INVERSE SETS AND INVERSE CORRESPONDENCES OVER INVERSE SEMIGROUPS

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ABSTRACT. This is a summary of the author's previous work. We introduce notions called inverse set and inverse correspondence over inverse semigroups. These are analogies of Hilbert C^*-modules and C^*-correspondences in the C^*-algebra theory. We show that inverse semigroups and inverse correspondences form a bicategory. In this bicategory, two inverse semigroups are equivalent if and only if they are Morita equivalent.

0. INTRODUCTION

This is a summary of the author's previous work [Uch24b]. The theory of inverse semigroups are closely related to the theory of C^*-algebras (for example, [Pat99, KS02, Exe08]). A *C^*-algebra* is a complex liner space equipped with a multiplication, an involution, and a complete norm which is compatible with the algebraic structures. In the theory of C^*-algebras, non-commutative and infinite-dimensional C^*-algebras often appear, but are generally difficult to investigate. Therefore, we construct C^*-algebras from some mathematical objects which are relatively easy to investigate, and study the C^*-algebras through their materials. Groups, étale groupoids, and inverse semigroups are used well as materials of C^*-algebras. Recent researches involve a categorical approach to the constructions of such C^*-algebras by using bicategory, which is a kind of category [BMZ13, AM16]. Albandik showed that the construction from étale groupoids to C^*-algebras forms a kind of functor (bifunctor) from the bicategory $\mathfrak{Gr}$ of étale groupoids to the bicategory $\mathfrak{Corr}$ of C^*-algebras [Alb15]. One might expect that the construction of C^*-algebras from inverse semigroups has a similar property. However, to the best knowledge of the author, any bicategory of inverse semigroups which corresponds to $\mathfrak{Gr}$ or $\mathfrak{Corr}$ has not been introduced in the theory of inverse semigroups. Therefore, the author introduced the bicategory $\mathfrak{IC}$ of inverse semigroups modeled on the bicategory $\mathfrak{Corr}$ in [Uch24b]. We also proved that two inverse semigroups are equivalent in this bicategory $\mathfrak{IC}$ if and only if they are strongly Morita equivalent.

1. INVERSE SEMIGROUPS

A *semigroup* is a set with an associative multiplication. A semigroup S is *regular* if for every $s \in S$ there exists an element $t \in S$ with $sts = s$ and $tst = t$. Such an element t is called a *generalized inverse* of s. A regular semigroup S is said to be *inverse* if each element has a unique generalized inverse. For an inverse semigroup S, we denote the generalized inverse of $s \in S$ as s^*.

An element e of a semigroup S is an *idempotent* if $ee = e$ holds. The set of all idempotents of S is denoted as $E(S)$.

Example 1.1. A discrete group is an inverse semigroup which has the unit as a unique idempotent.

Example 1.2. For topological spaces X and Y, a *partial homeomorphism* u from X to Y is a homeomorphism from an open subset D_u of X to an open subset R_u of Y. For a partial homeomorphism u from X to Y, we define a partial homeomorphism from Y to X, called an inverse of u, as the homeomorphism u^{-1} from R_u to D_u regarded as a partial homeomorphism from Y to X. We denote this partial homeomorphism by the same symbol u^{-1}. For topological spaces X_1, X_2, X_3 and partial homeomorphisms u_1 from X_1 to X_2, u_2 from X_2 to X_3, we define a composition u_2u_1 of u_1 and u_2 as the partial homeomorphism from X_1 to X_3 defined by $u_2u_1(x) := u_2(u_1(x))$ for $x \in D_{u_2u_1} := u_1^{-1}(D_{u_2})$. We denote the set of all partial homeomorphisms from X to Y as $I(X, Y)$. We abbreviate $I(X, X)$ to $I(X)$. The set $I(X)$ becomes an inverse semigroup with respect to the composition of partial homeomorphisms.

A subset I of a semigroup S is a *two-sided ideal* if $st \in I$ and $ts \in I$ hold for $s \in S$ and $t \in I$. A two-sided ideal of a semigroup becomes a subsemigroup. A two-sided ideal of an inverse semigroup becomes an inverse subsemigroup. We can prove the following proposition which is similar to [Pat99, Proposition 2.1.1] or [Law98, Theorem 3]:

Proposition 1.3. *Let S be a semigroup and I be a two-sided ideal of S. If I is an inverse subsemigroup of S, then for every $e \in E(S)$ and $f \in E(I)$, $ef = fe$ holds.*

This proposition plays an important role for proving Theorem 3.8.

Let S be an inverse semigroup. It is clear that $s^{**} = s$ for $s \in S$. We have $(st)^* = t^*s^*$ for $s, t \in S$ by using Proposition 1.3.

The following theorem is well-known as a characterization of inverse semigroups:

Theorem 1.4. *A regular semigroup S is inverse if and only if all idempotents of S commute.*

Proof. The only if part follows from Proposition 1.3. See [Law98, Theorem 3] for a proof of the if part. □

2. The bicategory $\mathfrak{Corr}$ of C^*-algebras

A *category* consists of collections of objects and morphisms; the composition $gf: x \to z$ is given for each two morphisms $f: x \to y$ and $g: y \to z$; the identity morphism 1_x is given for each object x. The following conditions are required to these structures:

(i) the associative law, that is, $h(gf) = (hg)f$ holds for $f: x \to y$, $g: y \to z$, $h: z \to w$.
(ii) the unit law, that is, $1_y f = f = f 1_x$ holds for $f: x \to y$.

Two objects x and y of a category are *isomorphic* if there exist $f\colon x \to y$ and $g\colon y \to x$ with $gf = 1_x$ and $fg = 1_y$.

We give two example of categories. For C^*-algebra A and B, a $*$-*homomorphism* $\sigma\colon A \to B$ is a linear map from A to B which preserves multiplications and involutions. C^*-algebras and $*$-homomorphisms form a category $\mathbf{C^*_{alg}}$ with respect to the usual composition of maps and the identity maps.

For semigroups S and T, a *semigroup homomorphism* $\theta\colon S \to T$ is a map from S to T which preserves multiplications. If S and T are inverse, then a semigroup homomorphism between them preserves the generalized inverses. Inverse semigroups and semigroup homomorphisms form a category **IS** with respect to the usual composition of maps and the identity maps. Two inverse semigroup are *isomorphic* if they are isomorphic in this category **IS**.

A *bicategory* introduced by Bénabou in [Bén67] equips 2-*arrows*, which are "morphisms between morphisms", in addition to objects and morphisms. For each two objects x and y of a bicategory, morphisms from x to y and 2-arrows between morphisms from x to y form a category. The morphisms $h(gf)$ and $(hg)f$, $1_y f$ and f, f and $f1_x$ are required to be isomorphic through some "natural" 2-arrows, instead of the associative law and the unit law. Two objects x and y of a bicategory are *equivalent* if there exist morphisms $f\colon x \to y$ and $g\colon y \to x$ such that gf is isomorphic to 1_x and fg is isomorphic to 1_y. See [Bén67] or [Lei98] for more details of the definition of bicategories.

As mentioned in Section 0, the bicategory $\mathfrak{Corr}$ of C^*-algebras appears in the theory of constructions of C^*-algebras. To define the morphisms of $\mathfrak{Corr}$, we see the definitions of Hilbert modules and C^*-correspondences. See [Lan95] for more details.

Let A be a C^*-algebra. An A-*action* on a complex linear space $\mathcal{E}$ is a bilinear map $\mathcal{E} \times A \to A; (\xi, a) \mapsto \xi a$ with $(\xi a)a' = \xi(aa')$ for $a, a' \in A$ and $\xi \in \mathcal{E}$. A *Hilbert A-module* $\mathcal{E}$ consists of an A-action on $\mathcal{E}$ and a map $\langle \cdot \,|\, \cdot \rangle_{\mathcal{E}}\colon \mathcal{E} \times \mathcal{E} \to A$ which satisfy similar conditions to Hilbert spaces. The map $\langle \cdot \,|\, \cdot \rangle_{\mathcal{E}}$ is so called A-valued inner product. If $A = \mathbb{C}$, a Hilbert A-module is nothing but a Hilbert space. In Section 3, we introduce the notion of *inverse S-set* as a set equipped with an action of an inverse semigroup S and an S-valued pairing.

For a C^*-algebra A and a Hilbert A-module $\mathcal{E}$, a linear map $\varphi\colon \mathcal{E} \to \mathcal{E}$ is *adjointable* if there exists a linear map $\psi\colon \mathcal{E} \to \mathcal{E}$ with $\langle \psi(\eta) \,|\, \xi \rangle_{\mathcal{E}} = \langle \eta \,|\, \varphi(\xi) \rangle_{\mathcal{E}}$ for $\xi, \eta \in \mathcal{E}$. The set $L(\mathcal{E})$ of all adjointable maps becomes a C^*-algebra with respect to the suitable structures. For C^*-algebras A and B, a *C^*-correspondence* $\mathcal{E}$ from A to B is a couple of a Hilbert B-module $\mathcal{E}$ and a $*$-homomorphism $\sigma_{\mathcal{E}}\colon A \to L(\mathcal{E})$. We denote it as $\mathcal{E}\colon A \to B$.

We give an example of C^*-correspondences. For a C^*-algebra B, the linear space B becomes a Hilbert B-module with respect to the B-action defined by the multiplication from the right side, and the inner product defined by $\langle b \,|\, b' \rangle_B := b^* b'$ for $b, b' \in B$. For every $b \in B$, the multiplication of b from the left side is an adjointable map λ_b on the Hilbert B-module B. The C^*-algebra B can be regarded as a C^*-subalgebra of $L(B)$ through the map $\lambda\colon B \to L(B); b \mapsto \lambda_b$. Thus a $*$-homomorphism $\sigma\colon A \to B$ induces a C^*-correspondence consisting of a Hilbert B-module B and a $*$-homomorphism $\sigma\colon A \to B \subset L(B)$. In this sense, C^*-correspondences can be regarded as a generalization of $*$-homomorphisms. We call the C^*-correspondence associated with the identity $*$-homomorphism on A the *identity correspondence*. In Section 4, we introduce adjointable maps on an inverse S-set $\mathcal{U}$ by using the S-valued pairing, and show that the set $L(\mathcal{U})$ of all adjointable maps becomes an inverse semigroup. For inverse semigroups S and T, the notion of inverse correspondence $\mathcal{U}$ is introduced as a couple of an inverse T-set $\mathcal{U}$ and a semigroup homomorphism $\theta_{\mathcal{U}}\colon S \to L(\mathcal{U})$ in Section 4. This is an analogy of C^*-correspondences in the theory of inverse semigroups.

Let A_i be a C^*-algebra with $i = 1, 2, 3, 4$, and $\mathcal{E}_i\colon A_i \to A_{i+1}$ be a C^*-correspondence with $i = 1, 2, 3$. For C^*-correspondences $\mathcal{E}_1$ and $\mathcal{E}_2$, we can define the C^*-correspondence $\mathcal{E}_1 \otimes \mathcal{E}_2$ from A_1 to A_3 called *(interior) tensor product*. C^*-algebras, C^*-correspondences, the tensor product, and the identity correspondences satisfy almost all of the conditions required to bicategories. However, they do not form a bicategory as discussed follows: The C^*-correspondences $(\mathcal{E}_1 \otimes \mathcal{E}_2) \otimes \mathcal{E}_3$ and $\mathcal{E}_1 \otimes (\mathcal{E}_2 \otimes \mathcal{E}_3)$ are isomorphic through the "natural" bijection $(\xi_1 \otimes \xi_2) \otimes \xi_3 \mapsto \xi_1 \otimes (\xi_2 \otimes \xi_3)$. This map is a 2-arrow which corresponds to the associative law. The C^*-correspondences $\mathcal{E}_1 \otimes A_2$ and $\mathcal{E}_1$ are isomorphic through the "natural" bijection $\xi_1 \otimes a_2 \mapsto \xi_1 a_2$. This map is a 2-arrow which corresponds to one of the two unit laws. The "natural" map $A_1 \otimes \mathcal{E}_1 \to \mathcal{E}_1; a_1 \otimes \xi_1 \mapsto a_1 \xi_1$ is injective, preserves the structures of C^*-correspondence, but is not surjective in general. This becomes an isomorphism if and only if $\mathcal{E}_1$ is *non-degenerate*, that is, it satisfies $\mathcal{E}_1 = \{\sigma_{\mathcal{E}_1}(a_1)(\xi_1) \mid a_1 \in A_1, \xi_1 \in \mathcal{E}_1\}$. By restricting the collection of morphisms to all non-generate C^*-correspondences, we obtain the bicategory $\mathfrak{Corr}$ of C^*-algebras. In Section 4, we introduce the property called non-degenerate for inverse correspondences, and show that inverse semigroups and non-degenerate inverse correspondences form a bicategory. We denote this bicategory as $\mathfrak{IC}$.

Rieffel introduced the equivalence relation between C^*-algebras called *strong Morita equivalence* [Rie74]. Two C^*-algebras are equivalent in the bicategory $\mathfrak{Corr}$ if and only if they are strongly Morita equivalent [EKQR06]. Steinberg introduced the equivalence relation between inverse semigroups also called strong Morita equivalence [Ste11]. He showed that the construction from inverse semigroups to C^*-algebras preserves strong Morita equivalence through the theory of groupoids and their C^*-algebras. The author showed that two inverse semigroups are equivalent in the bicategory $\mathfrak{IC}$ if and only if they are strongly Morita equivalent [Uch24b]. In forth coming paper [Uch24a], we will show that the construction from inverse semigroups to C^*-algebras forms a bifunctor from $\mathfrak{IC}$ to $\mathfrak{Corr}$. Because of these result, we can give another proof of the fact proved by Steinberg since every bifunctor preserves equivalences in bicategories.

3. Inverse sets and inverse semigroup $L(\mathcal{U})$ of adjointable maps

In this section, we introduce inverse sets and adjointable maps on them. Let S be an inverse semigroup.

Definition 3.1 ([Uch24b, Definition 2.2]). A *regular S-set* $\mathcal{U}$ is a set $\mathcal{U}$ equipped with a right S-action (that is, a map $\mathcal{U} \times S \to \mathcal{U}; (u, s) \mapsto us$ with

$(us)s' = u(ss')$ for $s, s' \in S$ and $u \in \mathcal{U}$) and a map $\langle \cdot \,|\, \cdot \rangle_{\mathcal{U}} \colon \mathcal{U} \times \mathcal{U} \to S$ called a *(right) pairing* on $\mathcal{U}$ which satisfy that

(R-i) $\langle u \,|\, u's \rangle_{\mathcal{U}} = \langle u \,|\, u' \rangle_{\mathcal{U}} s$,
(R-ii) $\langle u \,|\, u' \rangle_{\mathcal{U}}^* = \langle u' \,|\, u \rangle_{\mathcal{U}}$,
(R-iii) $u \langle u \,|\, u \rangle_{\mathcal{U}} = u$,

for every $u, u' \in \mathcal{U}$ and $s \in S$. An *inverse S-set* $\mathcal{U}$ is a regular S-set which satisfies that

(R-iv) $u \langle u' \,|\, u \rangle_{\mathcal{U}} = u$ and $u' \langle u \,|\, u' \rangle_{\mathcal{U}} = u'$ imply $u = u'$

for every $u, u' \in \mathcal{U}$.

As a first example, we regard an inverse semigroup S as an inverse S-set.

Example 3.2. We set a right action of S on S as the multiplication from the right side and define a map $\langle \cdot \,|\, \cdot \rangle_S \colon S \times S \to S$ by $\langle s \,|\, s' \rangle_S := s^* s'$ for every $s, s' \in S$. It is clear that this map satisfies (R-i) and (R-ii). The map $\langle \cdot \,|\, \cdot \rangle_S$ satisfies (R-iii) by the definition of the generalized inverse and satisfies (R-iv) since S is inverse. Thus S is an inverse S-set with respect to the above structures.

Example 3.3. We define a right action of $I(X)$ on $I(X, Y)$ by the composition from the right side and a pairing on $I(X, Y)$ by

$$\langle u_1 \,|\, u_2 \rangle_{I(X,Y)} := u_1^{-1} u_2$$

for $u_1, u_2 \in I(X, Y)$. We can see that the set $I(X, Y)$ becomes an inverse $I(X)$-set with respect to the above structures.

Let $\mathcal{U}$ and $\mathcal{V}$ be regular S-sets.

Definition 3.4 ([Uch24b, Definition 2.9])**.** A map $\varphi \colon \mathcal{U} \to \mathcal{V}$ is an *S-map* if $\varphi(us) = \varphi(u)s$ for $u \in \mathcal{U}$ and $s \in S$.

Definition 3.5 ([Uch24b, Definition 3.1])**.** A map $\varphi \colon \mathcal{U} \to \mathcal{V}$ is said to be *adjointable* if there exists a map $\psi \colon \mathcal{V} \to \mathcal{U}$ such that

$$\langle \psi(v) \,|\, u \rangle_{\mathcal{U}} = \langle v \,|\, \varphi(u) \rangle_{\mathcal{V}}$$

holds for every $u \in \mathcal{U}$ and $v \in \mathcal{V}$. Such a map ψ is said to be an *adjoint* of φ. We denote the set of all adjointable maps from $\mathcal{U}$ to $\mathcal{V}$ as $L(\mathcal{U}, \mathcal{V})$. We abbreviate $L(\mathcal{U}, \mathcal{U})$ as $L(\mathcal{U})$.

We can easily see that the set $L(\mathcal{U})$ becomes a semigroup with respect to the composition of maps. We give examples of adjointable maps:

Definition 3.6 ([Uch24b, Definition 3.4])**.** For $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we define a map $\omega_{v,u} \colon \mathcal{U} \to \mathcal{V}$ by

$$\omega_{v,u}(u') := v \langle u \,|\, u' \rangle_{\mathcal{U}}$$

for $u' \in \mathcal{U}$. We denote the set $\{\omega_{v,u} \mid u \in \mathcal{U}, v \in \mathcal{V}\}$ as $K(\mathcal{U}, \mathcal{V})$. We abbreviate $K(\mathcal{U}, \mathcal{U})$ as $K(\mathcal{U})$.

For $u \in \mathcal{U}$ and $v \in \mathcal{V}$, the map $\omega_{v,u}$ is an adjoint of $\omega_{u,v}$. Thus $K(\mathcal{U}, \mathcal{V})$ is a subset of $L(\mathcal{U}, \mathcal{V})$.

Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be regular S-sets. We can see

$$\varphi' \omega_{v,u} = \omega_{\varphi'(v),u}, \quad \omega_{w,v} \varphi = \omega_{w,\psi(v)}$$

for $u \in \mathcal{U}$, $v \in \mathcal{V}$, $w \in \mathcal{W}$, $\varphi \in L(\mathcal{U}, \mathcal{V})$, $\varphi' \in L(\mathcal{V}, \mathcal{W})$, where ψ is an adjoint of φ. These imply that $K(\mathcal{U})$ is a two-sided ideal of $L(\mathcal{U})$.

The following proposition plays an important role in the proofs of Theorem 3.8, 3.9, and Proposition 4.6:

Proposition 3.7 ([Uch24b, Proposition 3.12]). *For a regular S-set $\mathcal{U}$, the following are equivalent:*

(i) $u\langle u' \mid u\rangle_{\mathcal{U}} = u$ and $u'\langle u \mid u'\rangle_{\mathcal{U}} = u'$ imply $u = u'$ for every $u, u' \in \mathcal{U}$ (that is, $\mathcal{U}$ is an inverse S-set),
(ii) $\langle u \mid u\rangle_{\mathcal{U}} = \langle u' \mid u'\rangle_{\mathcal{U}} = \langle u \mid u'\rangle_{\mathcal{U}}$ implies $u = u'$ for every $u, u' \in \mathcal{U}$,
(iii) $u\langle u \mid u'\rangle_{\mathcal{U}} = u'\langle u' \mid u\rangle_{\mathcal{U}}\langle u \mid u'\rangle_{\mathcal{U}}$ for every $u, u' \in \mathcal{U}$,
(iv) $\omega_{u,u}$ and $\omega_{u',u'}$ commutes for every $u, u' \in \mathcal{U}$,

We give the properties of adjointable maps between inverse S-sets (see [Uch24b, Section 3]). Let $\mathcal{U}$, $\mathcal{V}$ be inverse S-sets and $\varphi\colon \mathcal{U} \to \mathcal{V}$ be an adjointable map. We can show that an adjoint of $\varphi\colon \mathcal{U} \to \mathcal{V}$ is unique. We denote the adjoint of φ as $\varphi^\dagger$. We can also see that φ becomes an S-map. For an inverse S-set $\mathcal{U}_i$ with $i = 1, 2, 3$ and an adjointable map $\varphi_i\colon \mathcal{U}_i \to \mathcal{U}_{i+1}$ with $i = 1, 2$, we have $\varphi_1^{\dagger\dagger} = \varphi_1$ and $(\varphi_2\varphi_1)^\dagger = \varphi_1^\dagger\varphi_2^\dagger$. An element $\varphi \in L(\mathcal{U})$ is said to be *self-adjoint* if $\varphi = \varphi^\dagger$ holds.

Theorem 3.8 ([Uch24b, Theorem 3.19, 3.30]). *For an inverse S-sets $\mathcal{U}$, the semigroups $K(\mathcal{U})$ and $L(\mathcal{U})$ are inverse.*

Sketch of proof. We can obtain $E(K(\mathcal{U})) = \{\omega_{u,u} \mid u \in \mathcal{U}\}$ by using Proposition 3.7 (ii). By Proposition 3.7 (iv), we see that all elements of $E(K(\mathcal{U}))$ commute. Thus $K(\mathcal{U})$ is inverse by Theorem 1.4.

We next prove that $L(\mathcal{U})$ is regular. Fix $\varphi \in L(\mathcal{U})$ and $u \in \mathcal{U}$. We already obtain that the two-sided ideal $K(\mathcal{U})$ of the semigroup $L(\mathcal{U})$ is inverse. Thus every idempotent of $L(\mathcal{U})$ and $\omega_{u,u}$ commute by Proposition 1.3. By using this fact and the fact that $\varphi^\dagger\varphi\omega_{u,u}$ and $\omega_{u,u}\varphi^\dagger\varphi$ are idempotents, we see that $\varphi^\dagger\varphi$ and $\omega_{u,u}$ commute. Since φ is an S-map, we have

$$\begin{aligned}\varphi(u) &= \varphi(u)\langle\varphi(u) \mid \varphi(u)\rangle_{\mathcal{U}} = \varphi(u\langle\varphi(u) \mid \varphi(u)\rangle_{\mathcal{U}}) = \varphi\left(u\left\langle u \,\middle|\, \varphi^\dagger\varphi(u)\right\rangle_{\mathcal{U}}\right)\\ &= \varphi\big(\omega_{u,u}\,\varphi^\dagger\varphi(u)\big) = \varphi\big(\varphi^\dagger\varphi\,\omega_{u,u}(u)\big) = \varphi\varphi^\dagger\varphi(u\langle u \mid u\rangle_{\mathcal{U}}) = \varphi\varphi^\dagger\varphi(u).\end{aligned}$$

Thus we have $\varphi = \varphi\varphi^\dagger\varphi$. By taking the adjoints, we obtain $\varphi^\dagger\varphi\varphi^\dagger = \varphi^\dagger$. Hence $\varphi^\dagger$ is a generalized inverse of φ.

We finally show that $L(\mathcal{U})$ is inverse. Let φ be an element of $L(\mathcal{U})$ and ψ_1, ψ_2 be adjoints of φ. We can easily see that $\varphi\psi_1$, $\varphi\psi_2$, $\psi_1\varphi$ and $\psi_2\varphi$ are idempotents. By using Proposition 1.3, we can prove that every idempotent of $L(\mathcal{U})$ is self-adjoint. This implies that

$$\begin{aligned}\psi_1 &= \psi_1\varphi\psi_1 = (\psi_1\varphi)^\dagger\psi_1 = (\psi_1\varphi\psi_2\varphi)^\dagger\psi_1\\ &= (\psi_2\varphi)^\dagger(\psi_1\varphi)^\dagger\psi_1 = \psi_2\varphi\psi_1\varphi\psi_1 = \psi_2\varphi\psi_1.\end{aligned}$$

We also have $\psi_2 = \psi_2\varphi\psi_1$ in a similar way. Thus $\psi_1 = \psi_2$ holds. ☐

We can show the following theorem in a similar way to the proof of Theorem 3.8:

Theorem 3.9 ([Uch24b, Theorem 3.31])**.** *For inverse S-sets $\mathcal{U}$ and $\mathcal{V}$, the set $L(\mathcal{U}, \mathcal{V})$ becomes an inverse $L(\mathcal{U})$-set with respect to the right $L(\mathcal{U})$-action defined by the composition from the right side and a pairing $\langle \cdot \,|\, \cdot \rangle_{L(\mathcal{U},\mathcal{V})} \colon L(\mathcal{U}, \mathcal{V}) \times L(\mathcal{U}, \mathcal{V}) \to L(\mathcal{U})$ defined by $\langle \varphi \,|\, \psi \rangle_{L(\mathcal{U},\mathcal{V})} := \varphi^\dagger \psi$. The set $K(\mathcal{U}, \mathcal{V})$ becomes an inverse $K(\mathcal{U})$-set with respect to the same structure.*

4. Inverse correspondences and the bicategory $\mathfrak{IC}$

We introduce inverse correspondence between inverse semigroups with the theory of C^*-correspondences in mind. Let S, T be inverse semigroups.

Definition 4.1 ([Uch24b, Definition 4.1, 4.5])**.** An *inverse correspondence* $\mathcal{U}$ from S to T is a couple of an inverse T-set $\mathcal{U}$ and a semigroup homomorphism $\theta_\mathcal{U} \colon S \to L(\mathcal{U})$. We denote it as $\mathcal{U} \colon S \to T$. An inverse correspondence $\mathcal{U}$ is said to be *non-degenerate* if $\mathcal{U} = \{\theta_\mathcal{U}(s)(u) \mid s \in S, u \in \mathcal{U}\}$ holds.

Definition 4.2. Let $\mathcal{U}$ and $\mathcal{V}$ be inverse correspondences from S to T. A map $\varphi \colon \mathcal{U} \to \mathcal{V}$ is an *isomorphism* if it is a bijective S-map such that $\langle \varphi(u) \,|\, \varphi(u') \rangle_\mathcal{V} = \langle u \,|\, u' \rangle_\mathcal{U}$ and $\varphi\big(\theta_\mathcal{U}(s)(u)\big) = \theta_\mathcal{V}(s)(\varphi(u))$ hold for $s \in S$ and $u, u' \in \mathcal{U}$. We say that $\mathcal{U}$ and $\mathcal{V}$ are *isomorphic* if there exists an isomorphism between them.

We give three examples of non-degenerate inverse correspondences.

Example 4.3. Let S, T be inverse semigroups and $\tau \colon S \to T$ be a semigroup homomorphism. We obtain an inverse T-set T as in Example 3.2. We can see that the subset $\mathcal{U}_\tau := \{\tau(s)t \mid s \in S, t \in T\}$ of T becomes an inverse T-set with respect to the same structures. We define a map $\theta_{\mathcal{U}_\tau} \colon S \to L(\mathcal{U}_\tau)$ by $\theta_{\mathcal{U}_\tau}(s)(u) := \tau(s)u$. We can see that $\theta_{\mathcal{U}_\tau}$ is a semigroup homomorphism and that the couple $\mathcal{U}_\tau$ and $\theta_{\mathcal{U}_\tau}$ becomes a non-degenerate inverse correspondence from S to T. We call the inverse correspondence $\mathcal{U}_{\mathrm{id}_S}$ from S to S associated with the identity map id_S on S the *identity correspondence*. This will be regarded as the identity morphism in the bicategory $\mathfrak{IC}$ later.

Example 4.4. For topological spaces X, Y, we obtain the inverse $I(X)$-set $I(X, Y)$ as in Example 3.3. The operation $\theta_{I(X,Y)}(v)$ to compose $v \in I(Y)$ from the left side is an adjointable map on $I(X, Y)$. The map $\theta_{I(X,Y)} \colon v \mapsto \theta_{I(X,Y)}(v)$ is a semigroup homomorphism from $I(Y)$ to $L(I(X, Y))$. We can see that the couple of the inverse $I(X)$-set $I(X, Y)$ and the semigroup homomorphism $\theta_{I(X,Y)}$ form a non-degenerate inverse correspondence from $I(Y)$ to $I(X)$.

Example 4.5. Let $\mathcal{U}, \mathcal{V}$ be inverse S-sets. We obtain the inverse $L(\mathcal{U})$-set $L(\mathcal{U}, \mathcal{V})$ as in Theorem 3.9. The operation $\theta_{L(\mathcal{U},\mathcal{V})}(\psi)$ to compose $\psi \in L(\mathcal{V})$ from the left side is an adjointable map on $L(\mathcal{U}, \mathcal{V})$. The map $\theta_{L(\mathcal{U},\mathcal{V})} \colon \psi \mapsto \theta_{L(\mathcal{U},\mathcal{V})}(\psi)$ is a semigroup homomorphism from $L(\mathcal{V})$ to $L(L(\mathcal{U}, \mathcal{V}))$. We can see that the couple of the inverse $L(\mathcal{U})$-set $L(\mathcal{U}, \mathcal{V})$ and the semigroup homomorphism $\theta_{L(\mathcal{U},\mathcal{V})}$ form a non-degenerate inverse correspondence from $L(\mathcal{V})$ to $L(\mathcal{U})$.

Let S_i be an inverse semigroup with $i = 1, 2, 3$ and $\mathcal{U} \colon S_1 \to S_2$, $\mathcal{V} \colon S_2 \to S_3$ be inverse correspondences. We introduce the inverse correspondence

$\mathcal{U}\otimes\mathcal{V}\colon S_1 \to S_3$ as follows: We define the set $\mathcal{U}\otimes\mathcal{V}$ as the quotient of the direct product $\mathcal{U}\times\mathcal{V}$ by the minimum equivalence relation $\sim$ such that $(us_2, v) \sim (u, \theta_{\mathcal{V}}(s_2)(v))$ holds for $u\in\mathcal{U}$, $s_2\in S_2$ and $v\in\mathcal{V}$. We denote the equivalence class of (u,v) as $u\otimes v$. We define a right action of S_3 on $\mathcal{U}\otimes\mathcal{V}$ as

$$(u\otimes v)s_3 := u\otimes(vs_3)$$

and a map $\langle\cdot\,|\,\cdot\rangle_{\mathcal{U}\otimes\mathcal{V}}\colon (\mathcal{U}\otimes\mathcal{V})\times(\mathcal{U}\otimes\mathcal{V})\to S_3$ as

$$\left\langle u'\otimes v' \,\middle|\, u\otimes v\right\rangle_{\mathcal{U}\otimes\mathcal{V}} := \left\langle v' \,\middle|\, \theta_{\mathcal{V}}(\langle u' \,|\, u\rangle_{\mathcal{U}})(v)\right\rangle_{\mathcal{V}}$$

for $u,u'\in\mathcal{U}$, $v,v'\in\mathcal{V}$ and $s_3\in S_3$.

Proposition 4.6 ([Uch24b, Proposition 4.11])**.** *The set $\mathcal{U}\otimes\mathcal{V}$ becomes an inverse S_3-set with respect to the above structures.*

Sketch of proof. We can easily see that the S_3-action and the pairing defined above are well-defined, and that $\mathcal{U}\otimes\mathcal{V}$ is a regular S_3-set. By using Proposition 3.7, we can prove that $\mathcal{U}\otimes\mathcal{V}$ is an inverse S_3-set. □

For $s_1\in S_1$, we define a map $\theta_{\mathcal{U}\otimes\mathcal{V}}(s_1)$ on $\mathcal{U}\otimes\mathcal{V}$ as

$$\theta_{\mathcal{U}\otimes\mathcal{V}}(s_1)(u\otimes v) := \theta_{\mathcal{U}}(s_1)(u)\otimes v.$$

for $u\in\mathcal{U}$ and $v\in\mathcal{V}$. We can see that this is an adjointable map on $\mathcal{U}\otimes\mathcal{V}$ and that the map $\theta_{\mathcal{U}\otimes\mathcal{V}}\colon S_1\to L(\mathcal{U}\otimes\mathcal{V}); s_1\mapsto\theta_{\mathcal{U}\otimes\mathcal{V}}(s_1)$ is a semigroup homomorphism.

Definition 4.7. For inverse correspondences $\mathcal{U}\colon S_1\to S_2$ and $\mathcal{V}\colon S_2\to S_3$, we call the couple of the inverse S_3-set $\mathcal{U}\otimes\mathcal{V}$ and the semigroup homomorphism $\theta_{\mathcal{U}\otimes\mathcal{V}}\colon S_1\to L(\mathcal{U}\otimes\mathcal{V})$ the *tensor product* of $\mathcal{U}$ and $\mathcal{V}$.

We can see that the tensor product of two non-degenerate inverse correspondences is non-degenerate.

Theorem 4.8 ([Uch24b, Theorem 5.12])**.** *Inverse semigroups and non-degenerate inverse correspondences form a bicategory with the tensor product as composition and the identity correspondences as identity morphisms. We denote this bicategory as $\mathfrak{IC}$.*

Sketch of proof. We only see that there exist isomorphisms which correspond to the associative law and the unit law. Let S_i be an inverse semigroup with $i=1,2,3,4$ and $\mathcal{U}_i\colon S_i\to S_{i+1}$ be an inverse correspondence with $i=1,2,3$. The map $\alpha\colon(\mathcal{U}_1\otimes\mathcal{U}_2)\otimes\mathcal{U}_3\to\mathcal{U}_1\otimes(\mathcal{U}_2\otimes\mathcal{U}_3)$ defined as

$$\alpha((u_1\otimes u_2)\otimes u_3) := u_1\otimes(u_2\otimes u_3)$$

and the map $\lambda\colon\mathcal{U}_1\otimes S_2\to\mathcal{U}_1$ defined as

$$\lambda(u_1\otimes s_2) := u_1 s_2$$

are isomorphisms. The map $\rho\colon S_1\otimes\mathcal{U}_1\to\mathcal{U}_1$ defined as

$$\rho(s_1\otimes u_1) := s_1 u_1$$

is an isomorphism since $\mathcal{U}_1$ is non-degenerate. □

We give an example of a kind of functor (bifunctor) to this bicategory $\mathfrak{IC}$:

Example 4.9. Let S_i be an inverse semigroup with $i = 1, 2, 3$ and $\tau_i \colon S_i \to S_{i+1}$ be a semigroup homomorphism with $i = 1, 2$. We obtain the inverse correspondences $\mathcal{U}_{\tau_1} \colon S_1 \to S_2$ and $\mathcal{U}_{\tau_2} \colon S_2 \to S_3$ as in Example 4.3. The tensor product $\mathcal{U}_{\tau_1} \otimes \mathcal{U}_{\tau_2}$ of these inverse correspondences is isomorphic to the inverse correspondence $\mathcal{U}_{\tau_2\tau_1}$ associated with the composition of the semigroup homomorphisms τ_1 and τ_2 through an isomorphism $\mathcal{U}_{\tau_1} \otimes \mathcal{U}_{\tau_2} \to \mathcal{U}_{\tau_2\tau_1}; u_1 \otimes u_2 \mapsto \tau_2(u_1)u_2$. The construction from semigroup homomorphisms τ to the associated inverse correspondences $\mathcal{U}_\tau$ form a bifunctor from the category **IS** to the bicategory $\mathfrak{IC}$.

The following theorem is one of the main results of [Uch24b].

Theorem 4.10 ([Uch24b, Theorem 5.16])**.** *Two inverse semigroups are equivalent in the bicategory* $\mathfrak{IC}$ *if and only if they are strongly Morita equivalent.*

We can prove this theorem in a similar way to [EKQR06, Lemma 2.4].

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