THE BIRMAN EXACT SEQUENCE FOR THE HYPERELLIPTIC MAPPING CLASS GROUP

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Abstract. In this survey paper, we will introduce the mapping class group and hyperelliptic mapping class group of an oriented topological surface and the Birman exact sequence associated to these groups. The mapping class group has been extensively studied in many areas of mathematics such as topology, geometry, geometric group theory, algebraic geometry, and number theory. The hyperelliptic mapping class group is a subgroup of the mapping class group determined by a fixed hyperelliptic involution of the surface. The involution yields symmetry on the surface, and the hyperelliptic mapping class group contains the information of the symmetry. It is known that the Birman exact sequence for the mapping class group does not split. In this notes, the analogous result for the hyperelliptic mapping class group will be introduced.

1. Introduction

In this survey paper, we will introduce a result (Theorem 4.2) on the hyperelliptic mapping class group and a key tool called the relative completion of a discrete group. The readers can find the detail of the result and its complete proof in the author's work [4].

Let $S_{g,n}$ be an oriented topological surface of genus g with n punctures. The mapping class group $\Gamma_{g,n}$ of $S_{g,n}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ fixing the punctures pointwise. Let $\pi_1(S_{g,n})$ be the fundamental group of $S_{g,n}$. Then there is a short exact sequence

$$1 \to \pi_1(S_g) \to \Gamma_{g,1} \to \Gamma_g \to 1, \tag{1}$$

where $S_g := S_{g,0}$ and $\Gamma_g := \Gamma_{g,0}$. The sequence (1) is called the Birman exact sequence. It is known that for $g \geq 2$, the sequence does not split. For example, see [1, Cor. 5.11]. Furthermore, there is a short exact sequence for $\Gamma_{g,n}$

$$1 \to \pi_1(S_{g,n}) \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1. \tag{2}$$

It follows from [3, Cor. 2] that the sequence (2) does not split for $g \geq 4$ and $n \geq 0$.

Let σ be a hyperelliptic involution of S_g. The hyperelliptic mapping class group Δ_g is the centralizer of the isotopy class $[\sigma]$ in Γ_g. Let $\Delta_{g,n}$ be the fiber product $\Delta_g \times_{\Gamma_g} \Gamma_{g,n}$. Then there is a short exact sequence

$$1 \to \pi_1(S_g) \to \Delta_{g,1} \to \Delta_g \to 1. \tag{3}$$

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It follows from [4, Thm. 3] that the sequence (3) does not split for $g \geq 3$. In this paper, we will briefly introduce an obstruction for the splitting of the sequence, which is constructed from a pair of certain mapping class elements in Δ_g.

2. Topology of $S_{g,n}$

Let S_g be a compact oriented topological surface of genus g. Let P be a subset of S_g consisting of n distinct points. Define $S_{g,n}$ as $S_g - P$. It is an oriented topological surface of genus g with n punctures. The fundamental group $\pi_1(S_{g,n}, p)$ of $S_{g,n}$ with base point p is the group of homotopy classes of loops in $S_{g,n}$ based at p. Changing the base point to another point q yields a natural isomorphism

$$\pi_1(S_{g,n}, p) \cong \pi_1(S_{g,n}, q),$$

which is unique up to a conjugation action by an element of $\pi_1(S_{g,n}, p)$. Therefore, we will omit the base point from the notation. Let $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ be the standard generators for $\pi_1(S_g)$ and $\gamma_1, \dots, \gamma_n$ the homotopy classes of loops surrounding the n punctures once. It has a minimal presentation given by

$$\pi_1(S_{g,n}) \cong \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_n = 1 \rangle .$$

For $n > 0$, it is a free group generated by $2g + n - 1$ elements. By the Hurewicz theorem, the natural map $\pi_1(S_{g,n}) \to H_1(S_{g,n}, \mathbb{Z})$ induces an isomorphism from the abelianization of $\pi_1(S_{g,n})$ to the homology group $H_1(S_{g,n}, \mathbb{Z})$. Denote the images of α_j and β_j in $H_1(S_g, \mathbb{Z})$ by a_j and b_j for $j = 1, \dots, g$. The abelianization $H_1(S_g, \mathbb{Z})$ is a free abelian group of rank $2g$.

2.1. **Symplectic group.**

The symplectic group $\mathrm{Sp}(2g; \mathbb{Z})$ is defined as

$$\left\{ M \in \mathrm{GL}(2g; \mathbb{Z}) | M^T J M = J \right\},$$

where $J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$ and I_g is the g-by-g identity matrix.

The group $H := H_1(S_g, \mathbb{Z})$ is equipped with the algebraic intersection paring $\langle\ ,\ \rangle$. The pairing $\langle\ ,\ \rangle$ is a non-degenerate bilinear alternating form, and H is a symplectic space of rank $2g$ with $\langle\ ,\ \rangle$. The elements $a_1, \dots, a_g, b_1, \dots, b_g$ form a symplectic basis for H. Then there is an isomorphism of the automorphism of H preserving $\langle\ ,\ \rangle$ with $\mathrm{Sp}(2g; \mathbb{Z})$:

$$\mathrm{Aut}(H, \langle\ ,\ \rangle) \cong \mathrm{Sp}(2g; \mathbb{Z}).$$

3. Mapping Class Groups

Assume that $2g - 2 + n > 0$. The mapping class group of $S_{g,n}$, denoted by $\Gamma_{g,n}$, is defined as the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ fixing the punctures pointwise:

$$\Gamma_{g,n} := \mathrm{Diff}^+(S_{g,n}) / \sim,$$

where $\sim$ denotes the isotopy relation. The group $\Gamma_{g,n}$ is independent of the choice of the subset P by the classification of surfaces. When $n = 0$, we simply denote $\Gamma_{g,0}$ by Γ_g. By filling a puncture, we obtain a surjection $\mathcal{F}\mathrm{orget} : \Gamma_{g,n+1} \to \Gamma_{g,n}$, called the forgetful map. Hence, by composing n forgetful maps, we obtain the surjection $\Gamma_{g,n} \to \Gamma_g$.

3.1. **Dehn twists.** The group Γ_g is finitely generated by the mapping class elements called the Dehn twists. Let d be a simple closed curve in S_g. Consider a tubular neighborhood N of d shown in Figure 1. A Dehn twist T_d about d is a left-twist map about d, fixing the boundary of N. More precisely, let A be the cylinder oriented outward given by $S^1 \times [0,1]$ with coordinates s and t, respectively. Let $T : A \to A$ be the twisting map sending $(s,t) \mapsto (s + 2\pi t, t)$. We see that T is an orientation-preserving diffeomorphism fixing the boundary of A pointwise. Choose an orientation-preserving diffeomorphism $\psi : A \to N$. Define a map $T_d : S_g \to S_g$ by sending

$$x \mapsto \begin{cases} \psi \circ T \circ \psi^{-1}(x) & \text{if } x \text{ is in } N, \\ x & \text{otherwise.} \end{cases}$$

The isotopy class of T_d does not depend on the choice of either N or ψ, and furthermore it

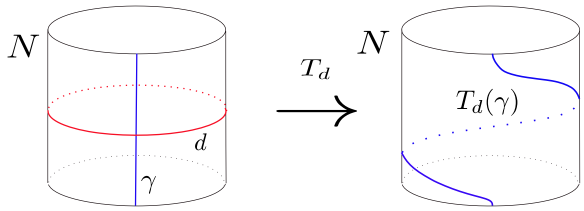

FIGURE 1. A Dehn twist

is independent of the choice of the simple curve d within its isotopy class. Thus, by abuse of notation, T_d also denotes its isotopy class in Γ_g.

A simple closed curve d in S_g is said to be separating if the surface obtained by cutting S_g along d is disconnected. Otherwise, it is said to be nonseparating. When $g = 1$, Γ_g is generated by the Dehn twists about α_1 and β_1 in the torus. For $g \geq 2$, the mapping class group Γ_g is finitely generated by the isotopy classes of Dehn twists about $2g+1$ nonseparating simple closed curves in S_g (see [1, Thm. 4.14]). Furthermore, it is also finitely presented [1, Thm. 5.3].

3.2. **Symplectic representation of $\Gamma_{g,n}$.** Each mapping class element $[\phi]$ in Γ_g induces an automorphism $\phi_* : H \to H$, which is independent of the choice of the representative of the class. The automorphism ϕ_* preserves the intersection pairing $\langle \; , \; \rangle$. Hence there is a representation

$$\rho_g : \Gamma_g \to \mathrm{Sp}(2g; \mathbb{Z}).$$

The homomorphism ρ_g is surjective for $g \geq 1$ [1, Thm. 6.4]. By composing with the forgetful map $\Gamma_{g,n} \to \Gamma_g$, we obtain a representation

$$\rho_{g,n} : \Gamma_{g,n} \to \mathrm{Sp}(2g; \mathbb{Z}).$$

This is called the symplectic representation of $\Gamma_{g,n}$.

3.3. **Torelli groups.** The Torelli group is defined as the kernel of the symplectic representation $\rho_{g,n}$: $T_{g,n} := \ker \rho_{g,n}$ and there is an exact sequence

$$1 \to T_{g,n} \to \Gamma_{g,n} \xrightarrow{\rho_{g,n}} \mathrm{Sp}(2g; \mathbb{Z}) \to 1.$$

It is an infinite-index subgroup of $\Gamma_{g,n}$. Therefore, there is no reason to believe that it carries the basic properties of $\Gamma_{g,n}$.

3.4. **The Birman exact sequences.** There is a natural injection $\mathcal{P}\mathrm{ush} : \pi_1(S_g) \hookrightarrow \Gamma_{g,1}$ called the push map. It is defined as follows. Let p be a point in S_g. Let d be a loop based at p. Let N be a tubular neighborhood of d with boundary curves x and y. Then the map $\mathcal{P}\mathrm{ush}$ is defined by sending the homotopy class $[d] \mapsto T_x T_y^{-1}$. It is not an obvious fact, but the map is injective. See the action of $\mathcal{P}\mathrm{ush}([d])$ on $\pi_1(S_g)$ in Figure 2:

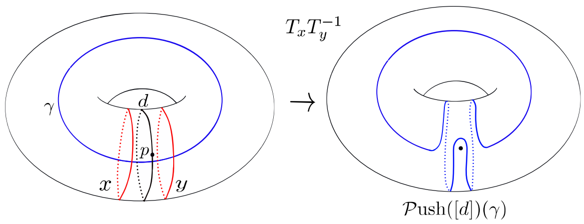

FIGURE 2. The point-pushing map $\mathcal{P}\mathrm{ush}$

Combining with the surjection $\mathcal{F}\mathrm{orget} : \Gamma_{g,1} \to \Gamma_g$, we obtain the sequence

$$1 \to \pi_1(S_g) \to \Gamma_{g,1} \to \Gamma_g \to 1. \tag{4}$$

This sequence is exact and it is called the Birman exact sequence. The sequence also extends to $\Gamma_{g,n}$ and there is the exact sequence

$$1 \to \pi_1(S_{g,n}) \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1.$$

Theorem 3.1 ([1, Cor. 5.11])**.** *If $g \geq 2$ and $n = 0$, the Birman exact sequence does not split.*

Furthermore, when $g \geq 4$ and $n \geq 0$, it follows from the author's work [3, Theorem 1] that the Birman exact sequence does not split either. This result is the profinite analogue of the Birman exact sequence obtained from the algebraic fundamental groups of the moduli of curves. The obstructions for the splitting of the Birman exact sequences lie in the Torelli groups of the mapping class groups.

4. Hyperelliptic Mapping Class Groups

We study a certain subgroup of the mapping class group that preserves symmetry on S_g. This symmetry is produced by an orientation-preserving diffeomorphism σ of order 2 of S_g fixing exactly $2g+2$ points. We call σ a hyperelliptic involution of S_g and may visualize σ as in Figure 3.

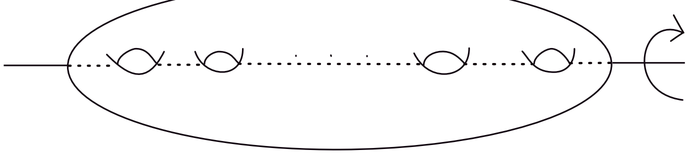

FIGURE 3. A hyperelliptic involution of S_g, rotation by π

Fix a hyperelliptic involution σ of S_g.

Definition 4.1. We define the hyperelliptic mapping class group Δ_g of S_g as

$$\Delta_g := \text{the centralizer of the isotopy class of } \sigma \text{ in } \Gamma_g.$$

We define the hyperelliptic mapping class group $\Delta_{g,n}$ as the fiber product of Δ_g and $\Gamma_{g,n}$ over Γ_g:

$$\Delta_{g,n} := \Delta_g \times_{\Gamma_g} \Gamma_{g,n},$$

where the surjection $\Gamma_{g,n} \to \Gamma_g$ is the forgetful map and $\Delta_g \to \Gamma_g$ is the natural inclusion.

4.1. **Generators.** A simple closed curve γ is said to be symmetric if $[\sigma(\gamma)] = [\gamma]$. The hyperelliptic mapping class group Δ_g can be generated by the Dehn twists about the $2g+1$ symmetric nonseparating simple closed curves in Figure 4:

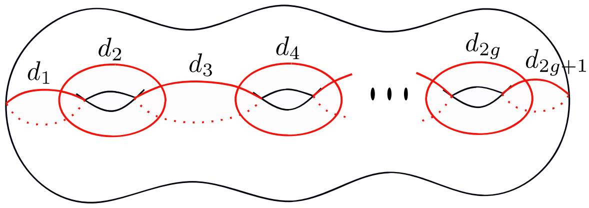

FIGURE 4. Dehn twists about symmetric nonseparating curves generating Δ_g

4.2. **The hyperelliptic Birman exact sequence.** By pulling back the Birman exact sequence for Γ_g (4) along the natural inclusion $\Delta_g \hookrightarrow \Gamma_g$, we obtain the exact sequence

$$1 \to \pi_1(S_g) \to \Delta_{g,1} \to \Delta_g \to 1, \tag{5}$$

which makes the diagram commute:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(S_g) \arrow[r] \arrow[d, equal] & \Delta_{g,1} \arrow[r] \arrow[d] & \Delta_g \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \pi_1(S_g) \arrow[r] & \Gamma_{g,1} \arrow[r] & \Gamma_g \arrow[r] & 1. \end{tikzcd}$$

Theorem 4.2 ([4, Cor. 4])**.** *If $g \geq 3$, the hyperelliptic Birman exact sequence (5) does not split.*

4.3. **Hyperelliptic Torelli group.** The obstruction for the splitting of the hyperelliptic Birman exact sequence comes from the intersection of Δ_g and T_g.

Definition 4.3. The hyperelliptic Torelli group $T\Delta_g$ is defined as

$$T\Delta_g := \Delta_g \cap T_g.$$

Although it is not known whether $T\Delta_g$ is finitely generated or not, we have the following remarkable result by Brendle, Margalit, and Putman.

Theorem 4.4 (Brendle-Margalit-Putman)**.** *If $g \geq 2$, then $T\Delta_g$ is generated by Dehn twists about symmetric separating curves.*

Remark 4.5. When $g = 2$, any two simple separating curves intersect at least 4 times (see Figure 5). On the other hand, when $g \geq 3$, there are disjoint symmetric separating curves as in Figure 6, which produce commuting Dehn twists in $T\Delta_g$.

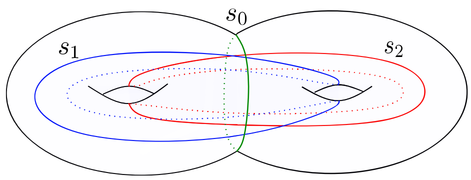

Figure 5. Symmetric separating curves in S_2

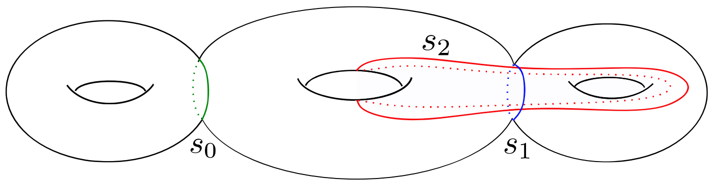

Figure 6. Symmetric separating curves in S_3

5. Relative completion of $\Delta_{g,n}$

Relative completion of a discrete group is a linearization. It is controlled by cohomology and so computable to some extent.

Definition 5.1. The relative completion of $\Delta_{g,n}$ with respect to $\rho : \Delta_{g,n} \to \mathrm{Sp}_{2g}(\mathbb{Q})$ is an extension of $\mathrm{Sp}_{2g}(\mathbb{Q})$ by a prounipotent $\mathbb{Q}$-group $\mathcal{V}_{g,n}$:

$$\begin{tikzcd}
 & T\Delta_g \arrow[r] \arrow[d] & \Delta_{g,n} \arrow[d, "\tilde{\rho}"] \arrow[dr, "\rho"] & & \\
1 \arrow[r] & \mathcal{V}_{g,n} \arrow[r] & \mathcal{D}_{g,n} \arrow[r] & \mathrm{Sp}_{2g}(\mathbb{Q}) \arrow[r] & 1,
\end{tikzcd}$$

satisfying the following universal property. If G is a proalgebraic $\mathbb{Q}$-group that is also an extension of $\mathrm{Sp}_{2g}(\mathbb{Q})$ by a prounipotent $\mathbb{Q}$-group U such that ρ factors through $G \to \mathrm{Sp}_{2g}(\mathbb{Q})$ with Zariski-dense image in G, then there exists a unique morphism $\phi : \mathcal{D}_{g,n} \to G$ of proalgebraic groups over $\mathbb{Q}$ such that the diagram

$$\begin{tikzcd}
\Delta_{g,n} \arrow[r, "\tilde{\rho}"] \arrow[d] & \mathcal{D}_{g,n} \arrow[d] \arrow[dl, "\phi"'] \\
G \arrow[r] & \mathrm{Sp}_{2g}(\mathbb{Q})
\end{tikzcd}$$

commutes.

By the Levi's theorem, the exact sequence

$$1 \to \mathcal{V}_{g,n} \to \mathcal{D}_{g,n} \to \mathrm{Sp}_{2g}(\mathbb{Q}) \to 1$$

splits, and hence there is an isomorphism $\mathcal{D}_{g,n} \cong \mathcal{V}_{g,n} \rtimes \mathrm{Sp}_{2g}(\mathbb{Q})$.

5.1. **The Key Exact Sequences of Completions.** Let $\mathcal{P}$ be the unipotent completion of $\pi_1(S_g)$ over $\mathbb{Q}$. It is a prounipotent $\mathbb{Q}$-group and there is the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S_g) \arrow[r] \arrow[d] & \Delta_{g,1} \arrow[r] \arrow[d, "\tilde{\rho}"] & \Delta_g \arrow[r] \arrow[d, "\tilde{\rho}"] & 1 \\
1 \arrow[r] & \mathcal{P} \arrow[r] \arrow[d, equal] & \mathcal{D}_{g,1} \arrow[r] & \mathcal{D}_g \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{V}_{g,1} \arrow[r] \arrow[u, hook] & \mathcal{V}_g \arrow[r] \arrow[u, hook] & 1,
\end{tikzcd}$$

where the rows are exact. A splitting of $\Delta_{g,1} \to \Delta_g$ induces that of $\mathcal{D}_{g,1} \to \mathcal{D}_g$ by the universal property, which by restricting to $\mathcal{V}_g$ yields a section of $\mathcal{V}_{g,1} \to \mathcal{V}_g$.

5.2. **The Key Exact Sequence of Graded Lie Algebras.** Let $\mathfrak{p}$ be the Lie algebra of $\mathcal{P}$ and $\mathfrak{v}_{g,n}$ the Lie algebra of $\mathcal{V}_{g,n}$. These are pronilpotent Lie algebras in the category of the mixed Hodge structures (MHSs), admitting weight filtrations $W_\bullet\mathfrak{p}$ and $W_\bullet\mathfrak{v}_{g,n}$, satisfying the property $\mathfrak{p} = W_{-1}\mathfrak{p}$ and $\mathfrak{v}_{g,n} = W_{-1}\mathfrak{v}_{g,n}$, respectively. Let $\mathrm{Gr}^W_\bullet\,\mathfrak{p}$ and $\mathrm{Gr}^W_\bullet\,\mathfrak{v}_{g,n}$ be the associated graded Lie algebras of $\mathfrak{p}$ and $\mathfrak{v}_{g,n}$, respectively. The bottom exact sequence of the diagram in §5.1 induces the exact sequence of pronilpotent Lie algebras

$$0 \to \mathfrak{p} \to \mathfrak{v}_{g,1} \to \mathfrak{v}_g \to 0.$$

Since the functor $\mathrm{Gr}^W_\bullet$ is exact in the category of MHSs, we have the exact sequence

$$0 \to \mathrm{Gr}^W_\bullet\,\mathfrak{p} \to \mathrm{Gr}^W_\bullet\,\mathfrak{v}_{g,1} \to \mathrm{Gr}^W_\bullet\,\mathfrak{v}_g \to 0.$$

Each graded quotient $\mathrm{Gr}^W_m\,\mathfrak{v}_{g,n} = W_m\mathfrak{v}_{g,n}/W_{m-1}$ is an $\mathrm{Sp}_{2g}(\mathbb{Q})$-module, and a section of $\mathcal{D}_{g,1} \to \mathcal{D}_g$ induces an Sp-module graded Lie algebra section of $\mathrm{Gr}^W_\bullet\,\mathfrak{v}_{g,1} \to \mathrm{Gr}^W_\bullet\,\mathfrak{v}_g$.

6. An obstruction for the hyperelliptic Birman exact sequence

In this section, we will briefly introduce an idea of the proof of Theorem 4.2. The detailed proof can be found in [4]. Let x be a section of the surjection $\Delta_{g,1} \to \Delta_g$. By the universal property of the relative completion, it induces a section $\tilde{x}$ of the surjective morphism $\mathcal{D}_{g,1} \to \mathcal{D}_g$. Furthermore, the section $\tilde{x}$ yields an Sp-module graded Lie algebra section $\mathrm{Gr}\,x$ of $\mathrm{Gr}^W_\bullet\,\mathfrak{v}_{g,1} \to \mathrm{Gr}^W_\bullet\,\mathfrak{v}_g$:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S_g) \arrow[r] \arrow[d] & \Delta_{g,1} \arrow[r] \arrow[d, "\tilde{\rho}"] & \Delta_g \arrow[r] \arrow[d, "\tilde{\rho}"] \arrow[l, bend right, "x"'] & 1 \\
1 \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{D}_{g,1} \arrow[r] & \mathcal{D}_g \arrow[r] \arrow[l, bend right, "\tilde{x}"'] & 1 \\
0 \arrow[r] & \mathrm{Gr}^W_\bullet\,\mathfrak{p} \arrow[r] & \mathrm{Gr}^W_\bullet\,\mathfrak{v}_{g,1} \arrow[r] & \mathrm{Gr}^W_\bullet\,\mathfrak{v}_g \arrow[r] \arrow[l, bend right, "\mathrm{Gr}\,x"'] & 0.
\end{tikzcd}$$

Therefore, if the bottom sequence does not admit such sections, then the top sequence does not split.

Let p be a point in S_g fixed by the hyperelliptic involution σ. Consider the two symmetric separating simple closed curves S_1 and S_2 bounding p in Figure 7. Let T_{S_1} and T_{S_2} be the separating twists about S_1 and S_2, respectively. Since S_1 and S_2 are disjoint, the Dehn twists T_{S_1} and T_{S_2} commute. This commutativity plays a key role in producing an obstruction for the

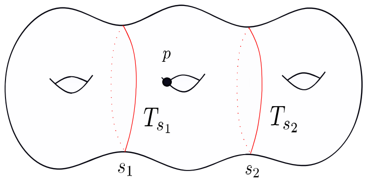

FIGURE 7. Two commuting separating Dehn twists

splitting of the hyperelliptic Birman exact sequence. In [2, §6], the obstruction obtained from such two commuting Dehn twists is computed using a homomorphism called the hyperelliptic Johnson homomorphism $T\Delta_g \to \mathrm{Hom}(H, L^3\pi_1(S_g)/L^4)$, where $L^\bullet\pi_1(S_g)$ is the lower central series of $\pi_1(S_g)$.

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