On the minimal 2-blocking sets in PG(5, 2) *

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1 Introduction

We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over the field of q elements $\mathbb{F}_q$. A j-space is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. In this paper, Π_k stands for a k-space in $\mathrm{PG}(r, q)$. We set $\Pi_k = \emptyset$ for $k < 0$. The 0-spaces, 1-spaces, 2-spaces, 3-spaces and $(r-1)$-spaces are called *points, lines, planes, solids* and *hyperplanes*, respectively. A set of points in $\mathrm{PG}(r, q)$ meeting every $(r-k)$-space is called a *k-blocking set* or a blocking set with respect to $(r-k)$-spaces [3]. A 1-blocking set is simply called a *blocking set.* A k-space in $\mathrm{PG}(r, q)$ is the smallest k-blocking set [5] and a k-blocking set containing a k-space in $\mathrm{PG}(r, q)$ is called *trivial.* A k-blocking set $\mathcal{B}$ is *minimal* if $\mathcal{B} \setminus \{P\}$ is no longer a k-blocking set for any point P of $\mathcal{B}$.

For an integer $r \geq 3$ and a prime power $q \geq 3$, a smallest non-trivial 1-blocking set $\mathcal{B}_0$ in a plane δ in $\mathrm{PG}(r, q)$ is also a smallest non-trivial 1-blocking set in $\mathrm{PG}(r, q)$. The speciality for the binary case is that a non-trivial 1-blocking set in $\mathrm{PG}(2, 2)$ does not exist.

Denote by $\mathrm{Cone}(\Pi_k, \mathcal{B})$ (or simply $\Pi_k\mathcal{B}$) a cone with vertex a k-space Π_k and base $\mathcal{B}$ in an s-space Δ skew to Π_k. Note that the cone is just $\mathcal{B}$ if Π_k is empty.

Govaerts and Storme proved the following.

Theorem 1.1 ([6]). (a) *Any smallest non-trivial* 1*-blocking set in PG*$(r, 2)$*,* $r \geq 3$*, is an elliptic quadric in a solid in PG*$(r, 2)$*.*

(b) *Every non-trivial minimal* 2*-blocking set in PG*$(3, 2)$ *is the complement of an elliptic quadric.*

(c) *Any smallest non-trivial k-blocking set in PG*$(r, 2)$*,* $r \geq 3$*, with* $2 \leq k \leq r-1$ *is Cone*$(\Pi_{k-3}, \mathcal{T})$ *where* $\mathcal{T}$ *is the set of 10 points consisting of the complement of an elliptic quadric in a solid* Δ*.*

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An elliptic quadric in PG$(3, 2)$ is a set of five points no four of which are coplanar, that is the only non-trivial minimal 1-blocking set in PG$(3, 2)$ up to projective equivalence. A natural question is to classify all non-trivial minimal k-blocking sets in PG$(r, 2)$ up to projective equivalence for $1 \le k \le r - 1$.

In this paper, the point P in PG$(r, 2)$ with coordinate vector $(p_0, p_1, \ldots, p_r)$ is denoted by $(p_0, p_1, \ldots, p_r)$ or simply $p_0 p_1 \ldots p_r$, and the number of 1's in $\{p_0, p_1, \ldots, p_r\}$ is called the *weight* of P. The hyperplane defined by the equation $a_0 x_0 + a_1 x_1 + \cdots + a_r x_r = 0$ is denoted by $[a_0 a_1 \ldots a_r]$. For two distinct points $P(p_0, p_1, \ldots, p_r)$ and $Q(q_0, q_1, \ldots, q_r)$ in PG$(r, 2)$, we denote the point $(p_0 + q_0, p_1 + q_1, \ldots, p_r + q_r)$ by $P + Q$.

Let $\boldsymbol{e}_i = 0 \cdots 010 \cdots 0$ be the point of PG$(r, 2)$ the only i-th entry of which is 1. We denote by $\mathbf{1}$ the point $11 \cdots 1$ and let $\mathcal{I}_r := \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{r+1}, \mathbf{1}\}$ in PG$(r, 2)$ with odd $r \ge 3$. Note that $\mathcal{I}_3 = \{1000, 0100, 0010, 0001, 1111\}$ is an elliptic quadric in PG$(3, 2)$. It is easy to see that $\mathcal{I}_r$ is a non-trivial 1-blocking set in PG$(r, 2)$ since r is odd. Since $\mathcal{I}_r$ meets the hyperplane $[\boldsymbol{e}_j + \mathbf{1}]$ in the point $\boldsymbol{e}_j$ and meets the hyperplane $[11 \cdots 1]$ in the point $\mathbf{1}$, $\mathcal{I}_r$ is minimal. Thus $\mathcal{I}_r$ is a non-trivial minimal 1-blocking set in PG$(r, 2)$ for odd $r \ge 3$.

Let $P_1, P_2, \ldots, P_{r+1}$ be $r + 1$ points of PG$(r, 2)$ in general position. We call the $(r + 2)$-set $\{P_1, P_2, \ldots, P_{r+1}, \sum_{i=1}^{r+1} P_i\}$ a *skeleton* in PG$(r, 2)$, which is also called a 'frame' [1]. Obviously, a skeleton in PG$(r, 2)$ is projectively equivalent to $\mathcal{I}_r$. Bono et al.[4] proved the following.

Theorem 1.2. *Let S be a non-trivial minimal 1-blocking set in PG$(r, 2)$, $r \ge 3$. Then, S is projectively equivalent to $\mathcal{I}_s$ in some s-space of PG$(r, 2)$ with odd $s \ge 3$.*

Corollary 1.3. *There are exactly $\lfloor (r - 1)/2 \rfloor$ non-trivial minimal 1-blocking sets up to projective equivalence in PG$(r, 2)$, $r \ge 3$.*

As a consequence of Theorem 1.2, there is only one non-trivial minimal 1-blocking set up to projective equivalence in PG$(4, 2)$, which is a skeleton in a solid. Bono et al.[4] also classified non-trivial minimal 2-blocking sets in PG$(4, 2)$ up to projective equivalence.

For t spaces $\chi_1, \ldots, \chi_t$, we denote by $\langle \chi_1, \ldots, \chi_t \rangle$ the smallest space containing $\chi_1, \ldots, \chi_t$. From Theorem 1.1, we get (a) of the following theorem.

Theorem 1.4 ([4])**.** (a) *Let S_{10} be the set of 10 points in a solid Δ in PG$(4, 2)$ which is the complement of a skeleton in Δ. Then, S_{10} is the smallest non-trivial 2-blocking set in PG$(4, 2)$.*

(b) *Let $S_{11} = \mathrm{Cone}(P, K)$ with a point P and a skeleton K in a solid Δ not containing P. Then, S_{11} is a non-trivial minimal 2-blocking set with size 11 in PG$(4, 2)$.*

(c) *Take two planes δ_1, δ_2 meeting in a point P in PG$(4, 2)$ and a point $Q_i \in \delta_i \setminus \{P\}$ for $i = 1, 2$. Let $S_{12} = (\delta_1 \setminus \{Q_1\}) \cup (\delta_2 \setminus \{Q_2\}) \cup \{Q_1 + Q_2\}$. Then, S_{12} is a non-trivial minimal 2-blocking set with size 12 in PG$(4, 2)$.*

(d) *Take three points* Q_1, Q_2, Q_3 *not on a line and a line* l *which is skew to the plane* $\langle Q_1, Q_2, Q_3\rangle$ *in* $PG(4,2)$*. Let* $\delta_i = \langle Q_i, l\rangle$ *for* $i = 1,2,3$ *and let* $P = Q_1 + Q_2 + Q_3$*. Then,* $S_{13} = \{P\} \cup \bigcup_{i=1}^{3}(\delta_i \setminus \{Q_i\})$ *is a non-trivial minimal* 2*-blocking set with size* 13 *in* $PG(4,2)$*.*

(e) *Take a skeleton* $\{Q_1, Q_2, Q_3, Q_4, P = \sum_{i=1}^{4} Q_i\}$ *in a solid* Δ *and a point* R_1 *out of* Δ*. Let* $l_1, \ldots, l_4$ *be the lines defined by* $l_1 = \{P, R_1, R_1' = P + R_1\}$ *and*

$$l_j = \{P, R_j = R_{j-1} + Q_{j-1}, R_j' = R_{j-1}' + Q_{j-1}\},\ j = 2,3,4.$$

Then, $S_{13}' = \bigcup_{i=1}^{4}(l_i \cup \{P + Q_i\})$ *is a non-trivial minimal* 2*-blocking set with size* 13 *in* $PG(4,2)$*.*

(f) *A parabolic quadric* $\mathcal{P}_4$ *is a non-trivial minimal* 2*-blocking set with size* 15 *in* $PG(4,2)$*.*

Theorem 1.5 ([4])**.** *There are exactly six non-trivial minimal* 2*-blocking sets in* $PG(4,2)$ *up to projective equivalence, which are described in Theorem 1.4.*

As a continuation of [4], we tried to find all non-trivial minimal 2-blocking sets in $\mathrm{PG}(5,2)$, and we obtained the following.

Theorem 1.6. *There are at least 84 non-trivial minimal* 2*-blocking sets in* $PG(5,2)$ *up to projective equivalence, whose sizes* n *satisfy* $13 \le n \le 20$*.*

2 How to search 2-blocking sets in PG$(5,2)$

Let B be a non-trivial minimal 2-blocking set in $\Sigma := \mathrm{PG}(5,2)$ with size n. For any point P of B, there exists a solid π such that $\pi \cap B = \{P\}$ since B is minimal. such a solid π is called a *tangent* of B at P. Let M_B be a $6 \times n$ matrix whose columns consist of the n points of B. We denote by rank B the rank of the matrix M_B. We may assume that rank $B = 6$ because of the following.

Lemma 2.1. *Let* S *be a non-trivial minimal* k*-blocking set in a* t*-space of* $PG(r,q)$ *with* $k < t < r$*. Then,* S *is also a non-trivial* k*-blocking set in* $PG(r,q)$*.*

Imamura [8] found that there is no non-trivial minimal 2-blocking set in $\Sigma = \mathrm{PG}(5,2)$ for $n \le 12$ by an exhaustive computer search. It can be proved that $n \le 21$, see Miura [9]. Hence, we may assume

$$13 \le n \le 21. \tag{2.1}$$

A hyperplane meeting B in exactly t points is called a *t-hyperplane*. A t-solid, a t-plane and so on are defined similarly. For a space Π in Σ, let

$$t_i(\Pi) = \max\{|S \cap B| \mid S \text{ is an } i\text{-space in } \Pi\}.$$

We simply denote by t_i for $t_i(\Sigma)$. Let Δ be a t_3-solid in Σ. Then, we have $t_3 \le 12$ since B is non-trivial. For any point P of $B \cap \Delta$, a tangent of B at P meets Δ in a

1-line or a 1-plane. Hence, each point of $B \cap \Delta$ is on a 1-line, giving $t_3 \leq 10$. Suppose $t = 10$. If the complement of B in Δ contains a line, then one can take a point of B which in not on a 1-line in Δ, a contradiction. Otherwise, B is the complement of a skeleton in Δ, and rank $B = 4$, a contradiction again. Thus, $t_3 \leq 9$. The following can be proved from the known results of binary linear codes [7].

Lemma 2.2. (a) $t_3 \geq 6$.

(b) $t_4 \geq 10$ *if* $n = 14$ *or* 15.

(c) $7 \leq t_3 \leq 9$ *if* $n \geq 16$.

We give the geometric description of the possible solid Δ with $t_3 = |\Delta \cap B|$.

Lemma 2.3. *If Δ is a 9-solid, then Δ satisfies one of the following conditions:*

(a) $\Delta \setminus B = \ell_1 \cup \ell_2$, *where ℓ_1 and ℓ_2 are skew lines;*

(b) $\Delta \cap B = (\delta_1 \cup \delta_2) \cap B$, *where δ_1 and δ_2 are 6-planes through a 3-line.*

Lemma 2.4. *If Δ is a 8-solid, then Δ satisfies one of the following conditions:*

(a) $\Delta \setminus B = \ell_1 \cup \ell_2 \cup \{P\}$, *where ℓ_1 and ℓ_2 are skew lines and $P = P_1 + P_2$ for some $P_1 \in \ell_1$ and $P_2 \in \ell_2$;*

(b) $\Delta \cap B = (\delta_1 \cup \delta_2) \cap B$, *where δ_1, δ_2 are a 6-plane and a 5-plane through a 3-line.*

Lemma 2.5. *If Δ is a 7-solid, then Δ satisfies one of the following conditions:*

(a) $\Delta \cap B = (\delta_6 \cap B) \cup \{P\}$, *where δ_6 is a 6-plane not containing a point P;*

(b) $\Delta \setminus B = \delta_0 \cup \{P\}$, *where δ_0 is a 0-plane not containing a point P;*

(c) $\Delta \cap B = (\delta_5 \cap B) \cup \{P_1, P_2\}$, *where δ_5 is a 5-plane meeting the line $\langle P_1, P_2 \rangle$ in a 0-point;*

(d) $\Delta \cap B$ *consists of three non-coplanar lines through a fixed point;*

(e) $\Delta \cap B = \ell_1 \cup \ell_2 \cup \{P\}$, *where ℓ_1 and ℓ_2 are skew lines and $P = P_1 + P_2$ for some $P_1 \in \ell_1$ and $P_2 \in \ell_2$.*

Lemma 2.6. *If Δ is a 6-solid, then Δ satisfies one of the following conditions:*

(a) $\Delta \cap B = (\delta_5 \cap B) \cup \{P\}$, *where δ_5 is a 5-plane not containing a point P;*

(b) $\Delta \setminus B = \delta_0 \cup \{P_1, P_2\}$, *where δ_0 is a 0-plane not containing the two points P_1, P_2;*

(c) $\Delta \cap B = K \cup \{P\}$, *where K is a skeleton and P is a point of $\Delta \setminus K$;*

(d) $\Delta \cap B = \ell_1 \cup \ell_2$, *where ℓ_1 and ℓ_2 are skew lines.*

Let H_1, H_2, H_3 be the hyperplanes through Δ. Without loss of generality, we may assume that $H_1 = [000010]$, $H_2 = [000001]$, $H_3 = [000011]$ and that

$$n_1 \geq n_2 \geq n_3,$$

where $n_i = |(H_i \setminus \Delta) \cap B|$ for $i = 1, 2, 3$. So, $n = t_3 + n_1 + n_2 + n_3$.

For example, assume $n = 13$ with $t_3 = 6$. Then, the possible (n_1, n_2, n_3)'s are

$$(3,3,1), (3,2,2), (4,3,0), (4,2,1), (5,2,0), (5,1,1), (6,1,0).$$

Since there are six possible 6-solids by Lemma 2.6, we checked 28 cases by a computer, giving no result. Next, assume $n = 13$ with $t_3 = 7$. Then, the possible (n_1, n_2, n_3)'s are

$$(2,2,2), (3,3,0), (3,2,1), (4,1,1), (4,2,0),$$

giving three non-trivial minimal 2-blocking sets. In this way, we found 84 non-trivial minimal 2-blocking sets in PG(5, 2) up to projective equivalence by an exhaustive computer search as Table 1.

Table 1: Possible $n = |B|$, (t_4, t_3, t_2) and the number of possible B

n	(t_4, t_3, t_2)	#
13	(9,7,6)	3
14	(10,8,6)	2
15	(10,7,5)	1
	(11,7,5)	1
	(10,8,6)	1
	(11,8,6)	6
	(11,9,6)	12

n	(t_4, t_3, t_2)	#
16	(10,7,5)	1
	(11,8,5)	1
	(12,8,5)	1
	(12,9,6)	30
17	(12,8,5)	3
	(13,8,5)	2
	(13,9,5)	4

n	(t_4, t_3, t_2)	#
18	(12,8,5)	6
	(12,8,6)	2
	(12,9,6)	1
	(14,9,5)	2
	(14,9,6)	1
19	(13,9,5)	3
20	(12,8,5)	1

3 A generalization

From Table 1, there are three non-trivial minimal 2-blocking sets of size 13 in PG(5, 2) up to projective equivalence. We give how to construct them in a geometric way.

We construct a minimal k-block B in $\Sigma = \mathrm{PG}(r, 2)$ from two k-blocks B_1 and B_2. For a point P_1 of B_1 and a set $T \subset B_2$, we denote by $(B_1; P_1) + (B_2; T)$ the set

$$(B_1 \setminus \{P_1\}) \cup (B_2 \setminus T) \cup \{P_1 + R \mid R \in T\}.$$

Lemma 3.1. *Let B_1, B_2 be k-spaces in $\Sigma = \mathrm{PG}(r, 2)$ with $B_1 \cap B_2 = \emptyset$, $r = 2k + 1$. Let P_1 be a point of B_1 and T be a subset of B_2 with $T \neq B_2$. Then, $B = (B_1; P_1) + (B_2; T)$ is a non-trivial minimal k-block in Σ.*

Example 3.1. Let B_1, B_2 be skew planes $\Sigma = \mathrm{PG}(5, 2)$. Take a point $P_1 \in B_1$ and a set T in B_2. Taking T as a point, a line and six points of B_2, we get three different non-trivial minimal 2-blocking sets of size 13 in PG(5, 2).

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