PERIODS OF EISENSTEIN SERIES OF WEIGHT 2 ON ARITHMETIC SUBGROUPS OF $\mathrm{SL}(2, \mathbb{Z})$

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1.

The key to various theorems about modular forms is the $\mathbb{R}$-linear map $f \mapsto \lambda_f$ defined as

$$\lambda_f\Big(\sum_{r\in\mathbb{P}^1(\mathbb{Q})} n_r \mathfrak{e}^r\Big) = \sum_{r\in\mathbb{P}^1(\mathbb{Q})} n_r \operatorname{Re}\int_r^\infty f(z)\,dz. \tag{1}$$

This map takes cusp forms in the space $M_2(\Gamma)$ of modular forms of weight 2 on a given subgroup of finite index in $\mathrm{SL}(2, \mathbb{Z})$ to group homomorphisms in $\mathrm{Hom}_\Gamma(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{R})$, the real vector space of Γ-invariant group homomorphisms on elements $\sum_r n_r \mathfrak{e}^r$ in the free $\mathbb{Z}$-module generated by the points in $\mathbb{P}^1(\mathbb{Q})$ whose degree $\sum_r n_r$ is zero. However, usually this map appears in a different formulation using Manin symbols and relative homology of the compactified Riemann surface associated to Γ. This map, in the form (1) or in the form of Manin symbols, is in particular the main tool for the computation of modular forms, but as pointed out in various talks by the author can be used for so much more (see [Skoa]). In our talk in this conference we proposed still another, even technically more convenient and computationally effective formulation of this map, and we proposed an extension of the theory surrounding this map to arbitrary Belyi pairs (i.e. pairs of compact Riemann surfaces and Belyi maps on them).

However, the map (1) (or rather its pendant using Manin symbols) seems to involve an annoyance in that they a priori do not admit to include Eisenstein series in the discussion, though the target spaces usually contain elements which appear to be somehow associated to

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Eisenstein series. In our talk we proposed a method for naturally extending the above map (and its proposed "new" formulation) to the subspace of Eisenstein series. The purpose of this note is to provide more details of this extension.

2.

It is a remarkable though trivial fact that

$$\frac{\partial}{\partial z} \log |\eta(z) y^{1/4}| = -\pi i E_2^*(z).$$

Here

$$\eta = q^{1/24} \prod_{n\geq 1}(1-q^n), \quad E_2^* = -\frac{1}{24} + \frac{1}{8\pi y} + \sum_{n\geq 1} \sigma_1(n) q^n$$

denote the Dedekind eta function and the non-holomorphic Eisenstein series of weight 2 on the full modular group, and we use $z = x+iy$ as variable in the complex upper half plane $\mathbb{H}$ and $q(z) = e^{2\pi i z}$. The identity itself is easy to verify e.g. as an identity of formal power series (the left hand side is essentially a Lambert series that can be expanded in terms of powers of q to yield the right hand side). What is remarkable is the fact that both $E_2^*\,dz$ and $\log|\eta(z)y^{1/4}|$ are modular, i.e. invariant with respect to the usual action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on the upper half plane via fractional transformations.

One finds similar identities also for subgroups of finite index in the modular group. For example, for any rational 2-vector $\mathbf{a} = (a_1, a_2)$ not in $\mathbb{Z}^2$, one has

$$\frac{\partial}{\partial z} H_{\mathbf{a}}(z) = \frac{\pi i}{2} \mathbb{B}_2(a_1) - \pi i \sum_{n\geq 1} \Big(\sum_{\substack{d\mid n \\ d\in a_1+\mathbb{Z}}} |d|\, e(a_2 \tfrac{n}{d}) \Big)\, q^n, \tag{2}$$

where

$$H_{\mathbf{a}} = \frac{1}{2}\mathbb{B}_2(a_1) \log|q| + \log \prod_{n\in a_1+\mathbb{Z}} \big|1 - q^{|n|} e(\mathrm{sign}(n) a_2)\big|,$$

and where we use $e(x) = e^{2\pi i x}$, the convention $\mathrm{sign}(0) = +1$, and $\mathbb{B}_2(a) = u^2 - u + \frac{1}{6}$ with $u = a - \lfloor a \rfloor$ denoting the fractional part of a. Here again the right hand side is a modular form of weight 2 (cf. [Gun62, § IV]— this time on the principal congruence subgroup $\Gamma(N)$), where N is the smallest positive integer such that $N\mathbf{a}$ is integral — and the left hand side before derivation is invariant under $\Gamma(N)$ [SE06, Thm. 6.1]. For verifying the latter the reader can rewrite the left hand side as

$$H_{\mathbf{a}} = \log \left| e^{-\pi a_1^2 y} \vartheta\left(z, \mathbf{a}\left[\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right]\right)/\eta(z) \right|.$$

with Jacobi's theta function $\vartheta(z,w)$ and the Dedekind eta-function, whose quotient satisfies the product formula

$$\vartheta(z,w)/\eta(z) = q^{1/12}\left(\zeta^{1/2}-\zeta^{-1/2}\right)\prod_{n\geq 1}\left(1-q^n\zeta\right)\left(1-q^n\zeta^{-1}\right)$$

with $\zeta(w) = e^{2\pi i w}$, and transforms like a Jacobi form of index $1/2$ under $\mathrm{SL}(2,\mathbb{Z})$ (see [SE06, § 6,7] for details).

We note that the left hand sides of the above identities before derivation are harmonic functions (i.e. smooth functions that are annihilated by $\frac{\partial^2}{\partial\bar{z}\partial z}$ (since the right hand sides are holomorphic, whence annihilated by $\frac{\partial}{\partial\bar{z}}$.

3.

If H is a harmonic function on the upper half plane then H is the real part of an antiderivative of the holomorphic function $f(z) := 2\frac{\partial}{\partial z}H(z)$. If the functions f is a modular form of weight 2 on a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ we may ask whether H is invariant under Γ as in the above examples. The answer is provided by the following theorem.

Theorem 1 ([Skob])**.** *Let f be a modular form of weight 2 on a subgroup Γ of finite index in the full modular group, and let F be an antiderivative of f. Then $\mathrm{Re}(F)$ is Γ-invariant if and only if f is an Eisenstein series whose values in the cusps are purely imaginary.*

For the proof of the theorem for general Γ we refer the reader to [Skob]. However, for Eisenstein series f of weight 2 on congruence subgroups Γ with purely imaginary values in the cusps, the Γ-invariance of H follows essentially from (2). Namely, every Eisenstein series on a congruence subgroup with purely imaginary values in the cusps is a real linear combination of the Eisenstein series on the right hand side of (2) (see e.g. [Hec27]), and the H_a are modular as remarked in Section 2.

Let f and F be as in the theorem. If s is a cusp, say $s = A\infty$ for some $A = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$ in $\mathrm{SL}(2,\mathbb{Z})$, and if $N \geq 1$ is an integer such that $AT^NA^{-1} \in \Gamma$, then $f|A(z) = f(Az)(cz+d)^{-2}$ possesses a Fourier expansion in terms of $q^{1/N} = e^{2\pi i z/N}$, i.e. $f|A(z) = \sum_{n\geq 0} c_n q^{n/N}$ with suitable complex numbers c_n. In other words,

$$\frac{2\pi i}{N}dF = c_0 d\log t + d\sum_{n\geq 1} c_n\frac{t^n}{n},$$

where $t(z) = e^{\frac{2\pi i}{N}A^{-1}z}$ (so that $d\log t = \frac{2\pi i}{N}\frac{dz}{(-cz+a)^2}$ for $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$). Accordingly,

$$F = c_0 A^{-1}z + c + \frac{N}{2\pi i}\sum_{n\geq 1} c_n \frac{t^n}{n}$$

for some constant c (depending on A but not on the choice of N).

Assume that $Re(F)$ is Γ-invariant. Then $Re(F)$ is in particular invariant under $AT^N A^{-1}$, which implies $\mathrm{Re}(c_0 N) = 0$, i.e. that c_0 is purely imaginary as claimed in the theorem.

Assume vice versa that c_0 is purely imaginary. Then we have

$$\mathrm{Re}\left(c_0 A^{-1}z\right) = ic_0 \,\mathrm{Im}\, A^{-1}z,$$

and therefore $\mathrm{Re}\, c$ does not depend on the choice of A but only on s (as every element of $\mathrm{SL}(2,\mathbb{Z})$ mapping ∞ to s is of the form $\pm A \left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$). We set

$$(\mathrm{Re}\, F)(s) := \mathrm{Re}\, c.$$

If $Re(F)$ is Γ-invariant then $(\mathrm{Re}\, F)(s)$ is Γ-invariant too, i.e. depends only on the orbit of s under Γ.

4.

Let $\mathcal{E}_2^{\mathrm{p.i.}}(\Gamma)$ denote the $\mathbb{R}$-linear subspace of all Eisenstein series of weight 2 on Γ whose values at the cusps are purely imaginary. As we have seen in the preceding paragraph, for any f in $\mathcal{E}_2^{\mathrm{p.i.}}(\Gamma)$, we can define a homomorphism of abelian groups $\lambda_f : \mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]^0 \to \mathbb{R}$ by setting

$$\lambda_f\big(\sum_s n_s \mathfrak{e}^s\big) := \sum_s n_s (2\,\mathrm{Re}\, F)(s), \tag{3}$$

where (again) the superscript 0 indicates the subspace of elements $D = \sum_s n_s \mathfrak{e}^s$ in $\mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]$ such that $\deg(D) = \sum_s n_s = 0$. Replacing F by $F + C$ for some constant C does not change the value on the right, and hence the given map λ_f does not depend on the choice of F. We inserted the factor 2 into the definition to account for the fact $\frac{\partial}{\partial z}\mathrm{Re}(2F) = f$.

The importance of this construction lies in the fact the the map $f \mapsto \lambda_f$ provides a natural extension of the map on cusp forms discussed in (1). Indeed, if f is a cusp form of weight 2 on Γ, and F an antiderivative of f, then the right hand side of (1) takes the form of the right hand side of (3), where, of course, s needs to run over all of $\mathbb{P}^1(\mathbb{Q})$. Note that the natural map $\nu : \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \to \mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]^0$ induces an injection $\nu^* : \mathrm{Hom}\,(\mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{R}) \to \mathrm{Hom}\,(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{R})$. For using (3) as extension of the map (1) to the whole space of modular forms of weight 2 on Γ we need the following theorem.

Theorem 2 ([Skob]). *For any subgroup* Γ *of finite index in* $\mathrm{SL}(2,\mathbb{Z})$*, the application* $f \mapsto \lambda_f$ *defines an* $\mathbb{R}$*-linear isomorphism* λ^Γ *of* $\mathcal{E}_2^{\text{p.i.}}(\Gamma)$ *onto* $\mathrm{Hom}\,(\mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{R})$.

Though we announce this theorem here we will not give any proof in this note. In fact, the proof is not yet completely written up but will eventually be published elsewhere (see [Skob]).

5.

The reader might find it amusing to check Theorem 2 numerically or even prove it for congruence subgroups Γ by direct computation.

For this we note, first of all that it suffices to prove the theorem for principal congruence subgroups $\Gamma(N)$. Indeed, if Γ contains $\Gamma(N)$, we have $\mathcal{E}_2^{\text{p.i.}}(\Gamma) \subseteq \mathcal{E}_2^{\text{p.i.}}(\Gamma(N))$, and λ^Γ equals the restriction of $\lambda^{\Gamma(N)}$ to $\mathcal{E}_2^{\text{p.i.}}(\Gamma)$ followed by the inverse of the natural injection of the real vector space $\mathrm{Hom}\,(\mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{R})$ into $\mathrm{Hom}\,(\mathbb{Z}[\Gamma(N)\backslash\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{R})$.

So assume that $\Gamma = \Gamma(N)$ for some N. It is well-known (see [Hec27, pp. 468]) that the space of Eisenstein series of weight 2 on $\Gamma(N)$ is spanned by the functions $E_{\mathbf{a}}$ on the right of (2), where $\mathbf{a} = (a_1, a_2)$ runs through a system I of representatives for all primitive elements in $\frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2$ modulo multiplication by ± 1 (where we call $\mathbf{a}+\mathbb{Z}^2$ primitive if $\gcd(N\mathbf{a}, N) = 1$). One has $\sum_{\mathbf{a}\in I} E_{\mathbf{a}} = 0$, and since $\mathrm{card}(I)$ equals the number ν_N of cusps of $\Gamma(N)$ there are no other linear relations among the $E_{\mathbf{a}}$. Here we use that the dimension of $\mathcal{E}_2^{\text{p.i.}}(\Gamma)$ equals $\nu(N) - 1$.

For proving Theorem 2 we thus need to show that the elements $\lambda_{\mathbf{a}} := \lambda_{E_{\mathbf{a}}}$ ($\mathbf{a} \in I$) span $\mathrm{Hom}\,(\mathbb{Z}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{R})$, and for this it suffices to show that $\sum_{\mathbf{a}\in I} x_{\mathbf{a}}\lambda_{\mathbf{a}} = 0$ implies that the $x_{\mathbf{a}}$ are all equal.

From the explicit formula for $H_{\mathbf{a}}$ (following (2)) we have $H_{\mathbf{a}}(\infty) = 0$ unless a_1 is integral when we have $H_{\mathbf{a}}(\infty) = \log|1 - e(a_2)|$. Since for any A in $\mathrm{SL}(2,\mathbb{Z})$, we have $H_{\mathbf{a}} \circ A = H_{\mathbf{a}A}$ (see [SE06, Thm. 6.1]), we find

$$H_{\mathbf{a}}(A\infty) = H_{\mathbf{a}A}(\infty) = \begin{cases} \log|1 - e(b_2)| & \text{if } \mathbf{a}A \equiv (0, b_2) \bmod \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

We leave it as an exercise to the ambitious reader to deduce from this that $\sum_{\mathbf{a}\in I} x_{\mathbf{a}}\lambda_{\mathbf{a}} = 0$ indeed implies that the $x\mathbf{a}$ are all equal, and thereby verifying Thm. 2 for the principle congruence subgroups.

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