \subseteq (E'_s)^W$. For each $k \in K$, let r^*_k be the pseudo-fusion of $\langle r_\ell \colon \ell \in I_k\rangle$ as in Lemma 3.12. For each $t \in T_*$ set $Z_t = Z^{\bar{r}}_t := \{k \in K \colon t \in r^*_k\}$. Note that $Z_s = K$. By recursion on the height, we construct $r^* := \lim^{\Xi,\bar{I}} \bar{r}$ in $\tilde{\mathbb{E}}$ with stem s such that, for any $t \in r^*$ above s, $\Xi(Z_t) \geq \delta_{|t|}$, where δ_n for $n \geq |s|$ is as in the proof of Lemma 3.12. Up to $|s|$, r^* is determined by s. Let $n \geq |s|$ and assume we have constructed r^* up to height n. It is enough to show how $\operatorname{succ}_{r^*}(t)$ is defined for any $t \in r^*$ at level n. Enumerate the finite set $\{\operatorname{succ}_{r^*_k}(t) \colon k \in Z_t\}$ as $\{a_j \colon j < m\}$. Consider the function $g \colon m \to [0,1]$ defined by

$$g(j) := \frac{\Xi(\{k \in Z_t \colon \operatorname{succ}_{r^*_k}(t) = a_j\})}{\Xi(Z_t)}$$

Then, by $b_*(t)$-co-bigness,

$$\left\| \left\{ t' \in \operatorname{succ}_{T_*}(t) \colon \frac{\Xi(Z_{t'})}{\Xi(Z_t)} \geq 1 - \frac{1}{n^2} \right\} \right\|_t \geq 1 + \frac{1}{|s|} + 2\log_{b_*(t)} |t| - \log_{b_*(t)} n^2 = 1 + \frac{1}{|s|},$$

so define $\operatorname{succ}_{r^*}(t)$ as this set. On the other hand, for $t' \in \operatorname{succ}_{r^*}(t)$,

$$\Xi(Z_{t'}) \geq \Xi(Z_t)\left(1 - \frac{1}{n^2}\right) \geq \delta_n \left(1 - \frac{1}{n^2}\right) = \delta_{n+1}.$$

It is clear that $r^* \in \tilde{\mathbb{E}}$ and $\operatorname{st}(r^*) = s$.

We now prove that this limit works. Work under the assumption for (1) and (2). For $i < i^*$ and $k \in K$, let r^i_k be the pseudo-fusion of $\langle r^i_\ell \colon \ell \in I_k\rangle$, $r^i := \lim^{\Xi,\bar{I}} \bar{r}^i$ and $Z^i_t := Z^{\bar{r}^i}_t$. So we have that $q \leq r^i$ for all $i < i^*$.

Let $t \in q$ above the stem. By Lemma 3.5 applied to the atoms of the field generated by $P \cup \{Z^i_t \colon i < i^*\}$ (there are at most $|P| 2^{i^*}$ many), there is some non-empty finite set $u_t \subseteq W$ satisfying (1) and, for $i < i^*$,

$$\frac{|Z^i_t \cap u_t|}{|u_t|} > \delta^i_{|t|} - \varepsilon' > 1 - \frac{1}{|s_i|} - \varepsilon'$$

where δ^i_n is as δ_n in the proof of Lemma 3.12 for s_i and $n \geq |s_i|$. Moreover, there is some $M > 0$ such that $|u_t| \leq M$ for all t as above (concretely, $M := M_{\varepsilon', |P| 2^{i^*}}$ as in Lemma 3.5). So pick $t \in q$ of large enough length such that $q|^t \in E_{t, M i^* + 1}$, and set $u := u_t$. Then, by Lemma 2.9, there is a lower bound q_1 of $\{q\} \cup \{r^i_k \colon i < i^*,\ k \in Z^i_t \cap u\}$. By using Lemma 3.12 (ii) $\left|\bigcup_{i<i^*} Z^i_t \cap u\right|$-many times, we can find some $q' \leq q_1$ such that, for any $i < i^*$ and $k \in Z^i_t \cap u$,

$$\frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} > 1 - \frac{1}{|s_i|}.$$

Then, for $i < i^*$,

$$\begin{aligned}\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} &> \frac{|Z^i_t \cap u|}{|u|}\left(1 - \frac{1}{|s_i|}\right) > \left(1 - \frac{1}{|s_i|} - \varepsilon'\right)\left(1 - \frac{1}{|s_i|}\right) \\ &= 1 - \frac{2}{|s_i|} - \varepsilon' + \frac{1}{|s_i|^2} + \frac{\varepsilon'}{|s_i|} > 1 - \frac{2}{|s_i|} - \varepsilon'.\end{aligned}$$

□

Theorem 3.8 follows directly by the previous theorem.

Proof of Theorem 3.8. For $s \in T_* \smallsetminus \{\langle\ \rangle\}$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$, define

$$Q_{s,\varepsilon} := \begin{cases} E'_s & \text{if } \dfrac{2}{|s|} \leq \varepsilon, \\ \{T_*|^s\} & \text{otherwise.} \end{cases}$$

It is easy to show that, for any $\varepsilon \in (0,1) \cap \mathbb{Q}$,

$$\bigcup_{s \in T_* \smallsetminus \{\langle\ \rangle\}} Q_{s,\varepsilon} \supseteq \bigcup \left\{ E'_s \colon s \in T_* \smallsetminus \{\langle\ \rangle\},\ |s| \geq \frac{2}{\varepsilon} \right\} \text{ is dense in } \tilde{\mathbb{E}}.$$

By checking (II) of Theorem 3.10, we show that $\langle Q_{s,\varepsilon} \colon s \in T_* \smallsetminus \{\langle\ \rangle\},\ \varepsilon \in (0,1) \cap \mathbb{Q} \rangle$ witnesses that $\tilde{\mathbb{E}}$ is σ-$\mathcal{Y}_*$-linked. Assume

- $(\Xi, \bar{I}) \in \mathcal{Y}_*$,
- $i^* < \omega$,
- $(s_i, \varepsilon_i) \in (T_* \smallsetminus \{\langle\ \rangle\}) \times ((0,1) \cap \mathbb{Q})$,
- $\bar{r}^i = \langle r^i_\ell \colon \ell \in W \rangle \in Q^W_{s_i,\varepsilon_i}$ for $i < i^*$,
- $P \in \mathbf{P}^\Xi$,
- $\varepsilon' > 0$, and
- $q \in \tilde{\mathbb{E}}$ stronger than $\lim^{\Xi,\bar{I}} \bar{r}^i$ for all $i < i^*$.

When $\varepsilon_i < \frac{2}{|s_i|}$ we are dealing with the singleton $Q_{s_i,\varepsilon_i} = \{T_*|^{s_i}\}$, for which the sequence $\bar{r}^i$ is constant and $\lim^{\Xi,\bar{I}} \bar{r}^i := T_*|^{s_i}$, so $q \leq T_*|^{s_i}$. Now, apply Theorem 3.13 to those $i < i^*$ such that $\frac{2}{|s_i|} \leq \varepsilon_i$, and find $q' \leq q$ in $\tilde{\mathbb{E}}$ and a finite non-empty $u \subseteq K$ satisfying (1) and (2) (for those i). Then, whenever $\frac{2}{|s_i|} \leq \varepsilon_i$,

$$\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} > 1 - \frac{2}{|s_i|} - \varepsilon' \geq 1 - \varepsilon_i - \varepsilon'.$$

On the other hand, whenever $\varepsilon_i < \frac{2}{|s_i|}$,

$$\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} = 1 > 1 - \varepsilon_i - \varepsilon'. \qquad \square$$

Remark 3.14. In [KST19], the function loss: $E' \to \mathbb{Q}$ is used most of the time, but it is not essential as seen above. Their definition of loss, adapted to this paper, is basically $\operatorname{loss}(p) = \frac{2}{|\operatorname{st}(p)|}$. Hence, in the proof of Theorem 3.8, $Q_{s,\varepsilon}$ refers to the set of $p \in E'$ with stem s such that $\operatorname{loss}(p) \leq \varepsilon$ (note that there are no such conditions when $\varepsilon < \frac{2}{|s|}$).

Remark 3.15. In [Uri24], Uribe-Zapata defined the notion *μ-intersection-linked* for posets, where μ is an infinite cardinal, using the intersection number from Kelley [Kel59]. Strictly speaking, this property should be part of Definition 3.6 (2) (μ-$\mathcal{Y}$-linked) for $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q} \rangle$, so that FS iterations of such posets have fam-limits [Uri23, CMU], but we excluded it for practicality. Moreover, we proved that, whenever $Q \subseteq \mathbb{P}$ is $(\Xi, \bar{I}, \varepsilon)$-linked, $K = \omega$ and $\lim_{k \to \infty} |I_k| = \infty$, Q has intersection number $\geq 1 - \varepsilon$ [CMU]. For this reason, we obtain this condition about the intersection number for free in many cases, e,g. for measure algebras and $\tilde{\mathbb{E}}$. But note that, for the later, Lemma 3.12 implies that E'_s has intersection number $\geq 1 - \frac{1}{|s|}$.

4 Uf-limits on intervals

Recall that an ultrafilter on a Boolean algebra can be seen as a fam taking values in $\{0,1\}$. In this sense, any ultrafilter has the uap.

We present a version of Definition 3.6 for ultrafilters,[3] which is the notion we call *ultrafilter-limits for intervals* in the Introduction.

Definition 4.1. Let $\mathbb{P}$ be a poset.

(1) Let D be an ultrafilter on $\mathcal{P}(K)$ for some non-empty set K, $\bar{I} = \langle I_k \colon k \in K\rangle$ a partition of a set W into finite sets, and $\varepsilon > 0$.

A set $Q \subseteq \mathbb{P}$ is $(D, \bar{I}, \varepsilon)^*$-*linked*[4] if there is a function $\lim\colon Q^W \to \mathbb{P}$ and a $\mathbb{P}$-name $\dot{D}'$ of an ultrafilter on $\mathcal{P}(K)$ extending D such that, for any $\bar{p} = \langle p_\ell \colon \ell \in W\rangle \in Q^W$,

$$\lim \bar{p} \Vdash \left\{ k \in K \colon \frac{|\{\ell \in I_k \colon p_\ell \in \dot{G}\}|}{|I_k|} \geq 1 - \varepsilon \right\} \in \dot{D}'. \tag{4.1.1}$$

(2) Let μ be an infinite cardinal, and let $\mathcal{D} \subseteq \mathcal{D}_*$, where $\mathcal{D}_*$ is the class of all pairs $(D, \bar{I})$ such that D is an ultrafilter on some $\mathcal{P}(K)$ (with $K \neq \emptyset$) and $\bar{I} = \langle I_k \colon k \in K\rangle$ is a pairwise disjoint family of finite non-empty sets.

The poset $\mathbb{P}$ is μ-$\mathcal{D}$-*linked*, witnessed by $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$, if:

  (i) Each $Q_{\alpha,\varepsilon}$ is $(D, \bar{I}, \varepsilon)^*$-linked for any $(D, \bar{I}) \in \mathcal{D}$.

  (ii) For $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\bigcup_{\alpha<\omega} Q_{\alpha,\varepsilon}$ is dense in $\mathbb{P}$.

(3) The poset $\mathbb{P}$ is *uniformly* μ-$\mathcal{D}$-*linked* if there is some $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$ as above, such that in (1) the name $\dot{D}'$ only depends on $(D, \bar{I})$ (and not on any $Q_{\alpha,\varepsilon}$, although we may have different limits on each $Q_{\alpha,\varepsilon}$).

We write σ-$\mathcal{D}$-*linked* when $\mu = \aleph_0$.

Remark 4.2. In (1) of Definition 4.1, if $\bar{I}$ is composed by singletons, say $I_k = \{k\}$, then Equation 4.1.1 is equivalent to

$$\lim \bar{p} \Vdash \{k \in K \colon p_k \in \dot{G}\} \in \dot{D}',$$

which means that Q *has* D-*limits* (cf. [GMS16, Mej24, CM24]).

In the case that $\mathcal{D}$ is the collection of all pairs $(D, \bar{I})$ such that D is an ultrafilter on $\mathcal{P}(\omega)$ and $\bar{I} = \langle \{k\} \colon k < \omega\rangle$, we obtain the notion of (uniform) μ-uf-lim-linked as in [Mej24, CM24], which is the notion of forcings with ultrafitler limits from [GMS16].

Example 4.3.

(1) Similar to Example 3.7 (1), all singletons are $(D, \bar{I}, \varepsilon)^*$-linked for any tuple $(D, \bar{I}, \varepsilon)$, and $\mathbb{P}$ is uniformly $|\mathbb{P}|$-$\mathcal{D}_*$-linked. In particular, Cohen forcing is σ-$\mathcal{D}_*$-linked.

[3] This may not equivalent to Definition 3.6 for fams with values in $\{0,1\}$, since they may not be extended to an ultrafilter (but to some fam) in the generic extension.

[4] We add the $*$ to avoid confusion with Definition 3.6 when D is intepreted as a fam.

(2) From [GMS16, BCM21] we have that several posets associated with localization and anti-localization are uniformly σ-uf-lim-linked. However, $\mathbb{E}$ and the localization posets are not σ-$\mathcal{D}_*$-linked because, similar to Example 3.7 (3), we have that σ-$\mathcal{D}_*$-linked poset do not increase $\mathrm{non}(\mathcal{E})$. The case of anti-localization posets is not clear (likewise in the case of fam-limits).

(3) Cardona and the author [CM24] presented uniformly σ-uf-lim-linked posets increasing $\mathrm{non}(\mathcal{MA})$ and $\mathrm{add}(\mathcal{SN})$, where $\mathcal{MA}$ denotes the ideal of meager-additive subsets of 2^ω, and $\mathcal{SN}$ is the ideal of strong measure zero subsets of 2^ω. Since $\mathrm{non}(\mathcal{MA}) \leq \mathrm{non}(\mathcal{E})$, many instances of the first poset cannot be σ-$\mathcal{D}_*$-linked neither σ-$\mathcal{Y}_*$-linked.

In contrast with Example 3.7 (2), it is unclear whether random forcing is σ-uf-lim-linked.

Similar to Theorem 3.10, we can characterize uniform μ-$\mathcal{D}$-linkedness as follows.

Theorem 4.4. *Let μ be a cardinal, $\mathbb{P}$ a poset, $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu,\ \varepsilon \in (0,1)\cap\mathbb{Q}\rangle$ a sequence of subsets of $\mathbb{P}$, $\mathcal{D} \subseteq \mathcal{D}_*$ and, for each $(D,\bar{I}) \in \mathcal{D}$, $\alpha<\mu$ and $\varepsilon \in (0,1)\cap\mathbb{Q}$, $\lim^{D,\bar{I}} \colon Q^W_{\alpha,\varepsilon} \to \mathbb{P}$ where $W := \bigcup_{k\in K} I_k$. Then, the following statements are equivalent.*

(I) *$\mathbb{P}$ is uniformly μ-$\mathcal{D}$-linked witnessed by $\langle Q_{\alpha,\varepsilon} \colon \alpha < \mu,\ \varepsilon \in (0,1)\cap\mathbb{Q}\rangle$.*

(II) *Definition 4.1 (2) (ii) holds and, for any*

- *$(D,\bar{I}) \in \mathcal{D}$,*
- *$i^* < \omega$,*
- *$(\alpha_i,\varepsilon_i) \in \mu \times ((0,1)\cap\mathbb{Q})$,*
- *$\bar{r}^i = \langle r^i_\ell \colon \ell \in W\rangle \in Q^W_{\alpha_i,\varepsilon_i}$ for $i<i^*$,*
- *$a \in D$, and*
- *$q \in \mathbb{P}$ stronger than $\lim^{D,\bar{I}} \bar{r}^i$ for all $i<i^*$,*

there are some $q' \leq q$ and $k \in a$ such that, for all $i<i^*$,

$$\frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} \geq 1-\varepsilon_i. \tag{4.4.1}$$

Proof. (I) $\Rightarrow$ (II): Assume (I) and the assumptions of (II). Let $\dot{D}'$ be as in Definition 4.1 (3), and let G be $\mathbb{P}$-generic over V such that $q \in G$, and $D' := \dot{D}'[G]$. In $V[G]$, since $q \leq \lim^{D,\bar{I}} \bar{r}^i$ for all $i<i^*$, by Equation 4.1.1

$$\{k \in K \colon f_i(k) \geq 1-\varepsilon_i\} \in D' \text{ where } f_i(k) := \frac{|\{\ell \in I_k \colon r^i_\ell \in \dot{G}\}|}{|I_k|}.$$

Therefore, $a \cap \bigcap_{i<i^*}\{k \in K \colon f_i(k) \geq 1-\varepsilon_i\} \in D'$, so this set is non-empty and contains some element k.

Back in V, find $q' \leq q$ forcing the above and such that either $q' \leq r^i_\ell$ or $q' \perp r^i_\ell$ for all $i<i^*$ and $\ell \in I_k$. Then, q' decides the value of $f_i(k)$ for all $i<i^*$, even more, q' forces

$$f_i(k) = \frac{\{\ell \in I_k \colon q' \leq r^i_\ell\}}{|I_k|}.$$

Then, Equation 4.4.1 follows.

(II) $\Rightarrow$ (I): Assume (II) and $(D,\bar{I}) \in \mathcal{D}$. Let G be $\mathbb{P}$-generic over V and work in $V[G]$. Consider the set

$$\Omega := \left\{(\alpha,\varepsilon,\bar{r})\colon \alpha<\mu,\ \varepsilon\in(0,1)\cap\mathbb{Q},\ \bar{r}\in Q^W_{\alpha,\varepsilon}\cap V,\ \lim^{D,\bar{I}}\bar{r}\in G\right\}.$$

For each $\alpha<\mu$, $\varepsilon\in(0,1)\cap\mathbb{Q}$ and $\bar{r}\in Q^W_{\alpha,\varepsilon}\cap V$, define

$$f_{\alpha,\varepsilon,\bar{r}}(k) := \frac{\{\ell\in I_k\colon r_\ell\in\dot{G}\}}{|I_k|},\qquad a_{\alpha,\varepsilon,\bar{r}} := \{k\in K\colon f_{\alpha,\varepsilon,\bar{r}}(k)\geq 1-\varepsilon\}.$$

To prove (I), it is enough to check that the family $D\cup\{a_{\alpha,\varepsilon,\bar{r}}\colon (\alpha,\varepsilon,\bar{r})\in\Omega\}$ has the finite intersection property. Indeed, assume

- $i^*<\omega$,
- $(\alpha_i,\varepsilon_i,\bar{r}^i)\in\Omega$ for $i<i^*$, and
- $a\in D$.

Back in V, let $q\in\mathbb{P}$ be a condition forcing the above such that, wlog, $q\leq\lim^{D,\bar{I}}\bar{r}^i$ for all $i<i^*$. Then, by (II), there exists some $q'\leq q$ in $\mathbb{P}$ and some $k\in a$ satisfying Equation 4.4.1 for all $i<i^*$. This density argument allows to find such a q' in G. Therefore, in $V[G]$, for any $i<i^*$,

$$f_{\alpha_i,\varepsilon_i,\bar{r}^i}(k)\geq\frac{|\{\ell\in I_k\colon q'\leq r^i_\ell\}|}{|I_k|}\geq 1-\varepsilon_i,$$

so $k\in a\cap\bigcap_{i<i^*}a_{\alpha_i,\varepsilon_i,\bar{r}^i}$. □

The purpose of this section is to prove the following.

Theorem 4.5. *Under Assumption 3.1, $\tilde{\mathbb{E}}$ is uniformly σ-$\mathcal{D}_*$-linked.*

The strategy to prove this theorem is similar to Theorem 3.8.

Theorem 4.6. *Let $K\neq\emptyset$, D an ultrafilter on $\mathcal{P}(K)$ and $\bar{I}=\langle I_k\colon k\in K\rangle$ a partition of a set W into finite sets. Then, for any $s\in T_*\smallsetminus\{\langle\,\rangle\}$ there is a function $\lim^{D,\bar{I}}\colon (E'_s)^W\to\tilde{\mathbb{E}}$ such that $\lim^{D,\bar{I}}\bar{r}$ has stem s for any $\bar{r}\in(E'_s)^W$, and satisfying: For any*

- *$i<i^*$*
- *$s_i\in T_*\smallsetminus\{\langle\,\rangle\}$,*
- *$\bar{r}^i=\langle r^i_\ell\colon \ell\in W\rangle\in(E'_{s_i})^W$ for $i<i^*$,*
- *$a\in D$, and*
- *$q\in\tilde{\mathbb{E}}$ stronger than $\lim^{D,\bar{I}}\bar{r}^i$ for all $i<i^*$,*

there are $q'\leq q$ in $\tilde{\mathbb{E}}$ and $k\in a$ such that, for any $i<i^*$,

$$\frac{|\{\ell\in I_k\colon q'\leq r^i_\ell\}|}{|I_k|}>1-\frac{1}{|s_i|}. \tag{4.6.1}$$

Proof. We proceed as in the proof of Theorem 3.13. Let $s \in T_*$. We first show how to define $\lim^{D,\bar{I}} \bar{r}$ for $\bar{r} = \langle r_\ell \colon \ell \in W\rangle \subseteq (E'_s)^W$. For each $k \in K$, let r^*_k be the pseudo-fusion of $\langle r_\ell \colon \ell \in I_k\rangle$ as in Lemma 3.12. For each $t \in T_*$ set $Z_t = Z^{\bar{r}}_t := \{k \in K \colon t \in r^*_k\}$. Define $r^* := \lim^{D,\bar{I}} \bar{r}$ such that, for $t \in T_*$,

$$t \in r^* \text{ iff } Z_t \in D.$$

Since the tree T_* is finitely branching, we obtain that, for any $t \in r^*$, $\{k \in K \colon \operatorname{succ}_{r^*}(t) = \operatorname{succ}_{r^*_k}(t)\} \in D$. This implies that r^* is a tree with stem s and

$$\|r^*\|_t \geq 1 + \frac{1}{|s|} + 2\log_{b_*(t)} |t|$$

for any $t \in r^*$ above s. Hence $r^* \in \tilde{\mathbb{E}}$.

We now prove that this limit works. Work under the assumptions of the bullet points. For $i < i^*$ and $k \in K$, let r^i_k be the pseudo-fusion of $\langle r^i_\ell \colon \ell \in I_k\rangle$, $r^i := \lim^{D,\bar{I}} \bar{r}^i$ and $Z^i_t := Z^{\bar{r}^i}_t$. So we have that $q \leq r^i$ for all $i < i^*$.

Pick $t \in q$ large enough such that $q|^t$ and $r^i|^t$ are in E_{t,i^*+1} for $i < i^*$ (which is fine because $t \in r^i$ for all $i < i^*$). Then $a \cap \bigcap_{i<i^*} Z^i_t \in D$, so this intersection is non-empty and we can pick some k in there, i.e. $k \in a$ and $t \in r^i_k$ for all $i < i^*$. Hence, by Lemma 2.9, there is a lower bound q_1 of $\{q\} \cup \{r^i_k \colon i < i^*\}$. Apply Lemma 3.12 (ii) i^*-many times to find $q' \leq q_1$ such that, for $i < i^*$,

$$\frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} > 1 - \frac{1}{|s_i|}. \qquad \square$$

Proof of Theorem 4.5. For $s \in T_* \smallsetminus \{\langle\,\rangle\}$ and $\varepsilon \in (0,1) \cap \mathbb{Q}$, define

$$Q'_{s,\varepsilon} := \begin{cases} E'_s & \text{if } \dfrac{1}{|s|} \leq \varepsilon, \\ \{T_*|^s\} & \text{otherwise.} \end{cases}$$

It is easy to show that, for any $\varepsilon \in (0,1) \cap \mathbb{Q}$,

$$\bigcup_{s \in T_* \smallsetminus \{\langle\,\rangle\}} Q'_{s,\varepsilon} \supseteq \bigcup \left\{ E'_s \colon s \in T_* \smallsetminus \{\langle\,\rangle\},\ |s| \geq \frac{1}{\varepsilon} \right\} \text{ is dense in } \tilde{\mathbb{E}}.$$

By checking (II) of Theorem 4.4, we show that $\langle Q'_{s,\varepsilon} \colon s \in T_* \smallsetminus \{\langle\,\rangle\},\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$ witnesses that $\tilde{\mathbb{E}}$ is σ-$\mathcal{D}_*$-linked. Assume

- $(D, \bar{I}) \in \mathcal{D}_*$,
- $i^* < \omega$,
- $(s_i, \varepsilon_i) \in (T_* \smallsetminus \{\langle\,\rangle\}) \times ((0,1) \cap \mathbb{Q})$,
- $\bar{r}^i = \langle r^i_\ell \colon \ell \in W\rangle \in Q'^W_{s_i,\varepsilon_i}$ for $i < i^*$,
- $a \in D$, and
- $q \in \tilde{\mathbb{E}}$ stronger than $\lim^{D,\bar{I}} \bar{r}^i$ for all $i < i^*$.

When $\varepsilon_i < \frac{1}{|s_i|}$ we are dealing with the singleton $Q'_{s_i,\varepsilon_i} = \{T_*|^{s_i}\}$, for which the sequence $\bar{r}^i$ is constant and $\lim^{D,\bar{I}} \bar{r}^i = T_*|^{s_i}$, so $q \leq T_*|^{s_i}$. Now, apply Theorem 4.6 to those $i < i^*$ such that $\frac{1}{|s_i|} \leq \varepsilon_i$, and find $q' \leq q$ in $\tilde{\mathbb{E}}$ and $k \in a$ satisfying Equation 4.6.1 (for those i). Then, whenever $\frac{1}{|s_i|} \leq \varepsilon_i$,

$$\frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} > 1 - \frac{1}{|s_i|} \geq 1 - \varepsilon_i.$$

On the other hand, whenever $\varepsilon_i < \frac{1}{|s_i|}$,

$$\frac{|\{\ell \in I_k \colon q' \leq r^i_\ell\}|}{|I_k|} = 1 > 1 - \varepsilon_i. \qquad \square$$

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