MODULAR FORMS WITH POLES ON HYPERPLANE ARRANGEMENTS

BRANDON WILLIAMS

This is a brief summary of a talk at the 2024 RIMS conference "Research on Automorphic Forms", based on the paper [4] which is joint work with Haowu Wang. It is a continuation of our earlier work [3] which describes the structure of the algebra of modular forms for certain orthogonal groups of signature $(2, n)$ related to root systems. Here, we consider algebras of meromorphic modular forms with certain particularly mild singularities.

1. Orthogonal modular forms

Let $(L, \langle -, - \rangle)$ be an even integral lattice (so L is a free abelian group of finite rank, and $\langle -, - \rangle$ is a nondegenerate, symmetric bilinear form on it) and assume further that L has signature $(2, \ell)$ for some $\ell \geq 3$. Let L' be the dual lattice

$$L' = \{r \in L_{\mathbb{Q}} = L \otimes \mathbb{Q} : \ \langle r, b \rangle \in \mathbb{Z} \text{ for all } b \in L\}.$$

Fix an orientation on $L_{\mathbb{R}} := L \otimes \mathbb{R}$, and define the set

$$\mathcal{A} := \Big\{ z \in L \otimes \mathbb{C} : \ \langle z, z \rangle = 0, \ \langle z, \overline{z} \rangle > 0, \ \{\mathrm{re}(z), \mathrm{im}(z)\} \text{ is oriented} \Big\}.$$

Then $\mathcal{D} := \mathcal{A}/\mathbb{C}^{\times}$ is an irreducible Hermitian symmetric space of Cartan type IV.

Let $\Gamma \leq \mathrm{O}^+(L)$ be the integral spinor kernel (equivalently, the subgroup of $\mathrm{O}(L)$ that preserves $\mathcal{D}$ by multiplication) or a finite-index subgroup in it. A modular form of weight k is a holomorphic function

$$f : \mathcal{A} \longrightarrow \mathbb{C}$$

that satisfies

$$f(\gamma z) = f(z) \text{ for all } \gamma \in \Gamma \quad \text{and} \quad f(tz) = t^{-k} f(z) \text{ for all } t \in \mathbb{C}^{\times}.$$

No growth condition has to be imposed since we assume L has rank at least 5.

We will always assume that the lattice L splits in the form

$$L = U \oplus U \oplus K(-1),$$

where U is the hyperbolic plane ($\mathbb{Z}^2$ with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$), where K is positive-definite, and where $K(-1)$ means that its bilinear form is multiplied by (-1). In this case, one can represent modular forms by their Fourier–Jacobi expansions:

$$f = \sum_{n=0}^{\infty} \phi_n(\tau, z) e^{2\pi i n w},$$

where $\tau, w \in \mathbb{H} = \{x + iy \in \mathbb{C} : \ y > 0\}$ and $z \in K \otimes \mathbb{C}$, and where each ϕ_n is a Jacobi form of index n attached to the lattice K. See [1] for details.

2. Hyperplane arrangements

We continue to use the setup and notation of the previous section.

Definition 1. A **rational quadratic divisor** $r^{\perp}$ is a divisor of the form

$$\{\mathbb{C}z : \ \langle z, r \rangle = 0\} \subseteq \mathcal{D}$$

for some $r \in L'$ with $\langle r, r \rangle > 0$.

Definition 2. A **hyperplane arrangement** $\mathcal{H}$ for Γ is a collection of rational quadratic divisors that satisfies:
(1) $\mathcal{H}$ is a finite union of Γ-orbits;
and every nonempty intersection $\mathcal{I} = r_1^{\perp} \cap ... \cap r_n^{\perp}$ in $\mathcal{H}$ satisfies the following conditions:
(2) $\dim \mathcal{I} > 1$;
(3) if $\dim \mathcal{I} = 2$, then $\mathcal{I}$ is anisotropic;
(4) if $\dim \mathcal{I} = 3$, then $\mathcal{I}$ contains no isotropic planes.

Here the dimension is taken in the naive sense (viewing $\mathcal{I}$ as a linear subspace of $L \otimes \mathbb{C}$). Items (2)-(4) are conditions "in codimension one". For example, if $\mathcal{H}$ is empty then these conditions are equivalent to requiring that the Baily–Borel boundary has no components of codimension one. The importance of this definition is due to Looijenga's [2] construction of a toroidal compactification of the arrangement complement $(\mathcal{D} \backslash \mathcal{H})/\Gamma$ generalizing the Baily–Borel compactification.

Theorem 3. *Let $\mathcal{H}$ be a hyperplane arrangement. For any $k \in \mathbb{Z}$, let $M_k^!(\Gamma; \mathcal{H})$ be the space of meromorphic modular forms of weight k whose poles are supported on $\mathcal{H}$. Then*

$$M^!(\Gamma; \mathcal{H}) = \bigoplus_{k \in \mathbb{Z}} M_k^!(\Gamma; \mathcal{H})$$

is a finitely-generated graded ring, and any $f \in M_k^!(\Gamma; \mathcal{H})$ has only poles of order at most k.
(In particular, $M_k^!(\Gamma; \mathcal{H}) = \{0\}$ for $k < 0$, and $M_0^!(\Gamma; \mathcal{H}) = \mathbb{C}$.)

3. Main results

In our paper [4] we consider lattices $L = U \oplus U \oplus K(-1)$, where K is a root lattice of ADE type of the form

$$K = A \oplus R = \Big(\bigoplus_{j=1}^{t} A_{m_j} \Big) \oplus R, \quad R \in \{A_m, D_m, E_6, E_7\}$$

subject to the constraints

$$\sum_{j=1}^{t}(m_j+1)+m \le 11 \quad \text{if } R = D_m; \tag{1}$$

$$\sum_{j=1}^{t}(m_j+1)+(m+1) \le 11 \quad \text{if } R = A_m; \tag{2}$$

$$\sum_{j=1}^{t}(m_j+1)+(m+2) \le 11 \quad \text{if } R = E_m. \tag{3}$$

(These constraints are necessary for our construction of generators of the algebra of modular forms.) For any such decomposition $L = U \oplus U \oplus A(-1) \oplus R(-1)$ (which need not be unique), we construct a hyperplane arrangement $\mathcal{H}_L = \mathcal{H}_A \cup \mathcal{H}_R$ for the discriminant kernel

$$\Gamma := \ker\Big(\mathrm{O}^+(L) \longrightarrow \mathrm{O}(L'/L)\Big)$$

as follows:

(i) If $A = \bigoplus_{j=1}^t A_{m_j}$, then $\mathcal{H}_A = \bigcup_{j=1}^t \mathcal{H}_{A_{m_j}}$, where $\mathcal{H}_{A_{m_j}}$ is the union of all rational quadratic divisors $r_j^\perp$, $r_j \in A_{m_j}$ of the smallest possible norm, $\langle r_j, r_j\rangle = \frac{m_j}{m_j+1}$.

(ii) $\mathcal{H}_R$ consists of a single orbit if $R \in \{A_8, A_9, A_{10}, D_9, D_{10}, D_{11}\}$, and is empty otherwise. In the nonempty case, there is a unique coset $\gamma \in R'/R$ that contains no short vectors, i.e.

$$\delta_\gamma := \min\{\langle x, x\rangle :\ x \in \gamma\} > 2,$$

and we define $\mathcal{H}_R = \bigcup r^\perp$ where the union runs over $r \in 2U \oplus R(-1)'$ with $r|_{R'} \in \gamma$ and $\langle r, r\rangle = \delta_\gamma - 2$. (For example, when $L = U \oplus U \oplus A_8(-1)$ with $A = \{0\}$ and $R = A_8$, the arrangment $\mathcal{H}_R$ is the Heegner divisor of discriminant $1/9$.)

The main result of [4] is:

Theorem 4 (Theorem 1.2 of [4])**.** *Let $K = A \oplus R$ be one of the 147 lattices satisfying Equations 1. Then $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_R$ is a hyperplane arrangement in the sense of Definition 2 and the algebra $M^!_*(\Gamma;\mathcal{H})$ is freely generated by $3 + \mathrm{rank}(K)$ meromorphic modular forms.*

In all cases, the generators of $M^!_*(\Gamma;\mathcal{H})$ naturally fall into three classes depending on their leading Fourier–Jacobi coefficient:
(i) Eisenstein type: the leading Fourier–Jacobi coefficient is the classical Eisenstein series E_4 or E_6 and it occurs in index zero.
(ii) Abelian type: the leading Fourier–Jacobi coefficient is an abelian (or multiply periodic) function. These are meromorphic generators associated to the component A.
(iii) Jacobi type: the leading Fourier–Jacobi coefficients are related to the Wirthmüller generators [5] of the algebra of Weyl-invariant weak Jacobi forms attached to the root lattice R.

For the precise definition of the generators and their construction (which in part is by cases) we refer to [4].

4. Two examples

We illustrate Theorem 4 with two examples, taking $A = \{0\}$ and R to be either the A_n root lattice ($n \le 10$) or the D_n root lattice ($n \le 11$). In the following tables, the generators colored black can be chosen to be holomorphic, while the generators colored red cannot.

Example 5. The A_n-tower: $L = U \oplus U \oplus A_n(-1)$ for $n \le 10$. We have Eisenstein-type generators of weights 4 and 6 and Jacobi-type generators of weights

$$12, 10, 9, 8, ..., 12 - (n+1).$$

n	weights	n	weights
1	$4, 6, 10, 12$	6	$4, 5, 6, 6, 7, 8, 9, 10, 12$
2	$4, 6, 9, 10, 12$	7	$4, 4, 5, 6, 6, 7, 8, 9, 10, 12$
3	$4, 6, 8, 9, 10, 12$	8	$3, 4, 4, 5, 6, 6, 7, 8, 9, 10, 12$
4	$4, 6, 7, 8, 9, 10, 12$	9	$2, 3, 4, 4, 5, 6, 6, 7, 8, 9, 10, 12$
5	$4, 6, 6, 7, 8, 9, 10, 12$	10	$1, 2, 3, 4, 4, 5, 6, 6, 7, 8, 9, 10, 12$

Example 6. The D_n-tower. Let $L = U \oplus U \oplus D_n(-1)$ for $4 \le n \le 11$. We have Eisenstein-type generators of weights 4 and 6 and Jacobi-type generators of weights

$$12 - n, 8, 10, 12 \text{ and } 18, 16, 14, ..., 26 - 2n.$$

n	weights	n	weights
4	$4, 6, 8, 8, 10, 12, 18$	8	$4, 4, 6, 8, 10, 10, 12, 12, 14, 16, 18$
5	$4, 6, 7, 8, 10, 12, 16, 18$	9	$3, 4, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18$
6	$4, 6, 6, 8, 10, 12, 14, 16, 18$	10	$2, 4, 6, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18$
7	$4, 5, 6, 8, 10, 12, 12, 14, 16, 18$	11	$1, 4, 4, 6, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18$

References

[1] Valery Gritsenko. Fourier-Jacobi functions in n variables. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 168(Anal. Teor. Chisel i Teor. Funktsiĭ. 9):32–44, 187–188, 1988.

[2] Eduard Looijenga. Compactifications defined by arrangements. II. Locally symmetric varieties of type IV. *Duke Math. J.*, 119(3):527–588, 2003.

[3] Haowu Wang and Brandon Williams. On some free algebras of orthogonal modular forms. *Adv. Math.*, 373:107332, 22, 2020.

[4] Haowu Wang and Brandon Williams. Modular forms with poles on hyperplane arrangements. *Algebr. Geom.*, 11(4):506–568, 2024.

[5] Klaus Wirthmüller. Root systems and Jacobi forms. *Compositio Math.*, 82(3): 293–354, 1992.

Lehrstuhl A für Mathematik, RWTH Aachen University, 52056 Aachen, Germany

Email address: `brandon.williams@matha.rwth-aachen.de`