Around the Banach-Tarski paradox and centroids of spherical or hyperbolic triangles

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Abstract

This is a survey of the Banach-Tarski paradox (§1–6) and a résumé of centroids of spherical or hyperbolic triangles (§7).

1 The Banach-Tarski paradox

Let X be a non-empty set and G a group acting on X. Then, for subsets E, $E' \subseteq X$ and an integer $m > 0$, E and E' are said to be *G-equidecomposable with m pieces* (denoted by $E \sim_G^m E'$) if there exist pairwise disjoint subsets $E_0, \ldots, E_{m-1} \subseteq E$, pairwise disjoint subsets $E'_0, \ldots, E'_{m-1} \subseteq E'$, and $h_0, \ldots, h_{m-1} \in G$ such that

$$E = E_0 \sqcup \cdots \sqcup E_{m-1}, \ \ E' = E'_0 \sqcup \cdots \sqcup E'_{m-1}, \ \ h_0(E_0) = E'_0, \ \ \ldots \ \ h_{m-1}(E_{m-1}) = E'_{m-1},$$

where $\sqcup$ means disjoint union. For a subset $E \subseteq X$ and integers k, $\ell > 0$, E is said to be *G-paradoxical with $k + \ell$ pieces* if there exist disjoint subsets A, $B \subseteq E$ such that

$$E = A \sqcup B, \ \ A \sim_G^k E, \ \ B \sim_G^\ell E.$$

For subsets E, $E' \subseteq X$, we denote $E \preceq_G E'$ if there exist a subset $E'_0 \subseteq E'$ and an integer $m > 0$ such that $E \sim_G^m E'_0$. The Banach-Tarski paradox is the following (AC means that the proof requires the axiom of choice. The proof of the existence of a free group does not requires the axiom of choice, but the proof of a paradox requires it, in general (Theorem 10 is an exception):

Theorem 1 (AC, [BT24])**.** *Let U, $V \subseteq \mathbb{R}^3$ be bounded sets with non-empty interior. Then there exists an integer $m > 0$ such that $U \sim_{SG_3(\mathbb{R})}^m V$, where $SG_n(\mathbb{R})$ is the group of all orientation-preserving isometries on $\mathbb{R}^n$.*

Sketch of proof. It is enough to prove $\mathbb{D}^3 \sqcup (\mathbb{D}^3 + \mathbf{a}) \preceq_{SG_3(\mathbb{R})} \mathbb{D}^3$ for some $\mathbf{a} \in \mathbb{R}^3$ with $\|\mathbf{a}\| > 2$. To prove it, it is enough to prove that $\mathbb{D}^3$ is $SG_3(\mathbb{R})$-paradoxical with $k + \ell$ pieces for some k, $\ell > 0$. To prove it, it is enough to prove that $\mathbb{S}^2$ is $SO_3(\mathbb{R})$-paradoxical with $k' + \ell'$ pieces for some k', $\ell' > 0$, where $SO_n(\mathbb{R})$ is the group of all $n \times n$ orthogonal matrices with determinant $+1$. To prove it, it is enough to prove that $\mathbb{S}^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical with $k'' + \ell''$ pieces for some k'', $\ell'' > 0$ and a countable subset $D \subseteq \mathbb{S}^2$. The subgroup F of $SO_3(\mathbb{R})$ generated by

$$f = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix} \text{ and } g = \frac{1}{5}\begin{pmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

is a free group, because of

$$5^{k_0} f^{\delta_0 k_0} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & \delta_0 \\ 0 & -\delta_0 & -2 \end{pmatrix}, \quad 5^{\ell_0} g^{\varepsilon_0 \ell_0} \equiv \begin{pmatrix} -2 & \varepsilon_0 & 0 \\ -\varepsilon_0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 5^{\ell_0 + k_0} g^{\varepsilon_0 \ell_0} f^{\delta_0 k_0} \equiv \begin{pmatrix} 0 & -2\varepsilon_0 & \varepsilon_0 \delta_0 \\ 0 & -1 & -2\delta_0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$5^{\ell_{m-1} + k_{m-1} + \cdots + \ell_0 + k_0} g^{\varepsilon_{m-1} \ell_{m-1}} f^{\delta_{m-1} k_{m-1}} \cdots g^{\varepsilon_0 \ell_0} f^{\delta_0 k_0} \equiv (-1)^{m-1} \begin{pmatrix} 0 & -2\varepsilon_{m-1} & \varepsilon_{m-1} \delta_0 \\ 0 & -1 & -2\delta_0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $m > 0$; k_i, $\ell_i > 0$; δ_i, $\varepsilon_i \in \{-1, 1\} \pmod 5$. F is F-paradoxical with $2+2$ pieces, where F acts on itself by left translation, because of

$$\begin{aligned} A \underset{\text{def}}{=} & (W_f \sqcup \{\mathbf{E}_3, f^{-1}, f^{-2}, \ldots\}) \sqcup \quad (W_{f^{-1}} \setminus \{f^{-1}, f^{-2}, \ldots\}) \sim_F^2 \\ \sim_F^2 & (W_f \sqcup \{\mathbf{E}_3, f^{-1}, f^{-2}, \ldots\}) \sqcup f \cdot (W_{f^{-1}} \setminus \{f^{-1}, f^{-2}, \ldots\}) = \quad F \end{aligned}$$

and

$$B \underset{\text{def}}{=} W_g \sqcup W_{g^{-1}} \sim_F^2 W_g \sqcup g \cdot W_{g^{-1}} = F,$$

where W_f (resp. $W_{f^{-1}}$, W_g, $W_{g^{-1}}$) is the set of reduced words whose leftmost is f (resp. f^{-1}, g, g^{-1}). The set $D = \{\mathbf{x} \in \mathbb{S}^2 : {}^{\exists} w \in F \setminus \{\mathbf{E}_3\}, w(\mathbf{x}) = \mathbf{x}\}$ is countable. Let $\tilde{M}$ be a choice set of the equivalence classes $(\mathbb{S}^2 \setminus D)/\sim_F^1$. Then $\mathbb{S}^2 \setminus D$ is F-paradoxical with $2 + 2$ pieces, because $A(\tilde{M}) \sim_F^2 F(\tilde{M}) = \mathbb{S}^2 \setminus D$ and $B(\tilde{M}) \sim_F^2 \mathbb{S}^2 \setminus D$. □

2 Without fixed points and local commutativity

Let H be a group acting on a non-empty set X.

Theorem 2 (AC, [Ada54, Dek56a])**.** *Then we have*

$$\text{(a)} \underset{\text{Adams}}{\overset{\text{(AC)}}{\not\rightleftarrows}} \text{(b)} \not\rightleftarrows \text{(c)} \underset{\substack{\text{Dekker} \\ \text{(AC)}}}{\rightleftarrows} \text{(d)}$$

for

(a) *H acts on X* without fixed points, *i.e., all non-identical elements of H have no fixed point in X,*
(b) *X has a H-*Hausdorff decomposition, *i.e., there exist pairwise disjoint sets A, B, $C \subseteq X$ such that*

$$X = A \sqcup B \sqcup C \text{ and } A \sim^1_H B \sim^1_H C \sim^1_H B \sqcup C \sim^1_H C \sqcup A \sim^1_H A \sqcup B,$$

(c) *X is H-paradoxical with $2+2$ pieces,*
(d) *the action of H on X is* locally commutative, *i.e., if two elements of H have a common fixed point in X then they are commutative.*

For $X = \mathbb{S}^{n-1}$ and $G = SO_n(\mathbb{R})$, does there exist a free subgroup $F \leq G$ of rank 2 acting on X without fixed points? It holds for neither $n \leq 2$ nor odd n because $SO_n(\mathbb{R})$ are commutative for $n \leq 2$ and

$$\begin{aligned}\det(g - \mathbf{E}_n) &= \det g \det(\mathbf{E}_n - g^{-1}) = 1 \cdot \det(\mathbf{E}_n - g^T) = \\ &= \det(\mathbf{E}_n - g) = (-1)^n \det(g - \mathbf{E}_n) = -\det(g - \mathbf{E}_n) \quad ({}^\forall g \in SO_n(\mathbb{R}))\end{aligned}$$

for odd n. So we have to be satisfied with the following theorem.

Theorem 3 ([Dek56b] for $n = 4$; [Bor83] for $n = 5$, $n = 6$, and yet another proof of $n = 6$; [DS83] for another proof of $n = 6$)**.** *For even $n \geq 4$, $SO_n(\mathbb{R})$ has a free subgroup acting on $\mathbb{S}^{n-1}$ without fixed points. For odd $n \geq 3$, $SO_n(\mathbb{R})$ has a free subgroup acting on $\mathbb{S}^{n-1}$ locally commutatively.*

Sketch of proof. It is enough to prove them for $3 \leq n \leq 6$, because if the group generated by f and g and by f' and g' are the groups for even n and even (resp. odd) n' respectively then the group generated by $f \oplus f'$ and $g \oplus g'$ is the group for even (resp. odd) $n + n'$. For $n = 3$, the group of the proof of Theorem 1 is valid. For $n = 4$, the group generated by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \text{ and } \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}$$

is valid, where $\cos\theta$ is transcendental. The proof is difficult for $n = 6$. For each non-empty reduced word w of letters f, g , f^{-1}, g^{-1}, the map

$$\begin{aligned} w : SO_6(\mathbb{R}) \times SO_6(\mathbb{R}) &\to SO_6(\mathbb{R}) \\ (\text{ex. } g^{-1}f^2gf : \quad (\sigma, \tau) &\mapsto \tau^{-1}\sigma^2\tau\sigma \ ) \end{aligned}$$

is dominant (i.e., the image of w is not included in any proper subvariety of $SO_6(\mathbb{R})$) because $SO_6(\mathbb{R})$ is semi-simple (i.e., it does not have a non-trivial connected commutative normal subgroup). So the composition $d \circ w : SO_6(\mathbb{R}) \times SO_6(\mathbb{R}) \to \mathbb{R}$ of dominant maps w and $d : SO_6(\mathbb{R}) \ni \mathbf{A} \mapsto \det(\mathbf{A} - \mathbf{E}_6) \in \mathbb{R}$ is also dominant. Hence

$(d \circ w)^{-1}(0) \subseteq SO_6(\mathbb{R}) \times SO_6(\mathbb{R})$ is a proper subvariety. Baire category theorem implies

$$\bigcup_{w}^{\substack{\text{non-empty} \\ \text{reduced}}} (d \circ w)^{-1}(0) \subsetneq SO_6(\mathbb{R}) \times SO_6(\mathbb{R}),$$

so we can fix $(\sigma, \tau) \in SO_6(\mathbb{R}) \times SO_6(\mathbb{R}) \setminus \bigcup_{w}^{\substack{\text{non-empty} \\ \text{reduced}}} (d \circ w)^{-1}(0)$. Then the group generated by σ and τ acts on $\mathbb{S}^5$ without fixed points (another proof: $\mathbb{Q}(\omega)[X, Y]/(X^3 - 7, Y^3 - 2, XY - \omega YX)$ is a division algebra where $\omega = e^{2\pi\sqrt{-1}/3}$, which also proves it for $n = 6$). The proof is also difficult for $n = 5$. Each non-trivial compact connected semi-simple Lie group G has a free subgroup F such that for closed subgroup $H \lneq G$ and non-empty reduced word $w \in F$,

$$w : G/H \ni gH \mapsto wgH \in G/H$$

has exactly $\chi(G/H)$ fixed pts. For $G = SO_5(\mathbb{R})$, $H = SO_4(\mathbb{R})$, we can regard $G/H = \mathbb{S}^4$ and $\chi(G/H) = 2$ (for $G = SO_6(\mathbb{R})$ and $H = SO_5(\mathbb{R})$, we can regard $G/H = \mathbb{S}^5$ and $\chi(G/H) = 0$, which is yet another proof for $n = 6$). □

3 On rational spheres

We consider groups of matrices with rational entries acting on rational spheres, because we expect stronger results and the existences of free groups imply paradoxes without the axiom of choice from the countability of rational spaces.

Problem 1. *For even $n \geq 4$, does $SO_n(\mathbb{Q})$ have a free subgroup acting on $\mathbb{S}^{n-1}$ without fixed points? For odd $n \geq 3$ and positive $q \in \mathbb{Q}$, does $SO_n(\mathbb{Q})$ have a free subgroup acting on $\mathbb{S}^{n-1}$ locally commutatively and acting on $(\sqrt{q}\mathbb{S}^{n-1}) \cap \mathbb{Q}^n$ without fixed points?*

It is also enough to prove them for $3 \leq n \leq 6$. We have partial answers.

Theorem 4 ([Sat95, Sat97, Sat98, Sat02])**.** *There exist free groups for the cases "yes" of the figure below.*

	$\sqrt{q} \in \mathbb{Q}$	$\sqrt{q} \notin \mathbb{Q}$
$n = 3$	*yes (1995)*	*yes (1998)*
$n = 4$	*yes (1997)*	
$n = 5$	*not yet*	*yes (2002)*
$n = 6$	*not yet*	

Sketch of proof. The groups generated by the following pairs are valid: For $n = 3$ and q with $\sqrt{q} \in \mathbb{Q}$,

$$\frac{1}{7}\begin{pmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{pmatrix} \text{ and } \frac{1}{7}\begin{pmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{pmatrix},$$

for $n=3$ and q with $\sqrt{q}\notin\mathbb{Q}$,

$$\frac{1}{1+b^2}\begin{pmatrix} 1+b^2 & 0 & 0 \\ 0 & 1-b^2 & -2b \\ 0 & 2b & 1-b^2 \end{pmatrix} \text{ and } \frac{1}{1+b^2}\begin{pmatrix} 1-b^2 & -2b & 0 \\ 2b & 1-b^2 & 0 \\ 0 & 0 & 1+b^2 \end{pmatrix},$$

for $n=4$,

$$\frac{1}{7}\begin{pmatrix} 2 & -6 & -3 & 0 \\ 6 & 2 & 0 & 3 \\ 3 & 0 & 2 & -6 \\ 0 & -3 & 6 & 2 \end{pmatrix} \text{ and } \frac{1}{7}\begin{pmatrix} 2 & 0 & -3 & -6 \\ 0 & 2 & -6 & 3 \\ 3 & 6 & 2 & 0 \\ 6 & -3 & 0 & 2 \end{pmatrix},$$

for $n=5$ and q with $\sqrt{q}\notin\mathbb{Q}$,

$$\frac{1}{1+b^2}\begin{pmatrix} 1+b^2 & 0 & 0 & 0 & 0 \\ 0 & 1-b^2 & -2b & 0 & 0 \\ 0 & 2b & 1-b^2 & 0 & 0 \\ 0 & 0 & 0 & 1-b^2 & -2b \\ 0 & 0 & 0 & 2b & 1-b^2 \end{pmatrix} \text{ and } \frac{1}{1+b^2}\begin{pmatrix} 1-b^2 & -2b & 0 & 0 & 0 \\ 2b & 1-b^2 & 0 & 0 & 0 \\ 0 & 0 & 1-b^2 & -2b & 0 \\ 0 & 0 & 2b & 1-b^2 & 0 \\ 0 & 0 & 0 & 0 & 1+b^2 \end{pmatrix},$$

where q can be square-free integer and b (with prime p) is s.t. $\left(\frac{q}{p}\right)=-1$, $p \mid 1+b^2$ for the case of $\sqrt{q}\notin\mathbb{Q}$. For

$$\phi = \begin{pmatrix} \phi_0^0 & \cdots & \phi_{n-1}^0 \\ \vdots & \ddots & \vdots \\ \phi_0^{n-1} & \cdots & \phi_{n-1}^{n-1} \end{pmatrix} \in SO_n(\mathbb{R}),$$

the vector

$$\mathbf{ax}(\phi) = \frac{1}{2^{\frac{n-1}{2}}(\frac{n-1}{2})!}\begin{pmatrix} \sum_{\mathfrak{s}\in\mathfrak{S}_{n-1}} \operatorname{sgn}\mathfrak{s} \prod_{r=0}^{\frac{n-3}{2}} (\phi_{(1+\mathfrak{s}(2r))\bmod n}^{(1+\mathfrak{s}(2r+1))\bmod n} - \phi_{(1+\mathfrak{s}(2r+1))\bmod n}^{(1+\mathfrak{s}(2r))\bmod n}) \\ \vdots \\ \sum_{\mathfrak{s}\in\mathfrak{S}_{n-1}} \operatorname{sgn}\mathfrak{s} \prod_{r=0}^{\frac{n-3}{2}} (\phi_{(n+\mathfrak{s}(2r))\bmod n}^{(n+\mathfrak{s}(2r+1))\bmod n} - \phi_{(n+\mathfrak{s}(2r+1))\bmod n}^{(n+\mathfrak{s}(2r))\bmod n}) \end{pmatrix} \in \mathbb{R}^n$$

is the axis of ϕ, in general, where n is odd. For example, for

$$\phi = \begin{pmatrix} \phi_0^0 & \phi_1^0 & \phi_2^0 \\ \phi_0^1 & \phi_1^1 & \phi_2^1 \\ \phi_0^2 & \phi_1^2 & \phi_2^2 \end{pmatrix} \in SO_3(\mathbb{R}),$$

the vector

$$\mathbf{ax}(\phi) = \begin{pmatrix} \phi_1^2-\phi_2^1 \\ \phi_2^0-\phi_0^2 \\ \phi_0^1-\phi_1^0 \end{pmatrix}$$

is the axis of ϕ, in general, and, for

$$\phi = \begin{pmatrix} \phi_0^0 & \phi_1^0 & \phi_2^0 & \phi_3^0 & \phi_4^0 \\ \phi_0^1 & \phi_1^1 & \phi_2^1 & \phi_3^1 & \phi_4^1 \\ \phi_0^2 & \phi_1^2 & \phi_2^2 & \phi_3^2 & \phi_4^2 \\ \phi_0^3 & \phi_1^3 & \phi_2^3 & \phi_3^3 & \phi_4^3 \\ \phi_0^4 & \phi_1^4 & \phi_2^4 & \phi_3^4 & \phi_4^4 \end{pmatrix} \in SO_5(\mathbb{R}),$$

the vector

$$\mathbf{ax}(\phi) = \begin{pmatrix} (\phi_1^2-\phi_2^1)(\phi_3^4-\phi_4^3)-(\phi_1^3-\phi_3^1)(\phi_2^4-\phi_4^2)+(\phi_1^4-\phi_4^1)(\phi_2^3-\phi_3^2) \\ (\phi_2^3-\phi_3^2)(\phi_4^0-\phi_0^4)-(\phi_2^4-\phi_4^2)(\phi_3^0-\phi_0^3)+(\phi_2^0-\phi_0^2)(\phi_3^4-\phi_4^3) \\ (\phi_3^4-\phi_4^3)(\phi_0^1-\phi_1^0)-(\phi_3^0-\phi_0^3)(\phi_4^1-\phi_1^4)+(\phi_3^1-\phi_1^3)(\phi_4^0-\phi_0^4) \\ (\phi_4^0-\phi_0^4)(\phi_1^2-\phi_2^1)-(\phi_4^1-\phi_1^4)(\phi_0^2-\phi_2^0)+(\phi_4^2-\phi_2^4)(\phi_0^1-\phi_1^0) \\ (\phi_0^1-\phi_1^0)(\phi_2^3-\phi_3^2)-(\phi_0^2-\phi_2^0)(\phi_1^3-\phi_3^1)+(\phi_0^3-\phi_3^0)(\phi_1^2-\phi_2^1) \end{pmatrix}$$

is the axis of ϕ, in general. They are useful to prove the theorem. □

4 Low dimensional case

The Banach-Tarski paradox does not hold for $n \leq 2$, because of the following theorem (see after Theorem 13, precisely).

Theorem 5. *$G_2(\mathbb{R})$ does not have a free subgroup of rank 2, where $G_n(\mathbb{R})$ is the group of all isometries on $\mathbb{R}^n$.*

Proof. For arbitrary f, $g \in G_2(\mathbb{R})$, their squares f^2, g^2 are in $SG_2(\mathbb{R})$ and their commutators $g^{-2}f^{-2}g^2f^2$ and $g^{-4}f^{-4}g^4f^4$ are translations, so their commutator

$$(g^{-4}f^{-4}g^4f^4)^{-1} \cdot (g^{-2}f^{-2}g^2f^2)^{-1} \cdot g^{-4}f^{-4}g^4f^4 \cdot g^{-2}f^{-2}g^2f^2$$

is the identity. □

But, by expanding the group, we have a new paradox.

Theorem 6 (AC, [Neu29])**.** *Let U, $V \subseteq \mathbb{R}^2$ be bounded sets with non-empty interior. Then there exists an integer $m > 0$ such that $U \sim^m_{SA_2(\mathbb{R})} V$, where $SA_n(\mathbb{R})$ is the group of all orientation and area-preserving affine transformations on $\mathbb{R}^n$.*

Sketch of proof. It is essentially enough to find a free subgroup of $SA_2(\mathbb{R})$. For example, the group generated by

$$\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \quad (\in SL_2(\mathbb{R}) \subseteq SA_2(\mathbb{R})),$$

where $SL_n(\mathbb{R})$ is the group of all $n \times n$ matrices with determinant $+1$. □

The group $SA_2(\mathbb{R})$ satisfies the following theorems.

Theorem 7. *$SA_2(\mathbb{R})$ has no free subgroup acting on $\mathbb{R}^2$ without fixed points.*

Proof. Assume that the group generated by f, $g \in SA_2(\mathbb{R})$ acts on $\mathbb{R}^2$ without fixed points. Then neither f, g, nor gf has a fixed point, so $\operatorname{tr} f = \operatorname{tr} f = \operatorname{tr} gf = 2$, hence matrix parts of f and g commute, so the matrix part of $g^{-1}f^{-1}gf$ is identity, hence $g^{-1}f^{-1}gf$ is a translation. In a similar fashion, $g^{-2}f^{-2}g^2f^2$ is also a translation. So,

$$(g^{-2}f^{-2}g^2f^2)^{-1} \cdot (g^{-1}f^{-1}gf)^{-1} \cdot g^{-2}f^{-2}g^2f^2 \cdot g^{-1}f^{-1}gf$$

is the identity. □

Theorem 8 ([Sat03]). *$SA_2(\mathbb{R})$ has a free subgroup acting on $\mathbb{R}^2$ locally commutatively.*

Sketch of proof. The group generated by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & \theta \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ \theta & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

is the group as desired, where θ is transcendental. □

There exist paradoxes on the line $\mathbb{R}^1$.

5 Some topics

There exists a paradox on the hyperbolic space.

Theorem 9 (AC, [Myc89, MT13]). *Mycielski barrel, the set of points whose distances from real and imaginary axes of Poincaré disc $\mathbb{H}^2 = \{z \in \mathbb{C} : |z| < 1\}$ are less than $\operatorname{arsinh}(1/\sqrt{\frac{1}{\alpha^2} - 2})$ including left and lower boundaries, is (isometries of $\mathbb{H}^2$)-paradoxical with $k + \ell$ pieces for some k, $\ell > 0$, where $0 < \alpha < \sqrt{2\sqrt{3} - 3}$.*

The following paradox does not depend on the axiom of choice.

Theorem 10 ([DF94]). *Let X be a Polish space, F a free group generated by homeomorphisms $f : X \to X$ and $g : X \to X$ such that F acts on a comeager subset $X_0 \subseteq X$ without fixed points. Then, there exist pairwise disjoint regular open subsets $A_0, \ldots, A_5 \subseteq X$, pairwise disjoint regular open subsets A'_0, A'_1, $A'_2 \subseteq X$, pairwise disjoint regular open subsets A'_3, A'_4, $A'_5 \subseteq X$ such that*

$$X = A_0 \vee \cdots \vee A_5, \ \ X = A'_0 \vee A'_1 \vee A'_2, \ \ X = A'_3 \vee A'_4 \vee A'_5, \ \ A_i \sim^1_F A'_i \ \ (0 \leq i \leq 5),$$

where $A \vee \cdots \vee B$ is the interior of the closure of the union $A \cup \cdots \cup B$.

In the classic Banach–Tarski paradox, the pieces could be moved continuously so that they remained disjoint throughout the transformation.

Theorem 11 (AC, [Wil05]). *Let U, $V \subseteq \mathbb{R}^3$ be bounded sets with non-empty interior. Then, there exist an integer $m > 0$, pairwise disjoint subsets $U_0, \ldots, U_{m-1} \subseteq U$, pairwise disjoint subsets $V_0, \ldots, V_{m-1} \subseteq V$, and continuous maps $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{m-1} : [0,1] \to SG_3(\mathbb{R})$ such that*

$$U = U_0 \sqcup \cdots \sqcup U_{m-1}, \ \ V = V_0 \sqcup \cdots \sqcup V_{m-1},$$

$$\tilde{\alpha}_i(0) = \mathrm{id}_{\mathbb{R}^3}, \ \ \tilde{\alpha}_i(1)(U_i) = V_i \ \ \text{for } 0 \leq i < m,$$

$$\tilde{\alpha}_i(t)(U_i) \cap \tilde{\alpha}_i(t)(U_j) = \emptyset \ \ \text{for } 0 \leq t \leq 1 \ \text{ and } 0 \leq i < j < m.$$

6 Non-existence of paradoxes

Until the previous section, we have considered a lot of paradoxical cases. On the contrary, in this section, we will consider a non-paradoxical case. A group G is said to be *amenable* if there exists $\mu : 2^G \to [0,1]$ such that $\mu(gA) = \mu(A)$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$ for disjoint subsets A, $B \subseteq G$ and $g \in G$ and $\mu(G) = 1$. The amenability means the existence of a left-invariant measure, which is not necessarily a right-invariant measure, but we can construct a left and right-invariant measure from it. We have the following theorem for an amenable group.

Theorem 12 (AC, [Ban23]). *If G is an amenable group of isometries of $\mathbb{R}^n$ (resp. $\mathbb{S}^{n-1}$), then there exists a finitely additive and G-invariant extension of Lebesgue measure λ to all subsets of $\mathbb{R}^n$ (resp. $\mathbb{S}^{n-1}$).*

The theorem above implies the following.

Theorem 13 (AC). *Lebesgue measure on $\mathbb{R}^2$ has an isometry-invariant and finitely additive extension to all subsets.*

Sketch of proof. $G_2(\mathbb{R})$ is solvable, so it is amenable. So we can apply Theorem 12. □

We used the axiom of choice to get Theorem 1, the Banach-Tarski paradox for $SG_3(\mathbb{R})$ and $\mathbb{R}^3$, but we also use the axiom of choice to get Theorem 13, which implies the impossibility of the Banach-Tarski paradox for $SG_2(\mathbb{R})$ and $\mathbb{R}^2$. In [TW16], 'Taking these facts and theorems into account, Mycielski [Myc06, MT∞b] suggested that the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + V{=}L(R)$ is a reasonable axiomatization of set theory (and hence mathematics) for use in science. Severely counterintuitive results such as the Banach-Tarski Paradox are false in this theory because all sets are Lebesgue measurable; there is enough choice so that Lebesgue measure works as expected; the universe does not contain any "unnecessary sets": It contains only the sets that must exist given that all reals and all ordinals are present; and all sets have the Property of Baire', where DC is the axiom of dependent choice and AD is the axiom of determinacy.

7 Centroids of spherical or hyperbolic triangles

In this section, we will discuss the current research of the author. The centroid of Euclidean triangle satisfies many properties:

(a) it is the concurrent point of 3 medians,
(a') it is the minimizer of the sum of (distances to vertices)^2,
(b) it is the gravity center,
(c) it is the area-trisecting point by connecting it to 3 vertices,
(d) it is the concurrent point of 3 pseudo-medians,

where a median is a line passing through a vertex and bisecting the opposite edge and a pseudo-median is a line passing through a vertex and bisecting the area. For a spherical (resp. hyperbolic) triangle, (a) = (a") but points (a), (b) [Sat2023, Sat2024a], (c) [Sat2024b], and (d) [Ako2009, Cas1889, Hor2015, Sat2024b] are pairwise disjoint, in general, where (c) is the same as above, in (a) and (d) the word "line" is replaced with "geodesic" in definitions of "median" and "pseudo-median", and

(a") the minimizer of the sum of $1 - \cos$(distances to vertices) (resp. $\cosh$(distances to vertices) $- 1$),
(b) the normalized (resp. pseudo-normalized) point of the gravity center of the triangle in Euclidean space $\mathbb{R}^3$ (resp. Minkowski space $\mathbb{R}^2 \times \mathbb{R}$).

Centroids (a) = (a") and (b) can be considered in higher dimensional simplex.

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