On numerical semigroups generated by 5 elements whose quotients by 2 are non-symmetric [1]

神奈川工科大学・基礎・教養教育センター　米田 二良
Jiryo Komeda
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology

Abstract

For a numerical semigroup H we generalize the concept of a pseudo-Frobenius number, which forms the set $\overline{PF^*(H)}$. Let H be non-symmetric and it is generated by three elements. Then we classify the elements of $\overline{PF^*(H)}$ into three kinds, which are regular, quasi-regular and irregular. We will show that a numerical semigroup $\tilde{H} = 2H + \langle n, n+2l \rangle$, which is generated by 5 elements, is obtained by a ramification point of some double covering of a curve if n is a sufficiently large odd integer and $l \in \overline{PF^*(H)}$ is regular.

1 Terminologies and introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, denoted by $g(H)$. In this paper H always stands for a numerical semigroup. We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\},$$

which is called the *conductor* of H. It is well-known that $c(H) \leqq 2g(H)$. H is said to be *symmetric* if $c(H) = 2g(H)$. We have $(c(H) - 1) + h \in H$ for any $h \in H$ with $h > 0$. The number $c(H) - 1$ is called the *Frobenius number* of H. An element $f \in \mathbb{N}_0 \backslash H$ is called a *pseudo-Frobenius number* of H if $f + h \in H$ for any $h \in H$ with $h > 0$. We denote by $PF(H)$ the set of pseudo-Frobenius numbers. We denote by $PF^*(H)$ the set of pseudo-Frobenius numbers distinct from the Frobenius number. We define the set $\overline{PF^*(H)}$ containing $PF^*(H)$ as follows:

$$\{l \in \mathbb{N}_0 \backslash H \mid c(H) - 1 - l \in \mathbb{N}_0 \backslash H\},$$

which is called *the closure of* $PF^*(H)$. We know that the cardinality of $\overline{PF^*(H)}$ is $2g(H) - c(H)$.

We explain numerical semigroups in connection with algebraic curves. A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0. For a pointed curve (C, P) we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

[1] This paper is an extended abstract and the details will be published (see [5])

where $k(C)$ is the field of rational functions on C. Then $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C. $H(P)$ is called *the Weierstrass semigroup* of P. A numerical semigroup H is said to be Weierstrass if there exists a pointed curve (C, P) with $H(P) = H$. A numerical semigroup H is said to be *of double covering type* , which is abbreviated to *DC*, if there exists a double covering of a curve with a ramification point P with $H(P) = H$. Hence, if H is DC, then it is Weierstrass. Let d_2 be the map from the set $\mathcal{H}$ of numerical semigroups to $\mathcal{H}$ itself defined by

$$d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\},$$

which is a numerical semigroup. Let $\pi : C \longrightarrow C'$ be a double covering of a curve with a ramification point P. Then we obtain $d_2(H(P)) = H(\pi(P))$.

2 Known Facts and Problem

Known Fact 1 (Classical). Any numerical semigroup generated by two elements is Weierstrass.

Known Fact 2 (Waldi [6]). Any numerical semigroup generated by three elements is Weierstrass.

Known Fact 3 (Buchweitz [1] (1980)). There exists a non-Weierstrass numerical semigroup H generated by nine elements.

Known Fact 4 ([3] (2013)). For any $l \geqq 6$ there exists a non-Weierstrass numerical semigroup H generated by l elements.

Problem. Is every numerical semigroup H generated by 4 or 5 elements Weierstrass?

We are interested in the case where H is a numerical semigroup generated by 5 elements.

3 Numerical semigroups generated by 5 elements

Notation. For any non-negative integers $a_1, a_2, \ldots, a_s$ we denote by $\langle a_1, a_2, \ldots, a_s \rangle$ the monoid generated by $a_1, a_2, \ldots, a_s$.

Fact ([3]). Let $l \geqq 2$ and n be an odd integer with $n \geqq 16l + 19$. We set

$$H = \langle 4, 6, 4l + 1, 4l + 3 \rangle.$$

Then $\tilde{H} = 2H + \langle n, n + 2 \cdot 2 \rangle$, which is generated by 6 elements, is non-Weierstrass. We note that $2 \in PF^*(H) = \{2, 4l - 3\}$.

In this paper we consider the case where H is a non-symmetric numerical semigroup generated by 3 elements. First, let $l \in PF^*(H)$. Let n be a sufficiently large odd integer.

Then is the numerical semigroup $2H + \langle n, n+2l\rangle$, which is generated by 5 elements, Weierstrass ?

Fact (Fröberg-Gottlieb-Häggkvist [2]). Let H be a numerical semigroup which is not symmetric. If H is generated by three elements, then the set $PF^*(H)$ consists of only one element.

Fact ([4]). Let H be a numerical semigroup generated by three elements which is not symmetric. We set $PF^*(H) = \{t\}$ and $c(H) = 2g(H) - r$. Let n be an odd integer larger than $2g(H) + 2r$. Then both $\tilde{H} = 2H + \langle n, n+2t\rangle$ and $\tilde{H}^* = 2H + \langle n, n+2(c(H)-1-t)\rangle$ are DC.

The above statement is Main Theorem in my RIM's talk on Feb. 2023.

Lemma. *Let H be a numerical semigroup with $PF^*(H) = \{t\}$. We set $c(H) = 2g(H) - r$. Let $l \in \overline{PF^*(H)}$ and n be an odd integer $\geqq c(H) + m(H) - 1$ where we set $m(H) = \min\{h \in H \mid h > 0\}$. In this case we obtain $g(2H + \langle n\rangle) = 2g(H) + \frac{n-1}{2}$. We set*

$$g(2H + \langle n, n+2l\rangle) = 2g(H) + \frac{n-1}{2} - s.$$

Then we have the following:
(i) $1 \leqq s \leqq r$.
(ii) $s = 1$ if and only if $l = t$.
(iii) $s = r$ if and only if $l = c(H) - 1 - t$.

Definition A. Let the notations and the assumptions be as in the above Lemma. We set

$$\overline{PF^*(H)} = \{l_1 = t > l_2 > \cdots > l_{r-1} > l_r = c(H) - 1 - t\}.$$

We note that for any $1 \leqq i \leqq r$ the equality $c(H) - 1 - l_i = l_{r+1-i}$ holds. We set

$$g(2H + \langle n, n+2l_i\rangle) = 2g(H) + \frac{n-1}{2} - d(i),$$

where we call $d(i)$ the *$d-$number* of l_i.

Lemma. *Let the notations and the assumptions be as in the above Definition. Then we obtain*

$$d(i) + d(r+1-i) \leqq r + 1$$

for any $1 \leqq i \leqq r$.

Definition. Let the notations and the assumptions be as in Definition A.
(i) l_i is said to be *regular* if $d(i) + d(r+1-i) = r+1$.
(ii) l_i is said to be *quasi-regular* if $d(i) + d(r+1-i) = r$.
(iii) l_i is said to be *irregular* if $d(i) + d(r+1-i) \leqq r - 1$.

Example. Let $H = \langle 4, 4s+1, 4(3s-3)+3\rangle$ with $s \geqq 3$. Then $g(H) = 6s - 3$ and $c(H) = 12s - 12 = 2g(H) - 6$. We have

$$\overline{PF^*(H)} = \{l_1 = 8s-2, l_2 = 8s-6, l_3 = 8s-10, l_4 = 4s-3, l_5 = 4s-7, l_6 = 4s-11\},$$

Then we have $d(1) = 1$, $d(2) = 2$, $d(3) = 3$, $d(4) = 2$, $d(5) = 4$ and $d(6) = 6$. Hence, l_1 and l_6 are regular, l_2 and l_5 are quasi-regular, and l_3 and l_4 are irregular.

Main Theorem A. *Let the notations and the assumptions be as in Definition A. Let n be an odd integer larger than $2g(H) + 2r$. Assume that H is Weierstrass. If l_i is regular, then both $2H + \langle n, n + 2l_i\rangle$ and $2H + \langle n, n + 2l_{r+1-i}\rangle$ are DC.*

The Main Theorem of RIM' s Talk on Feb. 2023 is derived from Main Theorem A.

Corollary. *Let the notations and the assumptions be as in Theorem A. Then both $2H + \langle n, n + 2t\rangle$ and $2H + \langle n, n + 2(c(H) - 1 - t)\rangle$ are DC where $PF^*(H) = \{t\}$.*

Proof. t and $c(H)-1-t$ are regular. qed

Proposition. *Let H be a non-symmetric numerical semigroup generated by three elements. Assume that $c(H) = 2g(H) - r$ with $1 \leqq r \leqq 3$. Then any $l \in \overline{PF^*(H)}$ is regular.*

Proposition. *Let H be a non-symmetric numerical semigroup with $m(H) = 3$ where we denote by $m(H)$ the minimum positive integer in H. Assume that $c(H) = 2g(H) - (2s + 1)$ with $s \geqq 1$. Then any $l \in \overline{PF^*(H)}$ is regular.*

Main Theorem B. *Let the notations and the assumptions be as in Definition A. Let n be an odd integer larger than $2g(H)+2r-2$. Assume that H is Weierstrass. If l_i is quasi-regular, then at least one of $2H + \langle n, n + 2l_i\rangle$ and $2H + \langle n, n + 2l_{r+1-i}\rangle$ is DC.*

Example. Let $H = \langle 4, 8s+3, 24s+1\rangle$. Then $g(H) = 12s+1$ and $c(H) = 24s-2 = 2g(H)-4$. We have $\overline{PF^*(H)} = \{l_1 = 16s+2, l_2 = 16s-2, l_3 = 8s-1, l_4 = 8s-5\}$ and $d(1) = 1$, $d(2) = 2$, $d(3) = 2$ and $d(4) = 4$. Hence, l_2 and l_3 are quasi-regular.

Main Theorem C. *Let m be an even integer and u be an integer with $4 \leqq m < u$ and $(m, u) = 1$. Let s be a positive integer. We set*

$$H = \left\langle m, u, (m-1)u - \frac{m(u-(2s+1))}{2} \right\rangle.$$

Assume that $\frac{m(u-(2s+1))}{2} > u$. Moreover, assume that $S(H) \ni iu$ for any $1 \leqq i \leqq m-2$ where $S(H) = \{s_i \mid i = 1, 2, \ldots, m-1\}$ with $s_i = \min\{h \in H \mid h \equiv i \bmod m\}$. Then any $l \in \overline{PF^*(H)}$ is regular. In this case, $c(H) = 2g(H) - (2s + 1)$. Moreover, we have $PF^*(H) = \{g(H) - s - 1 + sm\}$.

Corollary . *Let $s \geqq 2$. Let H be a non-symmetric numerical semigroup with $c(H) = 2g(H) - (2s + 1)$ which is generated by three elements.*
(i) If $m(H) = 4$, then any $l \in \overline{PF^*(H)}$ is regular.
(ii) If $m(H) = 6$ and $s \equiv 0$ or $2 \mod 3$, then any $l \in \overline{PF^*(H)}$ is regular.

Remark. Let $H = \langle 6, 19, 29\rangle$. Then we have the following:
(i) $m(H) = 6$ and $c(H) = 2g(H) - (2 \cdot 4 + 1)$. We note that $s = 4 \equiv 1 \bmod 3$.
(ii) Any $l \in \overline{PF^*(H)}$ except t and $c(H) - 1 - t$ with $PF^*(H) = \{t\}$ is irregular.

Remark. Let H be a non-symmetric numerical semigroup generated by three elements. Assume that $c(H) = 2g(H) - 4$. Then any element of $\overline{PF^*(H)}$ is regular or quasi-regular.

Problem. Are there a numerical semigroup H with $m(H) = 4$ and $c(H) = 2g(H) - 6$ generated by three elements and $l \in \overline{PF^*(H)}$ such that $2H + \langle n, n + 2l \rangle$ is not Weierstrass for sufficiently large odd number n?

References

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