STEPPING-UP SEMIPROPERNESS OF NAMBA FORCINGS

KENTA TSUKUURA

Abstract. We show that if both of $\mathrm{Nm}(\kappa,\lambda)$ and $\mathrm{Nm}(\lambda^+)$ are semiproper then $\mathrm{Nm}(\kappa,\lambda^+)$ is semiproper.

1. Introduction

In [3], we introduced Namba forcings $\mathrm{Nm}(\kappa,\lambda)$ over $\mathcal{P}_\kappa\lambda$ and studied these. We also showed that for regular cardinals $\aleph_2 \leq \kappa \leq \lambda$, the following holds.

(1) $\mathrm{Nm}(\kappa,\lambda)$ is semiproper if $\mathrm{Nm}(\kappa, 2^{\lambda^{<\kappa}})$ is semiproper.
(2) $\mathrm{Nm}(\kappa,\lambda)$ forces that $\dot{\mathrm{cf}}(\delta) = \omega$ for all $\delta \in [\kappa,\lambda] \cap \mathrm{Reg}$.

These properties and [2, Theorem 2.2 of Chapter XI] shows that $\mathrm{Nm}(\kappa,\lambda)$ and $\mathrm{Nm}(\lambda^+)$ are semiproper if $\mathrm{Nm}(\kappa,\lambda^+)$ is semiproper and $2^\lambda = \lambda^+$.

In this paper, we show the converse of this as follows.

Theorem 1.1. *Suppose that $\kappa \leq \lambda$ are regular cardinals greater than $\aleph_2$. If both of $\mathrm{Nm}(\kappa,\lambda)$ and $\mathrm{Nm}(\lambda^+)$ are semiproper then $\mathrm{Nm}(\kappa,\lambda^+)$ is semiproper.*

The structure of this paper is as follows. In Section 2, we recall the basic materials for Namba forcings. Section 3 is devoted to Theorem 1.1.

2. Preliminaries

We use [1] as a reference for set theory in general. In this paper, κ and λ denote regular cardinals greater than $\aleph_2$ unless otherwise stated. Reg is the class of regular cardinals.

A tree $p \subseteq [X]^{<\omega}$ is a set that is closed under the initial segment, that is, $s \in p$ and $l \sqsubseteq s$ implies $l \in p$. By $l \sqsubseteq s$, we mean that s end-extends l. Note that, when we consider a tree structure or an end-extension relation $l \sqsubseteq s$, we assume that X has an ordered relation and l and s are finite increasing sequences in the sense of a fixed order. In this paper, X is λ or $\mathcal{P}_\kappa\lambda$. These have natural orders.

2.1. **Namba forcings.** Namba forcing $\mathrm{Nm}(\lambda)$ is the set of all Namba trees over λ. Namba tree is a tree $p \subseteq [\lambda]^{<\omega}$ with the following conditions:

(1) p has a trunk $\mathrm{tr}(p)$, that is, $\mathrm{tr}(p)$ is the maximal $t \in p$ such that $\forall s \in p(s \subseteq t \vee t \subseteq s)$.
(2) For all $s \in p$, if $s \supseteq \mathrm{tr}(p)$ then $\mathrm{Suc}(s) = \{\xi < \lambda \mid s^\frown\langle\xi\rangle \in p\}$ is unbounded in λ.

$\mathrm{Nm}(\lambda)$ is ordered by inclusion.

Namba forcing $\mathrm{Nm}(\kappa,\lambda)$ is the set of all Namba trees over $\mathcal{P}_\kappa\lambda$. Then Namba tree is a tree $p \subseteq [\mathcal{P}_\kappa\lambda]^{<\omega}_{\subseteq}$ with the following conditions:

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(1) p has a trunk $\mathrm{tr}(p)$, that is, $\mathrm{tr}(p)$ is the maximal $t \in p$ such that $\forall s \in p(s \subseteq t \vee t \subseteq s)$.
(2) For all $s \in p$, if $s \supseteq \mathrm{tr}(p)$ then $\mathrm{Suc}(s) = \{a \in \mathcal{P}_\kappa\lambda \mid s^\frown\langle a\rangle \in p\}$ is unbounded in $\mathcal{P}_\kappa\lambda$.

Here, $[\mathcal{P}_\kappa\lambda]^{<\omega}_{\subseteq}$ is the set of all $\subseteq$-increasing finite set of subsets of $\mathcal{P}_\kappa\lambda$. $\mathrm{Nm}(\kappa,\lambda)$ is ordered by inclusion.

2.2. **Galvin games.** For an ideal I over Z and $A \in I^+$, the Galvin game $\Game(I, A)$ is a game of length ω with two players as follows:

Player I	$F_0 : A \to \omega_1$		$\cdots$	$F_i : A \to \omega_1$		$\cdots$
Player II		$\xi_0 < \omega_1$	$\cdots$		$\xi_i < \omega_1$	$\cdots$

Let $\xi = \sup_n \xi_n$. II wins if $\bigcap_{n<\omega} F_n^{-1}\xi \in I^+$. $\Phi_{\kappa,\lambda}$ is the statement that claims Player II has a winning strategy for $\Game(J^{\mathrm{bd}}_{\kappa\lambda}, A)$ for all $A \in (J^{\mathrm{bd}}_{\kappa\lambda})^+$. Φ_λ is the statement that claims Player II has a winning strategy for $\Game(J^{\mathrm{bd}}_{\lambda}, A)$ for all $A \in (J^{\mathrm{bd}}_{\lambda})^+$. Here, $J^{\mathrm{bd}}_{\kappa\lambda}$ and $J^{\mathrm{bd}}_{\lambda}$ are bounded ideals over $\mathcal{P}_\kappa\lambda$ and λ, respectively. The importance is as follows.

Lemma 2.1 (Shelah)**.** *The following are equivalent.*
(1) Φ_λ.
(2) $\mathrm{Nm}(\lambda)$ *is semiproper.*

Lemma 2.2 is an analogie of Lemma 2.1.

Lemma 2.2. *The following are equivalent.*
(1) $\Phi_{\kappa\lambda}$.
(2) $\mathrm{Nm}(\kappa,\lambda)$ *is semiproper.*

Proof. See [3]. □

3. Proof of Main Theorem

By Lemmas 2.1 and 2.2, for Theorem 1.1, it is enough to show

Lemma 3.1. $\Phi_{\kappa\lambda} \wedge \Phi_{\lambda^+} \to \Phi_{\kappa\lambda^+}$.

Proof. For $\lambda \leq \alpha < \lambda^+$, let $\varphi_\alpha : \alpha \to \lambda$ be a bijection. For an unbounded set $A \subseteq \mathcal{P}_\kappa\lambda^+$, $A^{\restriction\alpha} = \{\varphi_\alpha``(a \cap \alpha) \mid a \in A\}$ is unbounded in $\mathcal{P}_\kappa\lambda$. Let ws_α be Player II's winning strategy for $\Game(J^{\mathrm{bd}}_{\kappa\lambda}, A^{\restriction\alpha})$ for each α. Let ws^* be Player II's winning strategy for $\Game(J^{\mathrm{bd}}_{\lambda}, \lambda)$.

For each $a \in A^{\restriction\alpha}$, by the definition, there is a $b \in A$ such that $a = \varphi_\alpha``(b \cap \alpha)$. We fix a such b as $b_{a,\alpha}$. For an $F : A \to \omega_1$, define $F^\alpha : A^{\restriction\alpha} \to \omega_1$ by $F^{\restriction\alpha}(a) = F(b_{a,\alpha})$.

Let us define a strategy of Player II for $\Game(J^{\mathrm{bd}}_{\kappa\lambda^+}, A)$. Suppose that $F_0, ..., F_n$ be Player I's partial play. Note that, for each $\alpha < \lambda^+$, Player II knows $\xi_{i,\alpha} = \mathrm{ws}_\alpha(F_0^{\restriction\alpha}, ..., F_i^{\restriction\alpha})$ for each $i \leq n$. Then $H_i(\alpha) = \xi_{i,\alpha}$ defines a mapping $H_i : \lambda^+ \to \omega_1$. Let us instruct Player II to choose $\xi_n = \mathrm{ws}^*(H_0, ..., H_n)$, so we define $\mathrm{ws}(F_0, ..., F_n) = \xi_n$.

Suppose that Player I choosed $F_0, F_1, ...$ and Player II used ws. We claim that Player II wins. Let $\xi_\infty = \sup_i \xi_i$, here $\xi_n = \mathrm{ws}(F_0, ..., F_n)$. The goal is showing that $\bigcap_i F_i^{-1}\xi_\infty$ is unbounded.

For a $c \subseteq \mathcal{P}_\kappa\lambda^+$, there is an α such that $c \subseteq \alpha$ and $\alpha \in \bigcap_i H_i^{-1}\xi_\infty$. Note that $\bigcap_i H_i^{-1}\xi_\infty$ is unbounded subset of λ^+ since ws^* is a winning strategy. So we have

$$\bigcap_i (F_i^{\restriction\alpha})^{-1}(\sup_i \xi_{\alpha,i}) \subseteq \bigcap_i (F_i^{\restriction\alpha})^{-1}\xi_\infty.$$

Since ws_α is a winning strategy, these are unbounded subset of $\mathcal{P}_\kappa\lambda$. Note that $\bigcap_i (F_i^{\restriction\alpha})^{-1}\xi_\infty \subseteq A^{\restriction\alpha}$. Then there is an $a \in \bigcap_i (F_i^{\restriction\alpha})^{-1}\xi_\infty$ such that φ_α“$c \subseteq a$. Then $\xi_\infty > F_i^{\restriction\alpha}(a) = F_i(b_{a,\alpha})$. By the choice of $b_{a,\alpha}$, we have $b_{a,\alpha} \in A$ and $a = \varphi_\alpha$“$(b_{a,\alpha} \cap \alpha)$. Therefore

$$\varphi_\alpha\text{“}c = \varphi_\alpha\text{“}(c \cap \alpha) \subseteq a = \varphi_\alpha\text{“}(b_{a,\alpha} \cap \alpha).$$

In particular, $c \subseteq b_{a,\alpha} \in \bigcap_i F_i^{-1}\xi_\infty$, as desired. □

In [3], we show that the “local” semiproperness of Namba forcings can be characterized in terms of semistationary reflection principles.

Theorem 3.2 (Tsukuura [3]). *If $\alpha^\omega < \kappa$ for all $\alpha < \kappa$ then the following are equivalent.*

(1) $\mathrm{Nm}(\kappa, \lambda)$ *preserves the semistationarity of any subset of* $[\lambda]^\omega$.
(2) $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$.

$\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ is the statement that claims, for every semistationary subset $S \subseteq [\lambda]^\omega$, there is an $R \in \mathcal{P}_\kappa\lambda$ with the following properties:

(1) $\omega_1 \subseteq R \cap \kappa \in \kappa$
(2) $S \cap [R]^\omega$ is semistationary.

We call the local semiproperness of $\mathrm{Nm}(\kappa, \lambda)$ (1) of Theorem 3.2. Using this, we can obtain an analogie of Theorem 1.1 for the local semiproperness.

Theorem 3.3. *Suppose $\alpha^\omega < \kappa$ for all $\alpha < \kappa$. If* $\mathrm{Nm}(\kappa, \lambda)$ *and* $\mathrm{Nm}(\lambda^+)$ *are locally semiproper then* $\mathrm{Nm}(\kappa, \lambda^+)$ *is locally semiproper.*

Proof. By Theorem 3.2, $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ and $\mathrm{SSR}([\lambda]^+, {<}\lambda^+)$ holds. It is easy to see that $\mathrm{SSR}([\lambda^+]^\omega, {<}\kappa)$ holds. Again, by Theorem 3.2, $\mathrm{Nm}(\kappa, \lambda^+)$ is locally semiproper. □

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Department of Industrial & Systems Engineering, Faculty of Science and Engineering, Hosei University, 184-8584, Japan
Email address: `kenta.tsukura.85@hosei.ac.jp`