Towards relations between Bloch–Kato Selmer groups and chromatic Selmer groups

Florian Ito Sprung

June 28, 2024

Abstract

This paper explores Iwasawa theory and its role in the Birch and Swinnerton-Dyer conjecture, and the generalizations to higher weight modular forms, arriving at connections between Bloch–Kato Selmer groups and chromatic Selmer groups of the modular forms.

1 Review of Iwasawa Theory

Iwasawa Theory is a mysterious bridge between two mathematically faraway worlds, the analytic realm and the algebraic realm:

$$(analytic) \quad \xleftrightarrow{\text{Iwasawa Theory}} \quad (algebraic)$$

For the rest of this article, let p be an odd prime. Iwasawa looked at the following towers of number fields:

After adjoining successively large p-power roots of unity, we obtain a tower of extensions whose union is $\mathbb{Q}(\mu_{p^\infty})$ so that $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times \cong \mathbb{Z}_p \times \Delta$

$$\begin{tikzcd}
\mathbb{Q}(\mu_{p^\infty}) = \bigcup_n \mathbb{Q}(\mu_{p^n}) \arrow[d, dash, dotted] \arrow[dr, dash, "\cong\Delta"] \arrow[ddd, dash, bend right, "\mathbb{Z}_p"'] & \\
\mathbb{Q}(\mu_{p^{n+1}}) \arrow[d, dash, dotted] \arrow[dr, dash, "\cong\Delta"] & \mathbb{Q}_\infty \arrow[d, dash, dotted] \arrow[ddd, dash, bend left, "\cong\mathbb{Z}_p"] \\
\mathbb{Q}(\mu_{p^2}) \arrow[d, dash] \arrow[dr, dash, "\cong\Delta"] & \mathbb{Q}_n \arrow[d, dash, dotted] \\
\mathbb{Q}(\mu_p) \arrow[dr, dash, "\Delta"] & \mathbb{Q}_1 \arrow[d, dash] \\
& \mathbb{Q} = \mathbb{Q}_0
\end{tikzcd}$$

By basic Galois theory, we can fix these fields by Galois groups isomorphic to $\Delta \cong \mathbb{Z}/(p-1)\mathbb{Z}$. We then get the tower of number fields on the right. This tower is called the *cyclotomic $\mathbb{Z}_p$-extension.*

Given a $\mathbb{Z}$-module $\mathcal{X}$, its p-primary part $\mathcal{X}[p^\infty]$ is a $\mathbb{Z}_p$-module. For simplicity, let's suppose that $\mathcal{X} = \mathcal{X}[p^\infty]$. If the Galois group $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$ acts continuously on $\mathcal{X}$, then $\mathcal{X}$ becomes a $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$-module. This ring Λ is called the *Iwasawa algebra* and is also a power series ring

$\Lambda \cong \mathbb{Z}_p[[T]]$, and thus is a ring of (special) p-adically continuous functions. (This observation is due to Serre [Ser95].)

When $\mathcal{X}$ is a finitely generated torsion Λ-module, a nice theory has been developed. The toy version of this theory comes about when replacing the ring Λ by $\mathbb{Z}$: Recall that a finitely generated torsion $\mathbb{Z}$-module G, i.e. a finite abelian group, admits an exact sequence

$$0 \to \bigoplus_i \mathbb{Z}/p_i^{e_i}\mathbb{Z} \to G \to 0.$$

The most important invariant of G is its size $|G|$. Note that the ideal in $\mathbb{Z}$ generated by $|G|$ encodes this information as well. We call it the *characteristic ideal*: $(|G|) = (\prod_i p_i^{e_i}) \subset \mathbb{Z}$.

Now suppose $\mathcal{X}$ is a finitely generated torsion Λ-module. It turns out that $\mathcal{X}$ then admits an exact sequence

$$0 \to \bigoplus_i \Lambda/f_i\Lambda \to \mathcal{X} \to (finite) \to 0,$$

where we have chosen f_i so that $f_i | f_{i+1}$. These f_i are not uniquely determined, but the ideal that their product generates in Λ is. This is our characteristic ideal:

$$(g_{\mathcal{X}}) := (\prod_i f_i) \subset \Lambda.$$

Elements of the Iwasawa algebra also have two canonical invariants:

The p-adic Weierstrass Preparation Theorem states that for $g(T) \in \Lambda$, there are (uniquely determined) non-negative integers μ, λ so that

$$g(T) = p^{\mu}(T^{\lambda} + a_1 T^{\lambda-1} + \cdots + a_{\lambda})U(T),$$

where $a_i \in p\mathbb{Z}_p$, and $U(T) \in \Lambda^{\times}$ is a unit.

For a finitely generated torsion Λ-module $\mathcal{X}$, the integers μ and λ of the generator of $g_{\mathcal{X}}$ as above are called the *Iwasawa invariants* of $\mathcal{X}$.

2 Applications toward the Birch and Swinnerton-Dyer conjecture

Classical Iwasawa theory packages together into a finitely generated torsion Λ-module $\mathcal{X}$ the p-primary part of the class group of $\mathbb{Q}_n$, whose size is controlled by the Iwasawa invariants of $\mathcal{X}$ when n is sufficiently large. (More precisely, the formula says that letting $p^{e_n} := \#\mathrm{Cl}(\mathbb{Q}_n)[p^{\infty}]$, we have

$$e_n - e_{n-1} = \mu(p^n - p^{n-1}) + \lambda$$

for $n \gg 0$, [Iwa58].)

We would like to look at a different $\mathcal{X}$ that knows an analogous object in the context of an elliptic curve E over $\mathbb{Q}$ with a weight two modular form $f = \sum_n a_n q^n$ of level $\Gamma_0(N)$ attached to it. The definition of the Selmer group $\mathrm{Sel}(E/\mathbb{Q})$ involves local points of the elliptic curve at the various completions of $\mathbb{Q}$. Just like the class group, we can consider various Selmer groups $\mathrm{Sel}(E/\mathbb{Q}_n)$ along the cyclotomic tower, and package their p-primary parts into one object:

$$\mathcal{X} := \varprojlim \operatorname{Hom}(\operatorname{Sel}(E/\mathbb{Q}_n)[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p)$$

We assume that p is a prime of good reduction, i.e. the elliptic curve remains an elliptic curve modulo p, or equivalently, p does not divide N. We also assume that $p \neq 2$.

Definition 2.1. *A good prime p is called ordinary if $\operatorname{ord}_p(a_p) = 0$, and non-ordinary if $\operatorname{ord}_p(a_p) > 0$.*

Theorem 2.2. *[Kato4] Let $p \neq 2$ be good ordinary. Then under mild assumptions[1], $\mathcal{X}$ is finitely generated torsion.*

In particular, this guarantees that we can define a characteristic power series $g_{\mathcal{X}}(T)$. The behavior of this power series at $T = 0$ should know the $\mathbb{Q}$-rational points of E. In fact, it should know more than just the rational points, the Selmer group. The following used to be well-known to experts, before it was written down by Ralph Greenberg in [Gre99, Theorem 4.1]:

Theorem 2.3 (folkore, written down in [Gre99])**.** *Let p be good ordinary. If $\mathcal{X}$ is finitely generated torsion and $\operatorname{Sel}(E/\mathbb{Q})[p^\infty]$ is finite, then*

$$g_{\mathcal{X}}(0) \sim |\operatorname{Sel}(E/\mathbb{Q})[p^\infty]|_p \times |\textit{algebraic invariants}|_p,$$

where the symbol $\sim$ means 'up to a p-adic unit.'

The algebraic invariants appear mostly in the Birch and Swinnerton-Dyer formula, which is the conjectured expression of the leading Taylor coefficient of the appropriately normalized Hasse–Weil L-function in the full Birch and Swinnerton-Dyer conjecture:

Conjecture 2.4. *Let E be an elliptic curve over $\mathbb{Q}$. Let r be the Mordell–Weil rank of E, i.e. the integer r so that $E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus$ (finite), and let r_{an} be the order of vanishing of the Hasse–Weil L-function $L(E,s)$ at $s = 1$. Then:*

1. $r = r_{an}$, *and*
2. $\frac{L^{(r_{an})}(E,1)}{\operatorname{Reg}(E/\mathbb{Q}) \times \Omega} = \frac{\#Ш(E/\mathbb{Q}) \prod_l c_l}{\#E(\mathbb{Q})^2_{tors}}$

Here, Ω is the Néron period, $\operatorname{Reg}(E/\mathbb{Q})$ the regulator of the elliptic curve, $Ш(E/\mathbb{Q})$ is the Shafarevich–Tate group (conjectured to be finite), c_l denote the Tamagawa numbers, and $E(\mathbb{Q})_{tors}$ denotes the torsion points of the elliptic curve with coordinates in $\mathbb{Q}$. The expression on the right side in the second statement of the conjecture is called the Birch and Swinnerton-Dyer formula ("BSD-formula"), and in the setting of Theorem 2.3, it is known that $Ш(E/\mathbb{Q})[p^\infty] = \operatorname{Sel}(E/\mathbb{Q})[p^\infty]$.

On the analytic side, we can package the special values $L(E,1,\chi_{p^n})$ into a p-adic L-function L_p, normalized so that $L_p(0) \sim \left|\frac{L(E,1)}{\Omega}\right|_p$.

Conjecture 2.5 (Main Conjecture)**.** *We have $(g_{\mathcal{X}}) = (L_p)$ as ideals in Λ.*

The conjecture is largely known in the ordinary case, i.e. when $\operatorname{ord}_p(a_p) = 0$ [Kato4, SU14].

[1]The assumption is that the p-adic representation of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the automorphism group of the p-adic Tate module is surjective.

Corollary 2.6 (p-part of Birch and Swinnerton-Dyer formula)**.** *Assume that p is good ordinary, i.e.* $\mathrm{ord}_p(a_p) = 0$ *and that If $L(E,1) \neq 0$. Then under mild hypotheses*[2]*,*

$$|\textit{BSD-formula}|_p \sim g_{\mathcal{X}}(0) \sim L_p(0) \sim \left|\frac{L(E,1)}{\Omega}\right|_p .$$

If $\mathrm{ord}_p(a_p) > 0$, we are in the non-ordinary case. In this case, $\mathcal{X}$ is not torsion as a Λ-module, so the above strategy fails. However, we have the following:

Remedy 2.7. *[Kob03, Spr12] One can construct two analogues of $\mathcal{X}$, the chromatic (or signed) Selmer group duals $\mathcal{X}^\sharp$ and $\mathcal{X}^\flat$, and at least one of them is cotorsion.*

The idea behind the construction of these modified ('chromatic' or 'signed') Selmer groups is to cut the local condition at p in the definition of Selmer group in half, i.e. we don't include all rational points of the elliptic curve in the completion of $\mathbb{Q}$ and $\mathbb{Q}_n$ at p, but half of them. There are two natural ways to divide the rational points in half, the $\sharp$ points and the $\flat$ points.

These $\mathcal{X}^\sharp$ and $\mathcal{X}^\flat$ can be used to now formulate two main conjectures. The main conjectures involve corresponding analytic objects: We can associate two analogues of the above L_p, but these analogues have bad growth patterns and are not elements of Λ. However, one can factor out a 2×2 matrix $\mathcal{L}og_{a_p}(T)$ that encodes this bad growth (see [Spr23] for a simple description in terms of p-adic digits) and arrive at a pair of power series $L^\sharp$ and $L^\flat$ which are both in Λ, and then pose up to two main conjectures (we need the appropriate Selmer group to be Λ-cotorsion, i.e. the characteristic ideal to be well-defined):

Conjecture 2.8. *We have $(g_{\mathcal{X}^\sharp}) = (L^\sharp)$, and $(g_{\mathcal{X}^\flat}) = (L^\flat)$ as ideals in Λ.*

Either main conjecture would give the analogous consequence of the p-primary part of the BSD formula. At present, we only know one direction in the main conjecture, giving for example the following theorem:

Theorem 2.9. *[Spr24, Wan14] Let $E/\mathbb{Q}$ be an elliptic curve with square-free conductor N, and let p be an odd non-ordinary prime. If $L(E,1) \neq 0$, then*

$$\left|\frac{L(E,1)}{\Omega}\right|_p \leqslant \left|\#\text{Ш}(E/\mathbb{Q}) \prod_l c_l\right|_p .$$

3 Applications to Selmer groups of modular forms

Let $f = \sum_n a_n q^n$ be a newform of even weight k and level $\Gamma_0(N)$ and assume $p \nmid 2N$. Fix $\mathfrak{p}|p$ in $R := \mathcal{O}_{\mathbb{Q}((a_n)_n)}$. We denote by $R_\mathfrak{p}$ the valuation ring of the completion at $\mathfrak{p}$ of $\mathbb{Q}((a_n)_n)$. Recall that the local condition of the classical Selmer group was essentially the local points of the elliptic curve. In the three generalizations of Selmer groups below, the local condition at the prime p is changed.

As before, these Selmer groups are not nice to work with, but their Pontryagin duals are. We denote the Pontryagin dual of a Selmer group Sel^{name} by

$$\mathcal{X}^{name} := \mathrm{Hom}_{cont}(\mathrm{Sel}^{name}(f), \mathbb{Q}_p/\mathbb{Z}_p).$$

[2]other than the hypothesis on the surjectivity of the Galois representation on the automorphism group fo the p-adic Tate module, we also need that the residual Galois representation associated to E is irreducible and ramified at a prime q that divides N exactly, and another condition on its image, see [SU14, Theorem 3.29].

Type of Selmer Group	Condition on a_p	Local Condition	Remark
Bloch–Kato Selmer group $\mathrm{Sel}^{BK}(f)$		Involves one of Fontaine's period rings $\mathbb{B}_{cris}$	
Greenberg-type Selmer group $\mathrm{Sel}^{Gr}(f)$	$\mathrm{ord}_{\mathfrak{p}}(a_p) = 0$	Representation-theoretic	
Chromatic Selmer groups $\mathrm{Sel}^{\sharp}(f)$ **and** $\mathrm{Sel}^{\flat}(f)$	$\mathrm{ord}_{\mathfrak{p}}(a_p) > 0$	Involves certain integral modules appearing in p-adic Hodge theory called Wach modules	In the case of higher weight modular forms, these are due to Lei, Loeffler, and Zerbes [LLZ10]. The essential idea is to translate the conditions of [Spr12] and [Kob03] into p-adic Hodge theoretic modules

Table 1: Comparison of Different Selmer Groups

In the ordinary case, Longo and Vigni proved the following generalization of the main result of the previous section. Recall that $\sim$ means 'up to a p-adic unit.'

Theorem 3.1. *[LV21] If* $\mathrm{ord}_{\mathfrak{p}}(a_p) = 0$, $a_p \not\equiv 1 \pmod{\mathfrak{p}}$ *and*

$$|\mathrm{Sel}^{BK}(f)| < \infty,$$

then

$$(R_{\mathfrak{p}}/g_{\mathcal{X}^{Gr}}(0)) \sim |\mathrm{Sel}^{BK}(f)|_p \times (\textit{Tamagawa numbers}).$$

In the non-ordinary case unfortunately, we do not know if $\mathcal{X}^{\sharp}$ and $\mathcal{X}^{\flat}$ are cotorsion as Λ-modules:

Conjecture 3.2. $\mathcal{X}^{\sharp}$ *and* $\mathcal{X}^{\flat}$ *are cotorsion as* Λ*-modules.*

Recall that we denote by k the weight of f and that k is even.

Theorem 3.3. *[RSar, Theorem 3.2] Assume* $p > k > 3$. *If* $\mathrm{ord}_{\mathfrak{p}}(a_p) > 0$ *then*

$$(R_{\mathfrak{p}}/g_{\mathcal{X}^{\flat}}(0)) \sim |\mathrm{Sel}^{BK}(f)|_p \times (\textit{Tamagawa numbers}).$$

This statement corresponds to [RSar, Theorem 3.2, ii][3]. The notation is a bit different. We have that $i = 2$ corresponds to the choice of $\flat$-Selmer group[4]. There is a corresponding theorem for $\sharp$-Selmer groups (i.e. $i = 1$), but it requires an additional mild hypothesis on a_p.

4 Open Questions

The two theorems relate the Bloch-Kato Selmer group to the Greenberg-type Selmer groups and the chromatic Selmer groups. Is there a precise relation between $\mathcal{X}^{Gr}$ respectively $\mathcal{X}^{\sharp}$ and $\mathcal{X}^{\flat}$ and $\mathcal{X}^{BK}$?

There are two Selmer groups in the non-ordinary case. Analogously, is there a second Greenberg-type Selmer group in the ordinary case?

Is there an analogous theory for Siegel modular forms?

[3]which works under more general hypotheses, all of which are satisfied when $p > k > 3$.

[4]and the theorem works for various twists of the Selmer groups. In the notatiopn of [RSar, Theorem 3.2,ii], the theorem presented here concerns the twist by $s = k - 2$.

References

[Gre99] Ralph Greenberg. Iwasawa theory for elliptic curves. In *Arithmetic theory of elliptic curves (Cetraro, 1997)*, volume 1716 of *Lecture Notes in Math.*, pages 51–144. Springer, Berlin, 1999.

[Iwa58] Kenkichi Iwasawa. On some invariants of cyclotomic fields. *Amer. J. Math.*, 80:773–783; erratum: **81 (1958), 280**, ISSN = 0002–9327,1080–6377, MRCLASS = 10.66, MRNUMBER = 124317, MRREVIEWER = Y. Kawada, URL = http://links.jstor.org/sici?sici=0002–9327(195807)80:3<773:OSIOCF>2.0.CO;2–Xorigin=MSN,, 1958.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p-adiques et applications arithmétiques. III.

[Kob03] Shin-ichi Kobayashi. Iwasawa theory for elliptic curves at supersingular primes. *Invent. Math.*, 152(1):1–36, 2003.

[LLZ10] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Wach modules and Iwasawa theory for modular forms. *Asian J. Math.*, 14(4):475–528, 2010.

[LV21] Matteo Longo and Stefano Vigni. On Bloch-Kato Selmer groups and Iwasawa theory of p-adic Galois representations. *New York Journal of Mathematics*, 27:437–467, 2021.

[RSar] Jishnu Ray and Florian Sprung. On characteristic power series of dual signed Selmer groups. *Annales de l'Institut Fourier*, to appear.

[Ser95] Jean-Pierre Serre. Classes des corps cyclotomiques (d'après K. Iwasawa). In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 174, 83–93. Soc. Math. France, Paris, 1995.

[Spr12] Florian E. Ito Sprung. Iwasawa theory for elliptic curves at supersingular primes: a pair of main conjectures. *J. Number Theory*, 132(7):1483–1506, 2012.

[Spr23] Florian Sprung. La matrice de logarithme en termes de chiffres p-adiques. *Annales mathématiques du Québec*, pages 1–12, 2023.

[Spr24] Florian Sprung. On Iwasawa main conjectures for elliptic curves at supersingular primes: beyond the case $a_p = 0$. *Adv. Math.*, 449:Paper No. 109741, 2024.

[SU14] Christopher Skinner and Eric. Urban. The Iwasawa main conjectures for GL_2. *Invent. Math.*, 195(1):1–277, 2014.

[Wan14] Xin Wan. Iwasawa Main Conjecture for Supersingular Elliptic Curves, 2014. preprint, arXiv:1411.6352.