Fourier-Jacobi expansion of cusp forms generating quaternionic discrete series

Hiro-aki Narita

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Abstract

This article is the write-up of what the author talked about at the RIMS workshop on automorphic forms held in 2024, January. The main result is a theory of a Fourier-Jacobi expansion of cusp forms generating quaternionic discrete series representations for some general class of simple adjoint groups. This expansion respects the Heisenberg parabolic subgroup, whose unipotent radical is a Heisenberg group. This parabolic subgroup is uniformly explained in terms of the notion of cubic norm structures. Aaron Pollack has already given the terms of the Fourier expansion with the trivial central character, which are contributed by characters of the Heisenberg group. We prove that every term of non-trivial central characters for the Fourier-Jacobi expansions of cusp forms generating quaternionic discrete series for the general class mentioned above has no contribution by the discrete spectrum of the Jacobi group, which is a subgroup of the Heisenberg parabolic subgroup. This is obtained by showing that generalized Whittaker functions for the Schrödinger representations are zero under some assumption sufficient for our purpose. It is shown that such terms of the Fourier-Jacobi expansion turn out to be sums of the translates of Pollack's functions (generalized Whittaker functions) by the Weyl reflection for some root.

1 Background of this study

As the arithmetic of quadratic forms motivated number theoretic studies of holomorphic Siegel modular forms, some arithmetic studies of binary cubic forms have turned out to be significantly related to the arithmetic of automorphic forms on the exceptional group of type G_2 generating quaternionic discrete series representations. Among discrete series representations, quaternionic discrete series can be said to be accessible next to holomorphic discrete series. An important initiation of the studies of such automorphic forms is due to Gross-Gan-Savin [3]. However, in spite of the impact of [3], there had been no further essential progress for a long time. A crucial ingredient of [3] is Wallach's multiplicity free property of generalized Whittaker models for quaternionic discrete series representations but there had been no realization of such Whittaker models by functions, namely generalized Whittaker functions. This is a possible reason of such stopping of the progress. Here note that quaternionic discrete series of G_2 are known to admit no Whittaker models in the usual sense. The generalized Whittaker functions mentioned above are attached to non-degenerate characters of the unipotent radical of the Heisenberg parabolic subgroup, which is not a Borel subgroup.

A next essential progress has been given by Pollack [12]. Pollack has succeeded in explicitly determining generalized Whittaker functions mentioned above for any characters. More precisely he has obtained such formula for a wide class of simple groups of adjoint type including G_2, which will be denoted by $\mathcal{G}$. Starting from [12] Pollack has been producing many interesting papers e.g. [13], [14], [15] and [16] and there seems further preprints, which we can find in his website.

Now let us note that the unipotent radical of the Heisenberg parabolic subgroup is a Heisenberg group with one-dimensional center. Along this parabolic subgroup we can consider the Fourier expansion of automorphic form on the adele group $\mathcal{G}(\mathbb{A})$ as follows

$$F(g) = \sum_{\xi \in \mathbb{Q}} F_\xi(g) \quad (g \in \mathcal{G}(\mathbb{A})),$$

where F_ξ denotes the Fourier transformation of F with respect to the additive character of the center of the Heisenberg group parametrized by ξ. It is natural to call this expansion Fourier-Jacobi expansion. When F generates a quaternionic discrete series, the generalized Whittaker functions determined by Pollack contributes to the archimedean part of F_0. Our recent result taken up in this article contributes to the determination of F_ξ for non-zero ξs. An essential ingredient is an explicit determination of generalized Whittaker functions for Schrödinger representations of the Heisenberg group, which exhausts all irreducible unitary representations of the group together with the characters.

2 Preliminaries

A fundamental paper of quaternionic discrete series representations is due to Gross-Wallach [5]. We know that, in [5], there is the list of simple groups admitting quaternionic discrete series representations as follows:

- Classical groups
  - i) $SU(d,2)$ (type A)
  - ii) $SO(d,4)$ (type B or D)
  - iii) $Sp(1,d)$ (type C),
- Exceptional groups
  - i) G_2 (real rank 2)
  - ii) F_4 (real rank 4)
  - iii) $E_6,\ E_7,\ E_8$ (real rank 4).

For this article the targets of the groups are $SO(d,4)$ and all the exceptional groups above. These groups have maximal parabolic subgroups to be called Heisenberg parabolic subgroups. The unipotent radicals of these parabolic subgroups is two step nilpotent groups, which are naturally referred to as Heisenberg groups. The Heisenberg groups just mentioned are understood commonly in terms of cubic norm structures.

We now review the notion of cubic norm structures, following [12, Section 2.1]. A finite dimensional vector space J over a field F with char(F) $= 0$ is called a cubic norm structure if it has a cubic polynomial $N_J : J \to F$, quadratic polynomial map $\sharp : J \to J$ and the trace paring $(*,*) : J \otimes J \to F$ such that

- $N_J(1_J) = 1,\ 1_J^\sharp = 1_J,\ 1_J \times x = (1_J, x) - x\ \forall x \in J.$
- $(x^\sharp)^\sharp = N_J(x)x\ \forall x \in J.$
- $(x,y) = \frac{1}{4}(1_J,1_J,x)(1_J,1_J,y) - (1_J,x,y)\ \forall x,y \in J.$
- $N_J(x+y) = N_J(x) + (x^\sharp, y) + (x, y^\sharp) + N_J(y)\ \forall x,y \in J,$

where $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$ and there is the unique trilinear form $(*,*,*)$ satisfying $(x,x,x) = 6N_J(x)$.

Hereafter, by $\mathcal{G}$ we denote a simple group of adjoint type over $\mathbb{Q}$ whose real groups $\mathcal{G}(\mathbb{R})$ correspond to the list above except for $SU(d,2)$ and $Sp(1,d)$. As we have said the group $\mathcal{G}$ has a maximal parabolic subgroup called a Heisenberg parabolic subgroup. Let $\mathcal{N}$ be its unipotent radical, which has one-dimensional center, say $\mathcal{Z}$. This is a Heisenberg group, whose vector part$\mathcal{W}_J \simeq \mathcal{N}/\mathcal{Z}$ is defined as a $\mathbb{Q}$-algebraic variety characterized by the $\mathbb{Q}$-rational points

$$\mathcal{W}_J(\mathbb{Q}) := \mathbb{Q} \oplus J_{\mathbb{Q}} \oplus J_{\mathbb{Q}}^{\vee} \oplus \mathbb{Q},$$

where $J_{\mathbb{Q}}$ is a vector space over $\mathbb{Q}$ equipped with a cubic norm structure and $J_{\mathbb{Q}}^{\vee}$ the dual with respect to the trace paring. For each case of $\mathcal{G}$, $J := J_{\mathbb{Q}} \otimes \mathbb{R}$s are explicitly given in [12, Section 1.2]. For this we note that these Js are well known examples of Euclidean Jordan algebras, whose trace parings are positive definite.

3 Generalized Whittaker functions and Fourier-Jacobi models

In this section we work over real groups. We put $G := \mathcal{G}(\mathbb{R})$ and $N := \mathcal{N}(\mathbb{R})$ to discuss generalized Whittaker functions and Fourier-Jacobi models for quaternionic discrete series representations.

Let K be a maximal compact subgroup of G. According to [5, Proposition 4.1], K is isomorphic to $M \times SU(2)/\langle(\epsilon, -1)\rangle$, where M is explained by [5, Table 2.6] and ϵ is the unique element of order two in the center of M. Let π_n be a quaternionic discrete series representation with minimal K type τ_n given by the trivial extension of $2n$-th symmetric tensor representation of $SU(2)$, where we assume $n \geq \dim J + 1$.

Let η be a weekly cyclic representation of N (for a definition see [19, Definition 2.2]). This notion includes irreducible unitary representations of N. Let N_m be the maximal abelian subgroup of N whose vector part is canonically isomorphic to $J^{\vee} \oplus \mathbb{R} \hookrightarrow N/Z$ with $J^{\vee} := J_{\mathbb{Q}}^{\vee} \otimes \mathbb{R}$ and $Z := \mathcal{Z}(\mathbb{R})$. The central character of Z parametrized by $\xi \in \mathbb{R} \setminus \{0\}$ extends trivially to a character of N_m. We denote this by ψ_ξ and define

$$\eta_\xi := L^2\text{-Ind}_{N_m}^{N} \psi_\xi, \quad \eta_\xi^{\infty} := C^{\infty}\text{-Ind}_{N_m}^{N} \psi_\xi.$$

The former is an irreducible unitary representation known as a Schröodinger representation while η_ξ^{∞} is weekly cyclic but not unitary.

Definition. 3.1. *For a weekly cyclic representation η of N the image of*

$$\iota : \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^{\infty}\text{-Ind}_N^G \eta) \to \mathrm{Hom}_K(\tau_n, C^{\infty}\text{-Ind}_N^G \eta)$$

induced by the canonical K-equivariant inclusion $\tau_n \hookrightarrow \pi_n$ is called the space of generalized Whittaker functions for π_n with K-type τ_n. On the left hand sides above we consider intertwining operators as $(\mathfrak{g}, K)$-modules with the Lie algebra $\mathfrak{g}$ of G.

Let us introduce the generalized Schmid operator $D_{\eta,\tau}$ (cf. [19, Section 2.1]) to state the following:

Proposition. 3.2. *We have a linear bijection*

$$\mathrm{Im}(\iota) \simeq \{W \in C^{\infty}_{\eta,\tau_n}(N\backslash G/K) \mid D_{\eta,\tau_n} \cdot W = 0\},$$

where

$$C^{\infty}_{\eta,\tau_n}(N\backslash G/K) := \{W : G \overset{C^{\infty}}{\to} H_\eta \boxtimes V_n \mid W(ugk) = \eta(u) \boxtimes \tau_n(k)^{-1} W(g) \quad \forall (u, g, k) \in N \times G \times K\}.$$

This is stated as an injection in [19, Proposition 2.1] for discrete series representations in general and proved to be a bijection in [19, Theorem 2.4] for discrete series with Blattner parameters satisfying "far from the wall". However, in [9], we can prove the above bijection for quaternionic discrete series of $Sp(1, d)$, which does not satisfy the condition "far from the wall". In [11] we have recently extended it to quaternionic discrete series π_n of the general class taken up in this article.

Solving the differential equations arising from $D_{\eta,\tau_n} \cdot W = 0$ we can explicitly determine generalized Whittaker functions under assuming the separation of variables explained soon. By the left N_m-equivariance and the right K-equivariance with respect to ψ_ξ and τ_n the generalized Whittaker functions are determined by the coordinate

$$(a, b) \times Z(= X + \sqrt{-1}Y) \times w \in (\mathbb{R} \oplus J) \times H_J \times \mathbb{R}_{>0},$$

where we note that $N_m\backslash N \simeq \mathbb{R} \oplus J$, H_J denotes the tube domain for the Euclidean Jordan algebra J (corresponding to the Levi subgroup for the Heisenberg parabolic subgroup) and $\mathbb{R}_{>0}$ is viewed as the connected component for the center of the Levi subgroup of the Heisenberg parabolic subgroup.

With the standard basis $\{x^{n+\nu}y^{n-\nu} \mid -n \leq \nu \leq n\}$ express the generalized Whittaker function W as

$$W(g) = \sum_{\nu=-n}^{n} \phi_\nu(g)\frac{x^{n+\nu}}{(n+\nu)!}\frac{y^{n-\nu}}{(n-\nu)!} \quad (g \in G).$$

Theorem. 3.3. *For each ν we assume the separation of variables $\phi_\nu(a,b,X,Y,w) = F_\nu(a,b,X,Y)H_\nu(w)$. Up to scalars we explicitly determine the generalized Whittaker function of moderate growth with respect to w as follows:*

i) Suppose $\xi > 0$. We have $\phi_\nu \equiv 0$ for $-n \leq \nu \leq n-1$ and

$$\phi_n = N_J(Y)^{-\frac{n}{2}}w^{n+2}e^{-2\pi w^2\xi}\exp(-2\pi\sqrt{-1}(a^2N_J(Z) + a(b,Z^\sharp) + \frac{1}{2}(b, b\times Z))).$$

ii) Suppose $\xi < 0$. We have $\phi_\nu \equiv 0$ for $-(n-1) \leq \nu n$ and

$$\phi_{-n} = N_J(Y)^{-\frac{n}{2}}w^{n+2}e^{2\pi w^2\xi}\exp(-2\pi\sqrt{-1}(a^2N_J(\bar{Z}) + a(b,\bar{Z}^\sharp) + \frac{1}{2}(b, b\times \bar{Z}))).$$

Here Y denotes the imaginary part of $Z \in H_J$.

Based on this result we deduce a general consequence on the Fourier-Jacobi model for π_n. To define this notion we introduce the Jacobi group

$$R_J := L_J^0 \ltimes N$$

with the connected component of the identity for the semisimple part of the Levi subgroup of the Heisenberg parabolic subgroup. From this explicit formula we know that there is no generalized Whittaker functions for η_ξ since W above can not be $L^2(\mathbb{R}\oplus J)$-valued. As a consequence of this we have obtained the following result.

Theorem. 3.4. *For any irreducible unitary representation ρ_ξ with the central character parametrized by $\xi \in \mathbb{R}\setminus\{0\}$ we have*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^\infty_{\mathrm{mod}}\text{-}\mathrm{Ind}_{R_J}^G\rho_\xi) = \{0\},$$

where $C^\infty_{\mathrm{mod}}\text{-}\mathrm{Ind}_{R_J}^G\rho_\xi$ denotes the representation of G with representation space given by moderate growth sections for $C^\infty\text{-}\mathrm{Ind}_{R_J}^G\rho_\xi$.

The notion above of the intertwining operators are naturally referred to as the Fourier-Jacobi models. For this we cite [2], [6], [7] and [8] et al.

4 Fourier-Jacobi expansion

For this section we need to review the Weyl group of $\mathcal{G}$. Regarding the G_2-case, the root system of G_2 has the set $\{\alpha,\beta\}$ of simple roots, where α (respectively β) denotes the long root (respectively short root). We let w_α and w_β be the simple reflections for α and β respectively. For a general $\mathcal{G}$ we refer to [17, Section 2.2]. For this we recall that there is the $\mathbb{Z}/3$-grading of the Lie algebra $\mathfrak{g}_J$ (this notation follows [12]) defined over some general ground field (cf. [12, Section 4.2]), from which we know without difficulty that the algebraic group of type G_2 can be embedded into a general $\mathcal{G}$ and that there is the $\mathfrak{sl}_3$-factor in the $\mathbb{Z}/3$-grading. Then we see that

$$w_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL_3$$

corresponds to w_α (denoted by s_α in [17]). Here we can view w_{12} as an element of $\mathcal{G}(\mathbb{Q})$ since we verify that w_{12} defines an adjoint action on the Lie algebra using its $\mathbb{Z}/3$-grading. In fact, w_{12} defines an element of the Weyl group of G given by the reflection of the simple root corresponding to the root vector $E_{12} \in \mathfrak{sl}_3$ (the Lie algebra of SL_3). There is also an element of Weyl group of G corresponding to w_β in G_2 explicitly given in [17, p145] (denoted by s_β there). This coincides with J_2 given in [12, Section 2.2]. In G we also use the same notation w_α and w_β. In [17, Section 2.2] it is remarked that the Weyl group of G is generated by w_α, w_β and the Weyl group for the automorphism groups of J preserving the cubic form of J up to scalars.

In addition we introduce a $\mathbb{Q}$-algebraic group $\mathcal{R}_J$ with the group $\mathcal{R}_J(\mathbb{R}) = R_J$ of real points. This is to be called a Jacobi group, which is given by the semi-direct product of $\mathcal{N}$ with the connected component $\mathcal{L}_J$ of the identity for the semisimple part of the Levi subgroup of the Heisenberg parabolic subgroup of $\mathcal{G}$. We also need the $\mathbb{Q}$-subgroup $\mathcal{N}_m$ of $\mathcal{N}$ whose vector part is defined by $J_{\mathbb{Q}}^\vee \oplus \mathbb{Q} \hookrightarrow \mathcal{W}_J(\mathbb{Q})$.

4.1 Fourier-Jacobi expansion of general automorphic forms

Let F be an automorphic form on $\mathcal{G}(\mathbb{A})$. For a character χ of $\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})$, F_χ denotes the Fourier transformation of F by χ. We note that χ is parametrized by $(\alpha, \beta, \gamma, \delta) \in \mathbb{Q} \oplus J_{\mathbb{Q}} \oplus J_{\mathbb{Q}}^\vee \oplus \mathbb{Q} = \mathcal{W}_J(\mathbb{Q})$.

Theorem. 4.1. *Let F_ξ be the ξ-term of the Fourier-Jacobi expansion of a general automorphic form F for $\xi \in \mathbb{Q} \setminus \{0\}$. We have*

$$F_\xi(g) = \sum_{\nu \in \mathbb{Q}} \sum_{\chi'_\xi} F_{\chi_\xi}(w_\alpha n((\nu, 0, 0, 0), 0)g) = \sum_{\chi''_\xi} \Theta_{\chi''_\xi}(g)$$

with

$$\Theta_{\chi''_\xi}(g) := \sum_{\gamma \in \mathcal{N}_m(\mathbb{Q})\backslash\mathcal{N}(\mathbb{Q})} F_{\chi''_\xi}(w_\alpha \gamma g),$$

where χ'_ξ (respectively χ''_ξ) runs over characters of $\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})$ parametrized by $(\xi, \beta, \gamma, \delta) \in \mathcal{W}_J(\mathbb{Q})$ (respectively $(\xi, 0, \gamma', \delta') \in \mathcal{W}_J(\mathbb{Q})$). The function $\Theta_{\chi''_\xi}$ above is left $\mathcal{R}_J(\mathbb{Q})$-invariant.

The proof of this begins with the expansion of F_ξ by characters of $\mathcal{N}_m(\mathbb{Q})\backslash\mathcal{N}_m(\mathbb{A})$ as follows:

$$F_\xi(g) = \sum_{\nu \in \mathbb{Q}} \sum_{\psi_\xi^{(b)}} F_{\psi_\xi^{(b)}}(n((\nu, 0, 0, 0))g),$$

where

- $\psi_\xi^{(b)}$ ranges over characters of $\mathcal{N}_m(\mathbb{Q})\backslash\mathcal{N}_m(\mathbb{A})$ parametrized by $(0, \beta, 0, 0) \in \mathcal{W}_J(\mathbb{Q})$ and ξ with $\beta \in J_{\mathbb{Q}}$,
- $F_{\psi_\xi^{(b)}}$ denotes the Fourier transformation of F with respect to $\psi_\xi^{(b)}$.

We now note that the w_α-conjugate of $\mathcal{N}$ exchanges the center of $\mathcal{N}$ and the last entry of $\mathcal{W}_J$ and that $w_\alpha \mathcal{N}_m w_\alpha^{-1} = \mathcal{N}_m$. Let us introduce another character $\psi_{\xi,\alpha}^{(b)}$ of $\mathcal{N}_m(\mathbb{Q})\backslash\mathcal{N}_m(\mathbb{A})$ defined by

$$\psi_{\xi,\alpha}^{(b)}(u) := \psi_\xi^(b)(w_\alpha u w_\alpha^{-1}) \quad (\forall u \in \mathcal{N}_m(\mathbb{A})).$$

We then see that $F_{\psi_\xi^{(b)}}(w_\alpha g) = F_{\psi_{\xi,\alpha}^{(b)}}(g)$ is trivial on the center $\mathcal{Z}(\mathbb{A})$ of $\mathcal{N}(\mathbb{A})$ and that this admits the expansion with respect to the characters of $\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})$. We therefore reach the expansion of F_ξ as in the theorem. The left $\mathcal{R}_J(\mathbb{Q})$-invariance of $\Theta_{\chi''}$ is based on a fundamental property of theta series attached to the Weil representations (cf. [18]), which is naturally exteded from the Schrödinger representations as is well known.

4.2 Fourier-Jacobi expansion for the case of cusp forms quaternionic discrete series

Now recall that $\mathcal{W}_J(\mathbb{Q})$ is equipped with a symplectic form (cf. [12, Section 2.2]). We can define the symplectic group $Sp(\mathcal{W}_J)$ over $\mathbb{Q}$ and the meteplectic group $Mp(\mathcal{W}_J)$, i.e. the non-split two fold cover of $Sp(\mathcal{W}_J)$. We can then consider the pull-back $\widetilde{\mathcal{L}_J}$ of $\mathcal{L}_J$ by the covering map $Mp(\mathcal{W}_J) \to Sp(\mathcal{W}_J)$, where note that $\mathcal{L}_J$ preserves the symplectic form on $\mathcal{W}_J$ and can be thus regarded as a subgroup of $Sp(\mathcal{W}_J)$.

Let $L^2_\xi(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$ be the space of square-integrable functions on $\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A})$ with the character of $\mathcal{Z}(\mathbb{A})$ (i.e. central character) parametrized by $\xi \in \mathbb{Q}\setminus\{0\}$. It is shown that we have a decomposition

$$L^2_\xi(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A})) = L^2(\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A})) \otimes L^2_\xi(\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})),$$

where

- $L^2_\xi(\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A}))$ denotes the L^2-space of $\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})$ with the central character parametrized by $\xi \in \mathbb{Q}\setminus\{0\}$, and
- $L^2(\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A}))$ denotes the space of square-integrable functions on $\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A})$ having the multiplier system coinciding with that of the restriction of the Weil representation to $\widetilde{\mathcal{L}_J}(\mathbb{A})$, which can be genuine or ordinary depending on $\mathcal{L}_J$.

Noting $L^2(\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A}))$ decomposes into a direct sum of the discrete spectrum $L^2_d(\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A}))$ and the continuous spectrum $L^2_c(\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A}))$. We then see that $L^2_\xi(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$ admits a decomposition

$$L^2_\xi(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A})) = L^2_{\xi,d}(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A})) \oplus L^2_{\xi,c}(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$$

with

$$L^2_{\xi,d}(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A})) = L^2_d(\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A})) \otimes L^2_\xi(\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})),$$
$$L^2_{\xi,c}(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A})) = L^2_c(\widetilde{\mathcal{L}_J}(\mathbb{Q})\backslash\widetilde{\mathcal{L}_J}(\mathbb{A})) \otimes L^2_\xi(\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})).$$

Definition. 4.2. *The space $L^2_{\xi,d}(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$ (respectively $L^2_{\xi,c}(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$) is called the discrete spectrum of $L^2_\xi(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$ (respectively the continuous spectrum of $L^2_\xi(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$).*

Theorem. 4.3. *Suppose that F be a cusp form generating a quaternionic discrete series representation at the archimedean place. Let $\xi \in \mathbb{Q}\setminus\{0\}$.*

i) We have

$$F_\xi(g) = \sum_{\nu\in\mathbb{Q}}\sum_{\chi_\xi} F_{\chi_\xi}(w_\alpha n((\nu,0,0,0),0)g),$$

where χ_ξ ranges over characters of $\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})$ parametrized by $(\xi,\beta,\gamma,\delta) \in \mathcal{W}_J(\mathbb{Q})$ of rank four satisfying the negativity with respect to Freudenthal's quartic form (cf. [12, Corollary 1.2.3]).

ii) Suppose that F generates a K-type (τ,V) of the quaternionic discrete series as a K-module. For each fixed $g \in \mathcal{G}(\mathbb{A})$, as a function in $h \in \mathcal{G}_J(\mathbb{A})$, we have

$$(F_\xi(hg), v) \in L^2_{\xi,c}(\mathcal{R}_J(\mathbb{Q})\backslash\mathcal{R}_J(\mathbb{A}))$$

for $v \in V$ with the K-invariant inner product $(*,*)$ of V.

The first assertion is a consequence of Theorem 4.1 and an explicit formula for the generalized Whittaker functions by Pollack [12, Theorem 1.2.1]. The second assertion is proved based on Theorem 3.4. In fact, if F_ξ has non-zero contribution from the discrete spectrum, what this causes to the archimedean component implies the non-zero existence of the moderate growth Fourier-Jacobi model for a quaternionic discrete series representation π_n, which contradicts to Theorem 3.4.

5 Concluding remarks

i) The full detail of this write-pu is included in [11]. Beside it [11] proves that cusp forms constructed by Pollack [14] generate quaternionic discrete series representation, and takes up the non-adelic formulation of the Fourier-Jacobi expansion.

ii) The idea to use an element of the Weyl group to understand the Fourier-Jacobi expansion is also found in [10]. This deals with the Fourier-Jacobi expansion of general cusp forms on $Sp(2,\mathbb{R})$, including generic cusp forms. Recently the author has rewritten the paper so that it also includes the adelic formulation of such expansion.

iii) As for the cases of $SU(d,2)$ and $Sp(1,d)$ we refer to [4] and [9]. It can be remarked that automorphic forms generating quaternionic discrete series for these two cases have unique features, which should called Köcher principle for the case of $Sp(1,d)$ (cf. [9]) and the anti-Köcher principle in terms of signature for the case of $SU(2,2)$ (cf. [4]). We can say that Theorem 4.3, particularly its second assertion has added another unique feature for automorphic forms generating quaternionic discrete series representations. In fact, Fourier-Jacobi coefficients of holomorphic Siegel cusp forms are Jacobi cusp forms, which are naturally considered to sit inside the discrete spectrum of a Jacobi group (indeed this is proved in [1, Section 4] for the case of degree two). By [10] dealing with the case of the symplectic group of degree two, we have known that, in order to understand the Fourier-Jacobi expansions of cusp forms, we need to study the continuous spectrum as well as the discrete spectrum for the square-integrable automorphic forms on a Jacobi group. However, the case of this write-up need only the continuous spectrum as we have seen above. This is totally beyond our expectation.

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Hiro-aki Narita
Department of Mathematics
Faculty of Science and Engineering
Waseda University
3-4-1 Ohkubo, Shinjuku, Tokyo 169-8555, Japan
E-mail address: hnarita@waseda.jp