Automorphism groups of zeropotent algebras of dimension 3 *

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1 Introduction

Let A be a (not necessarily associative) algebra over a field K. The automorphism group $G(A)$ of A is the group of bijective linear mappings f on A satisfying $f(xy) = f(x)f(y)$ for all $x, y \in A$.

An algebra A is zeropotent if $x^2 = 0$ for any $x \in A$. A zeropotent algebra is anti-commutative, that is, $xy = -yx$ for all $x, y \in A$. Zeropotent algebras of dimension 3 over an algebraically closed field are completely classified by Kobayashi et al. [1]. We can determine the automorphism groups of zeropotent algebras of dimension 3. In this article, we choose a typical algebra from them and calculate its automorphism group.

2 Zeropotent algebras of dimension 3

Let A be a zeropotent algebras of dimension 3 over K. Let $E = \{e, f, g\}$ be a basis of A. Because A is anti-commutative, the multiplication table of A on E is given by

$$\begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix},$$

where

$$\begin{cases} \gamma = fg = a_{11}e + a_{12}f + a_{13}g \\ \beta = ge = a_{21}e + a_{22}f + a_{23}g \\ \alpha = ef = a_{31}e + a_{32}f + a_{33}g \end{cases}$$

for $a_{ij} \in K$. We call

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}. \end{pmatrix}$$

*This is a preliminary report and will not be published elsewhere in this current form.

the *structural matrix* of A because it determines the structure of the algebra A (we use the bold symbol $\boldsymbol{A}$ for the structural matrix).

Let T be a linear mapping on A. We use the bold symbol $\boldsymbol{T}$ for the representation matrix of T on the basis E. By Proposition 2.1 in [1], we have

Theorem 2.1. *T is an automorphism of A, if and only if $|\boldsymbol{T}| \neq 0$ and*

$$\boldsymbol{A} = \frac{1}{|\boldsymbol{T}|}\boldsymbol{T}^t \boldsymbol{A}\boldsymbol{T}. \tag{1}$$

Up to isomorphism, we have 9 families of nonzero zeropotent algebras of dimension 3 over an algebraically closed field K with $\mathrm{char}(K) \neq 2$:

$$Z_1, Z_2, Z_3, \{Z_4(k)\}_{k\in H}, Z_5, Z_6, \{Z_7(k)\}_{k\in H}, Z_8 \text{ and } Z_9$$

defined by the structural matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively, where H is a subset of K such that

$$K = H \cup -H, \ H \cap -H = \{0\}.$$

See Kobayashi et al. [1].

3 Circle groups

For $\sigma \in K$ define the multiplication $*_\sigma$ on $K^2 = K \times K$ by

$$(x, y) *_\sigma (z, u) = (xu + yz, -\sigma xz + yu)$$

for $(x, y), (z, u) \in K^2$. It is easy to see that $(K^2, *_\sigma)$ is a monoid with identity element $(0, 1)$.

The mapping $\phi_\sigma : K^2 \to K$ defined by

$$\phi_\sigma(x, y) = \sigma x^2 + y^2 \quad ((x, y) \in K \times K),$$

is a homomorphism of multiplicative monoids, and the kernel

$$C_\sigma = \phi_\sigma^{-1}(1) = \{(x, y) \in K \times K \mid \sigma x^2 + y^2 = 1\}$$

forms a group.

If $\sigma = 0$, then

$$C_0 = K \times \{1, -1\},$$

and the multiplication $* = *_0$ is given by

$$(x, \epsilon) * (y, \delta) = (\delta x + \epsilon y, \epsilon\delta)$$

for $x, y \in K$ and $\epsilon, \delta \in \{1, -1\}$.

If $\sigma \neq 0$ and the square root $\sqrt{\sigma}$ is in K, C_σ is isomorphic to the circle group $(C, *)$:

$$C = C_1 = \{(x, y) \in K^2 \mid x^2 + y^2 = 1\}$$

$$(x, y) * (z, u) = (xu + yz, -xz + yu) \quad ((x, y), (z, u) \in C)\,,$$

through the isomorphism $\psi_\sigma : C_\sigma \to C$ given by

$$\psi_\sigma(x, y) = (\sqrt{\sigma}x, y) \quad ((x, y) \in C_\sigma)\,.$$

4 Determination of the automorphism groups

We can determine the automorphism group of each zeropotent algebra classified above using Theorem 2.1. Here, we choose the algebra $Z_7(k)$ defined by the structural matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with parameter $k \in H$.

Let T be a linear mapping represented by

$$\boldsymbol{T} = \begin{pmatrix} a & b & c \\ p & q & r \\ s & t & u \end{pmatrix}$$

on a basis $E = \{e, f, g\}$. By Theorem 2.1, T is an automorphism if and only if $|\boldsymbol{T}| \neq 0$ and

$$\begin{aligned} |T|\boldsymbol{A} &= (aqu + brs + cpt - cqs - art - bpu) \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \\ \boldsymbol{T}^t\boldsymbol{A}\boldsymbol{T} &= \begin{pmatrix} a^2 + p^2 + s^2 + kap & ab + pq + st + kaq & ac + pr + su + kar \\ ab + pq + st + kbp & b^2 + q^2 + t^2 + kbq & bc + qr + tu + kbr \\ ac + pr + su + kcp & bc + qr + tu + kcq & c^2 + r^2 + u^2 + kcr \end{pmatrix}. \end{aligned}$$

Solving this equation, T is an automorphism if and only if

$$\begin{cases} b = -p = \frac{ka \,\pm\, h}{2} \\ q = \frac{(2-k^2)a \,\mp\, kh}{2} \\ u = 1 \\ c = r = s = t = 0, \end{cases}$$

where $h = \sqrt{(k^2-4)a^2+4}$. Hence, we see that the automorphism group of $Z_7(k)$ is the linear group

$$G = \left\{ \boldsymbol{T}(a,h) \,\middle|\, a, h \in K, h^2 = (k^2-4)a^2+4 \right\},$$

where

$$\boldsymbol{T}(a,h) = \frac{1}{2}\begin{pmatrix} 2a & ka+h & 0 \\ -ka-h & (2-k^2)a-kh & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

When $k = 2$, we have

$$G = \left\{ \boldsymbol{T}(a,h) \,\middle|\, a, h \in K, h = \pm 1 \right\},$$

where

$$\boldsymbol{T}(a,h) = \begin{pmatrix} a & a+h & 0 \\ -a-h & -a-2h & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Because

$$\boldsymbol{T}(a,h)\boldsymbol{T}(b,j) = \boldsymbol{T}(-aj-bh-hj, -hj)$$

for $a, b \in K$ and $h, j \in \{1, -1\}$, we see that G is isomorphic to C_0 via isomorphism:

$$(x,h) \mapsto (x+h, -h).$$

If $k \neq \pm 2$, letting $\sigma = 4 - k^2$, G is isomorphic to the ellipse

$$E_\sigma : x^2 + \sigma y^2 = 4,$$

with multiplication $*$ given by

$$(x,y) * (z,u) \;=\; \frac{1}{4}\left(k(xz-\sigma yu) + \sigma(xu+yz),\, k(xu+yz) - (xz - \sigma yu)\right).$$

It is isomorphic to the group C_σ through the transformation

$$x \mapsto \frac{-x+ky}{4},\; y \mapsto \frac{kx+\sigma y}{4}.$$

Thus, G is isomorphic to the circle group C as stated in Section 3.

References

[1] Y. Kobayashi, K. Shirayanagi, S. Takahasi, M. Tsukada, Classification of three-dimensional zeropotent algebras over an algebraically closed field, Communication in Algebra, vol. 45, 2017, 5037–5052.