A Novel Systematic Rotation Method to Color Planar Graphs

Weiguo Xie[1,*] and Andrew Bowling[1]

[1]Swenson College of Science & Engineering
University of Minnesota Duluth
Duluth, MN 55812, USA

*Corresponding author email: xiew@umn.edu

Abstract

The problem of finding four-colorings of plane graphs is one of the most famous in graph theory. Alfred Kempe believed that a systematic method of exchanging colors on what are now called Kempe chains would enable the coloring of any planar map. While Kempe's proof using this fact was flawed, the general premise is quite useful in the study of plane graph coloring. Specifically, we use his idea to develop several effective deterministic methods for coloring plane graphs. We present results on the efficacy of our algorithms on a variety of graphs.

Key Words: Four Color Problem, cubic map, Kempe chains, Errera map, coloring algorithms

AMS Subject Classification: 05C15.

1 Introduction

The Four Color Theorem was an object of intense study during the 19th and 20th centuries. The Four Color Theorem states that the regions of a plane graph (that is, a graph drawn in the plane with no two edges crossing) can be colored with four or fewer colors so that no two regions sharing a boundary line have the same color. Such a coloring is a *proper region coloring.* Francis Guthrie conjectured this to be true in 1852, but it took until 1976 for a proof to be widely accepted [1, 9]. Even so, all accepted proofs have major portions that necessitate the use of computers, and therefore these proofs cannot be checked by hand. The machine-checkable proofs found in [1, 8] both translate to efficient polynomial time algorithms for four-coloring plane graphs; however the algorithms are quite slow in practice. Further attempts like those in [7] attain near-linear time efficiency in practice, but they are not provably correct and utilize randomness to attain their relative efficiency. Here we will study the deterministic coloring algorithms developed in [10, 11, 12, 14] and present new results on simulations for these algorithms.

2 Definitions

Our algorithms will make use of Kempe chains, first utilized by Alfred Kempe in [5]. An AB *Kempe chain* is a maximal connected set of regions of G such that every region has color either A or B. Given a proper region coloring (or a proper partial region coloring), exchanging the colors A and B on an AB Kempe chain results in a new proper (partial) region coloring.

The terminology we present next is from [6], with some alterations based on further work in [13, 14]. We will assume in the following definitions that the exterior region R is the region we are attempting to assign a color, and that only five neighbors of R are colored. We will also assume that the four colors appear on regions adjacent with R so that no two consecutive regions (ignoring uncolored regions) have the same color. We will call these colored regions adjacent to R *boundary regions.* Figure 1 will be used for reference.

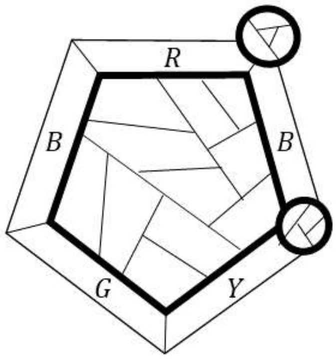

Figure 1: A partially colored graph where the exterior region has 5 colored neighbors, with no two consecutive colored regions of the same color. Some of the unmarked regions may be colored, but not those adjacent to the exterior region.

- The boundary region situated between two boundary regions of the same color is called the *top region.* In Figure 1, this is the boundary region labeled R.
- A Kempe chain containing both the top region and the boundary region positioned two spaces counterclockwise from the top region is called the *left-hand circuit.* If such a Kempe chain does not exist, we will refer to the Kempe chain using these colors and starting at the top region as a *broken left-hand circuit.* In Figure 1, this will be an RG Kempe chain.
- Similarly, a Kempe chain containing both the top region and the boundary region positioned two spaces clockwise from the top region is called the *right-hand circuit* (analogously, *broken right-hand circuit*).
- A Kempe chain beginning at the boundary region counterclockwise to the top region and whose other color is that of the region two spaces clockwise from the top region (in Figure 1, B and Y) is called the *left-hand chain.*
- A Kempe chain beginning at the boundary region clockwise to the top region and whose other color is that of the region two spaces counterclockwise from the top region is called the *right-hand chain.*

- The Kempe chain containing the two boundary regions not directly clockwise or counter-clockwise to the top region is called the *end tangent chain*. In this case, this would be a GY Kempe chain. It is possible that these two regions are not connected by a Kempe chain; in this case, we still refer to the Kempe chain using these colors and containing at least one of these regions as an end tangent chain.

The following operations will be utilized in our algorithms.

- ℓ (for left): Exchanging colors on the left-hand chain
- r (for right): Exchanging colors on the right-hand chain
- t_ℓ (for top-left): Exchanging colors on the left-hand circuit (or the broken left-hand circuit)
- t_r (for top-right): Exchanging colors on the right-hand circuit (or the broken right-hand circuit)
- e (for end): Exchanging colors on the end tangent chain

Given a partial coloring c from a subset of the regions of a graph to the colors $\{R, B, Y, G\}$ and some operation σ, we will refer to $\sigma(c)$ as the resulting coloring after applying σ to c. In addition, given two operations σ_1, σ_2, we will refer to $\sigma_2\sigma_1$ as the result of applying first σ_1, then σ_2.

As in [14], we say a coloring c is at *impasse* if $c, \ell(c)$, and $r(c)$ each have a left-hand and right-hand circuit. If a coloring is not at impasse, it can easily be used to obtain a color for the exterior region. We also only use the operation ℓ on colorings having a left-hand circuit, and similarly r is only applied to colorings having a right-hand circuit. Given this terminology, we can explore several systematic approaches to plane graph coloring.

3 Systematic Coloring Algorithms

In practice we color the vertices of maximal planar graphs instead of coloring regions of cubic maps. By principles of plane duality, these are equivalent problems. In each of our coloring algorithms, we begin by obtaining a *smallest-last* ordering of the vertices of a maximal planar graph as described in [7]. This ensures that when coloring a vertex, at most 5 of its neighbors have previously been colored. In terms of regions and region colorings, the smallest-last ordering ensures that when coloring a given region, at most 5 neighboring regions have already been colored. If the region has at most 4 colored neighbors, or otherwise if it has 5 colored neighbors with an arrangement of colors different from that in Figure 1, we can use known methods to color the region; see [3] for an outline of these methods. To color regions with 5 colored neighbors situatated as in Figure 1, we used the following algorithms, also discussed in [13]:

Alogirthm 0: The Basic Rotation Algorithm. Algorithm 0 uses only operations ℓ and t_ℓ. We start with a coloring c_0 as in Figure 1. If the coloring c_i has a left-hand circuit, we let $c_{i+1} = \ell(c_i)$. Otherwise, we apply t_ℓ to c_i and return this coloring, as this removes a color from the boundary regions and allows us to obtain a coloring for the exterior region. We observe that this algorithm is summarized as $c_i = \ell^i(c_0)$. Therefore, this algorithm terminates if and only if $\ell^i(c_0)$ has no left-hand circuit for some i. Algorithm 0 is quite effective; however, there is a historical map and coloring c_0 for which Algorithm 0 does not terminate. This map is the Errera map, introduced by Alfred Errera in [2]. In fact, $\ell^{20}(c_0) = c_0$. The coloring c_0 is illustrated in Figure 2 as presented by Kittell in [6].

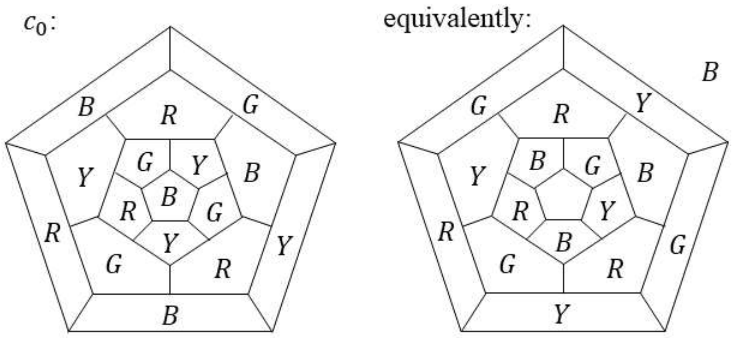

Figure 2: A coloring of the Errera map for which Algorithm 0 does not terminate, with the uncolored region drawn as the exterior region. An equivalent drawing with the uncolored region as the pentagon at the center is also drawn.

With this key example in mind, we will explore several different modifications of Algorithm 0 that avoid the pitfalls presented by the Errera map.

Alogrithm 1: Rotation with Errera Fix. In Algorithm 1, we run Algorithm 0 for a large number of times. If the algorithm does not terminate, or if we return to our original coloring, we implement the additional operations e and $e\ell$. It was shown in [14] that for the Errera map and certain variations that this resolves impasse. There are however maps and colorings for which Algorithm 1 does not resolve impasse; an example is given in Figure 3.

Algorithm 2: Rotation with Multistart. For Algorithm 2, we use the principles of multistart to find a pentagon which is more amenable to coloring by Algorithm 0. Let R_1 be the region we are attempting to color. We use ℓ up to 20 times to color R_1. If this fails, then we apply ℓ enough times so that the top region has at most 4 colored neighbors. We call the top

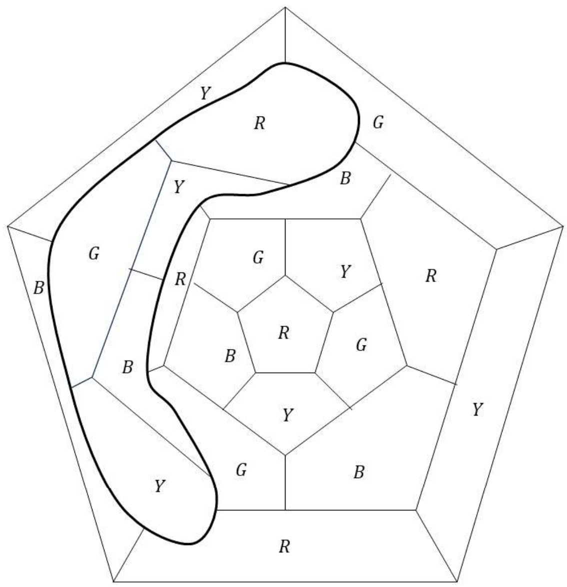

Figure 3: A graph for which Algorithm 1 does not resolve impasse.

region R_2. We assign to R_1 the color of R_2, and then uncolor R_2. Finally, we attempt to color R_2 with up to 20 applications of ℓ. If we either cannot find a neighbor of R_1 that has only four colored neighbors or cannot color R_2 after this exchange, then Algorithm 2 does not succeed. While we did not encounter a graph for which Algorithm 2 failed in our simulations, there are graphs and colorings where this is possible.

In addition to the multistart algorithm described above, we have also designed and implemented a more general multistart algorithm that can change to any pentagon. However, this necessitates obtaining a new smallest-last ordering and restarting the coloring process each time we run into an obstacle, reducing the efficiency of this algorithm.

Algorithm 3: Alternating e and ℓ. This last approach is motivated by a more general consideration of Algorithm 1. In Algorithm 1, we use ℓ many times; if this fails we try either e or $e\ell$, noting that the final colorings in both cases are $e\ell^n(c_0)$ for some n. Algorithm 3 applies color exchanges in order $\{e, \ell, r, e, \ell, e, \ell, r, e, \ell, ...\}$. This systematically tests whether each coloring $\ell^n(c_0)$ and $e\ell^n(c_0)$ is at impasse. When a coloring that is not at impasse is encountered, a coloring of the exterior region is produced. This has not failed to color any of the graphs in our testing suite.

4 Results

Algorithms 0 through 3 have proven effective on many graphs. The following result on Heawood's historical counterexample in [4] to Kempe's color-swapping argument, here referred to as the Heawood map, has been previously proven by Weiguo Xie in [11, 12]:

Theorem 4.1. *The Heawood map can be* 4*-colored by Algorithm* 0*.*

We add to this the following additional result, which was also presented in [13, 15]:

Theorem 4.2. *The Errera map can be* 4*-colored by Algorithms* 1 *through* 3*.*

We encountered in Figure 3 a graph with a coloring such that Algorithm 1 did not succeed. However, it can be colored by other algorithms:

Theorem 4.3. *The graph in Figure 3 can be* 4*-colored by Algorithms* 2 *and* 3*.*

At this point we have tested over $175,000,000$ graphs having up to $34,110$ regions generated using the methods described in [7]. Of these, over 99.998% of graphs were successfully 4-colored by Algorithm 0. Of those for which Algorithm 0 failed, all but 16 graphs were 4-colorable by Algorithm 1. Notably, neither Algorithm 2 nor Algorithm 3 have failed to color a graph in our simulations.

In conclusion, our method of systematic rotation is powerful enough to successfully four-color a large number of plane graphs, and we believe that further insight into the algorithm and the graphs for which certain algorithms fail will provide deeper insight into the coloring of plane graphs.

Data Availability Statements The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Kenneth Appel and Wolfgang Haken. Every planar map is four colorable. *Illinois J. Math*, 21:429–567, 1977.

[2] Alfred Errera. *Du coloriage des cartes et de quelques questions d'ánalysis situs.* PhD thesis, Université Livre de Bruxelles, 1921.

[3] Ellen Gethner, Bopanna Kallichanda, Alexander S Mentis, Sarah Braudrick, Sumeet Chawla, Andrew Clune, Rachel Drummond, Panagiota Evans, William Roche, and Nao Takano. How false is kempe's proof of the four color theorem? part ii. *Involve*, 2(3):249–266, 2009.

[4] Percy John Heawood. Map colour theorem. *Quart J. Pure Appl. Math*, 24:332–338, 1890.

[5] A. B. Kempe. On the geographical problem of the four colours. *American Journal of Mathematics*, 2(3):193–200, 1879.

[6] Irving Kittell. A group of operations on a partially colored map. *Bulletin of the American Mathematical Society*, 41(6):407–413, 1935.

[7] Craig A. Morgenstern and Henry D. Shapiro. Heuristics for rapidly four-coloring large planar graphs. *Algorithmica*, 6(6):869–891, 1991.

[8] Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas. The four-colour theorem. *Journal of Combinatorial Theory, Series B*, 70(1):2–44, 1997.

[9] Robin Wilson. *Four Colors Suffice*. Princeton University Press, Princeton, NJ, revised color edition, 2014.

[10] Weiguo Xie. A novel method to prove the four color theorem. Preprint on Authorea at https://doi.org/10.22541/au.166490983.39190127/v1, October 2022.

[11] Weiguo Xie. A systematic rotation method to color the historic heawood map by four colors. Preprint on Authorea at https://doi.org/10.22541/au.166931559.90016969/v1, November 2022.

[12] Weiguo Xie. To color the historic heawood map with four colors using a systematic rotation method. Presented at 54th Southeastern International Conference on Combinatorics, Graph Theory, & Computing, March 2023.

[13] Weiguo Xie and Andrew Bowling. An effective rotational algorithm for coloring planar graphs. Presented at 1st International Mathematics Conclave, November 2023.

[14] Weiguo Xie and Andrew Bowling. To color the errera map and its variations using four colors. In Christopher Briggs, editor, *Proceedings of the Ninth Annual Exchange of Mathematical Ideas*, pages 13–37, August 2023.

[15] Weiguo Xie and Andrew Bowling. To color the errera map and its variations using four colors: Idea i. Presented at the Ninth Annual Excahnge of Mathematical Ideas, August 2023.