A LOCAL TRACE FORMULA FOR SOME SPHERICAL VARIETIES

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1. Introduction

This short note is based on a talk given at the RIMS conference "Research on automorphic forms", held at Kyoto in January 2024. It summarizes the main results of my joint work with Raphaël Beuzart-Plessis in [4]. I would like to thank the organizers for the kind invitation. Since this is a short note, I refer the reader to [4] for some undefined notation.

Let F be a local non-Archimedean field of characteristic 0, G be a reductive group defined over F, $H \subset G$ be a unimodular subgroup and $\xi : H(F) \to \mathbb{C}^\times$ be a smooth unitary character. Let $L^2(H(F)\backslash G(F), \xi)$ be the space of $\varphi : G(F) \to \mathbb{C}^\times$ that transform by left multiplication by $H(F)$ according to the character ξ (i.e. $\varphi(hg) = \xi(h)\varphi(g)$ for $(h, g) \in H(F) \times G(F)$) and whose norm is square-integrable on $H(F)\backslash G(F)$. The natural action of $G(F)$ on $L^2(H(F)\backslash G(F), \xi)$ by right translation is a unitary representation and for $f \in C_c^\infty(G(F))$, we define by integration an operator $R(f)$ on $L^2(H(F)\backslash G(F), \xi)$. This operator is associated to the following kernel function [1]

$$K_f(x, y) = \int_{H(F)} f(x^{-1}hy)\xi(h)dh, \quad x, y \in G(F).$$

Formally, the trace of the operator $R(f)$ should be given by the integral of $K_f(x, x)$ over $x \in H(F)\backslash G(F)$. However, neither of these two expressions are well-defined in general. The main result of [4] is to define some canonical regularizations of the integral of K_f over the diagonal for certain triples (G, H, ξ) (essentially associated to symmetric varieties that we name "coregular") and to express the resulting distribution on $G(F)$ as a sum (or integral) of contributions naturally generalizing the weighted orbital integrals of Arthur [1]. This can be

[1]for simplicity we assume the center of G is trivial in this note

considered as the geometric side of a local trace formula for the corresponding unitary representations $L^2(H(F)\backslash G(F),\xi)$. We plan to develop in a subsequent paper a general spectral expansion for those trace formulas.

In the so-called *group-case*, corresponding to $G = H \times H$ with H embedded diagonally in the product, we recover the geometric side of Arthur local trace formula [1]. We actually also consider an enhancement of the previous setting where we fix an extra automorphism θ of the triple (G,H,ξ) and we formally try to compute the trace of the composition $R(f)\circ\theta$. This can be more naturally formulated using the notion of twisted spaces due to Labesse [6]. In the group-case again, we recover the geometric side of the local twisted trace formula due to Waldspurger [8].

2. The trace formula

2.1. The coregular variety.

Definition 2.1. *Let $X = H\backslash G$ be a homogeneous G-variety with H reductive. We say that X is* ***coregular*** *if there exists an non-empty open subset $U \subset X \times X$ such that for every $x \in U$, the stabilizer $G_x \subset G$ of x for the diagonal action contains regular elements.*

In Section 3 of [4], we give various alternative characterizations of coregular homogeneous G-varieties. Technically, the most important for us is the following property (where $G_{rs} \subset G$ denotes the open locus of regular semisimple elements and D^G, D^H stand for the usual Weyl discriminants):

A homogeneous G-variety $X = H\backslash G$ is coregular if and only if $H \cap G_{rs}$ is nonempty and the function $h \in H(F)\cap G_{rs}(F) \mapsto \frac{D^H(h)^2}{D^G(h)}$ is locally bounded on $H(F)$.

In this note for simplicity we will restrict to the cases when $X = H\backslash G$ is coregular and symmetric (i.e. $H = (G^\iota)^\circ$ for some involutation ι of G). We would like to point out that in [4] we actually considered a more general case which is the Whittaker induction of symmetric coregular varieties. Examples of coregular symmetric varieties are the group case (i.e. $G = H\times H$), Galois symmetric varieties (i.e. $G = Res_{E/F}H$ where E/F is a quadratic extension) or $\mathrm{Sp}_{2n}\backslash\mathrm{GL}_{2n}$. However, many other natural examples of homogeneous varieties such as $O_n\backslash\mathrm{GL}_n$, $\mathrm{GL}_n \times \mathrm{GL}_n\backslash\mathrm{GL}_{2n}$ or $\mathrm{SO}_n^{\mathrm{diag}}\backslash(\mathrm{SO}_n \times \mathrm{SO}_{n+1})$ are not coregular. Examples of Whittaker induction of coregular symmetric varieties are Shalika model or unitary Shalika model.

2.2. **Truncation.** Fix a triple (G, H, ξ) as in the previous subsection. The first step is to truncate the (usually not convergent) integral

$$I(f) = \int_{H(F)\backslash G(F)} K_f(x, x)dx$$

in a meaningful way. For this we introduce a sequence of truncation functions $(\kappa_Y)_Y$ indexed by points Y in a certain affine space [2]. Roughly speaker this affine space (denoted by $\mathcal{A}_{P_0,\iota}$) is the -1-eigenspace of $\mathcal{A}_{P_0}$ under the action of ι where $\mathcal{A}_{P_0} := \mathrm{Hom}(X^*(P_0), \mathbb{R})$ and P_0 is a minimal ι-split parabolic subgroup of G.

In Section 3 of [4], we defined a map $H_X : H(F)\backslash G(F) \to \mathcal{A}_{P_0,\iota}$ which is an analogue of the natural map $G(F) \to \mathcal{A}_{P_0}^+$ induced by the Cartan decomposition in the group case. The definition of such a map is non-trivial and we have used some results of Delorme [5] about the neighborhoods at infinity for symmetric varieties. We refer the reader to Section 3 of [4] for details.

Then for $Y \in \mathcal{A}_{P_0,\iota}^+$ that is "sufficiently positive", we denote by κ_Y the characteristic function of the image in $H(F)\backslash G(F)$ of the set

$$\{x \in H(F)\backslash G(F) \mid H_X(x) \in Y + {}^+\mathcal{A}_{P_0,\iota}\}.$$

Note that the above set is a local analogue of the global Siegel domain.

Then the trace formula is to study the asymptotic behavior of the expression

$$I_Y(f) := \int_{H(F)\backslash G(F)} K_f(x, x)\kappa_Y(x)dx$$

when $Y \xrightarrow{P_0} \infty$. Note that because of the truncation function κ_Y the integrand in the above expression is compactly supported and hence the integral is well defined.

2.3. **The trace formula.** Let $\Gamma(H)$ (resp. $\Gamma_{ell}(H)$) be the set of regular semisimple (resp. regular elliptic) conjugacy classes in $H(F)$. These two sets can be naturally equipped with measures.

For $t \in \Gamma(H)$, that we identify with a representative in $H(F)$, we denote by H_t, G_t the neutral components of the centralizers of t in H, G respectively. Then, for t in general position G_t is a maximal torus of G by the coregular assumption. For $f \in C_c^\infty(G(F))$ and $Y \in \mathcal{A}_{P_0,\iota}$, define

$$J_Y(f) = \int_{\Gamma(H)} D^H(t)\xi(t)J_Y(t, f)dt$$

[2]For the definition of our truncation functions, we do not need to assume (G, H) is coregular. It works for all the symmetric varieties.

where $J_Y(t,f)$ denotes some kind of "weighted orbital integral". More precisely, $J_Y(t,.)$ is a distribution of the form

$$J_Y(t,f) = \int_{G_t(F)\backslash G(F)} f(g^{-1}tg)v_{\iota,Y}(g)dg$$

where the function $g \mapsto v_{\iota,Y}(g)$ is a certain weight function very similar to the one appearing in the definition of Arthur's weighted orbital integrals as $v_{\iota,Y}(g^{-1}tg)$ is given by the volume of the convex hull of a certain family $(-H_{\overline{Q},\iota}(g)+Y_Q)_Q$ where Q runs over the minimal ι-split parabolic subgroups of G containing t.

Remark 2.2. *In the Whitaker induced case the definition of the weight is more complicated and in that case the weighted orbital integral is at some singular semisimple conjugacy classes. We refer the reader to Section 4 of* [4] *for the definition of the weight in this case and for various properties of the singular weighted orbital integral. We just wanted to emphasize one important property we proved in Theorem 4.8 of* [4] *which states that the singular weighted orbital integral of a matrix coefficient of a discrete series is equal to the regular germ of the Harish-Chandra character of that discrete series.*

Theorem 2.3. *(Theorem 6.5 of* [4]*) Let $0 < \epsilon < 1$ and fix $f \in C_c^\infty(G(F))$. Then, for any $k > 0$, we have*

$$|I_Y(f) - J_Y(f)| \ll N(Y)^{-k}$$

for every $Y \in \mathcal{A}_{P_{0,\iota}}$ with $d(Y) > \epsilon N(Y)$ [3]*. Moreover, the function $Y \in \mathcal{A}_{P_{0,\iota}} \mapsto J_Y(f)$ is a polynomial-exponential function in a suitable sense (see Section 2.9 of* [4]*). If the variety $X = H\backslash G$ is tempered (see Section 3.2 of* [4]*), then the same statement holds for functions f in the Harish-Chandra Schwartz space $\mathcal{C}(G(F))$.*

In the group case the above theorem recovers the geometric side of the local trace formula proved by Arthur in [1]. Also as we mentioned in the introduction, in [4] we actually considered the case of twisted space. Under the twisted setting, in the group case the above theorem recovers the geometric side of the local twisted trace formula proved by Waldspurger in [8]. Finally our results also include the Whittaker induced case.

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[3]Here $d(Y)$ (resp. $N(Y)$) is the depth (resp. norm) of Y defined in Section 3.8 of [4]

3. Applications

3.1. **A simple local trace formula and the multiplicity formula.** Most applications of our trace formula comes from a simple version obtained by specializing it to the case of *strongly cuspidal* test functions. More precisely, we recall following [10] that a function $f \in C_c^\infty(G(F))$ is said to be *strongly cuspidal* if for every proper parabolic subgroup $Q = LV \subset G$ we have

$$\int_{V(F)} f(lu)du = 0, \quad \text{for every } l \in L(F).$$

It is then shown in *loc. cit.* that the regular semisimple weighted orbital integrals of a strongly cuspidal function f don't depend on any choice (except that of a Haar measure on $G(F)$) and that, correctly normalized by certain signs, they define a function

$$\Theta_f : G_{rs}(F) \to \mathbb{C}$$

which is $G(F)$-invariant by conjugation and a *quasi-character*. For every strongly cuspidal test function $f \in C_c^\infty(G(F))$ we set

$$I_{geom}(f) := \int_{\Gamma_{ell}(H)} D^H(t)\Theta_f(t)\xi(t)dt.$$

Theorem 3.1. *(Theorem 7.2 of* [4]*) Let $f \in C_c^\infty(G(F))$ be a strongly cuspidal function. Then we have*

$$\lim_{Y \xrightarrow{P_0} \infty} I_Y(f) = I_{geom}(f),$$

in particular the limit exists. Furthermore, if the pair (G, H) is tempered then the same holds for strongly cuspidal test functions $f \in \mathcal{C}(G(F))$.

As a corollary of the above theorem we can also obtain general integral formulas for the multiplicities

$$m_{H,\xi^{-1}}(\pi) = \dim(\mathrm{Hom}_{H(F)}(\pi, \xi^{-1})).$$

More precisely, for π an irreducible representation of $G(F)$, let Θ_π be the Harish-Chandra character. We can define an expression $m_{geom,H}(\pi)$ similar to $I_{geom}(f)$ by formally replacing Θ_f by Θ_π. Then, we have the following.

Theorem 3.2. *(Theorem 7.4 of* [4]*) Assume that π is supercuspidal and the multiplicity $m_{H,\xi^{-1}}(\pi)$ is finite. Then, we have*

$$m_{H,\xi^{-1}}(\pi) = m_{geom,H}(\pi). \tag{3.1}$$

If the pair (G,H) is tempered, π is square-integrable and the multiplicity $m_{H,\xi}(\pi)$ is finite, then the equality (3.1) *also holds.*

In the case of Galois models or the Shalika model, the above corollary recovers one of the main results in [2] and [3] respectively. Actually for Galois models associated to classical groups, we can also deduce new results from the analog of the above corollary in certain twisted situations as explained in more details below.

3.2. **The local multiplicity problem for the Galois models.** Let E/F be a quadratic extension, H be a reductive group defined over F, χ be a character of $H(F)$ and $G = Res_{E/F}H_E$. The model (G,H,χ) is the so-called Galois model. In [9], Prasad made a general conjectural regarding the multiplicity of Galois model. In this paper, we will study the case when H is a classical group.

Let H be a quasi-split special orthogonal group or a symplectic group and $G = Res_{E/F}H_E$. If H is the even special orthogonal group, let H_0 be a quasi-split special orthogonal group that is not a pure inner form of H and such that $G = Res_{E/F}H_E = Res_{E/F}H_{0,E}$ (i.e. the determinants of the quadratic forms defining H and H_0 belong to the same square class in $E^\times/(E^\times)^2$ but belong to different square classes in $F^\times/(F^\times)^2$). If $H = \mathrm{Sp}_{2n}$ or SO_{2n}, let χ be the trivial character on H (and H_0 if $H = \mathrm{SO}_{2n}$). If $H = \mathrm{SO}_{2n+1}$, let $\chi \in \{1, \eta_n\}$ where η_n is the composition of the Spin norm character of SO_{2n+1} with the quadratic character $\eta_{E/F}$.

Our first result is a necessary condition for a discrete L-packet to be distinguished.

Theorem 3.3. *(Theorem 9.2 of* [4]*) Let $H = \mathrm{Sp}_{2n}, \mathrm{SO}_{2n}$ or SO_{2n+1}, $G = Res_{E/F}H$, $\chi = 1$ if $H = \mathrm{Sp}_{2n}$ or SO_{2n}, and $\chi \in \{1, \eta_n\}$ if $H = \mathrm{SO}_{2n+1}$. Let $\Pi_\phi(G)$ be a discrete L-packet of $G(F)$ and $\Pi_\phi(G')$ be the endoscopic transfer of the L-packet to the general linear group $G' = \mathrm{GL}_a(E)$ (here $a = 2n$ if $H = \mathrm{SO}_{2n}$ or SO_{2n+1} and $a = 2n+1$ if $H = \mathrm{Sp}_{2n}$). Then the packet $\Pi_\phi(G)$ is distinguished (i.e. $m(\pi,\chi) \neq 0$ for some $\pi \in \Pi_\phi(G)$) only if $\Pi_\phi(G')$ is (H',χ')-distinguished. Here $H' = \mathrm{GL}_a(F)$, $\chi' = 1$ if $\chi = 1$ and $\chi' = \eta_n' := \eta_{E/F} \circ \det$ if $\chi = \eta_n$.*

Our second result is to compute the summation of the multiplicities over certain discrete L-packets. Assume that $\Pi_\phi(G')$ is $(\mathrm{GL}_a(F),\chi')$-distinguished. By Theorem 4.2 of [7], $\Pi_\phi(G')$ is of the form

$$\Pi_\phi(G') = (\tau_1 \times \cdots \times \tau_l) \times (\sigma_1 \times \bar{\sigma}_1) \times \cdots \times (\sigma_m \times \bar{\sigma}_m)$$

where

- τ_i is a discrete series of $\mathrm{GL}_{a_i}(E)$ that is conjugate self-dual. Moreover, if $(H,\chi)=(\mathrm{SO}_{2n+1},\eta_n)$, τ_i is self-dual of symplectic type; otherwise, τ_i is self-dual of orthogonal type.
- σ_j is a discrete series of $\mathrm{GL}_{b_i}(E)$ that is NOT conjugate self-dual. Moreover, if $(H,\chi)=(\mathrm{SO}_{2n+1},\eta_n)$, σ_j is self-dual of symplectic type; otherwise, σ_j is self-dual of orthogonal type.
- τ_i,σ_j are all distinct.
- $\sum_{i=1}^{l} a_i+2\sum_{j=1}^{m} b_j=a$.

We will consider the special case when $m=0$. The general case will be consider in our future paper. When $m=0$, $\Pi_\phi(G')$ appears discretely in the L^2 space of the Galois model $(\mathrm{GL}_a(E),\mathrm{GL}_a(F),\chi')$.

Theorem 3.4. *(Theorem 9.3 of* [4]*) With the notation above, if H is the symplectic group or the odd special orthogonal group, we have*

$$\sum_{\pi\in\Pi_\phi(G)} m(\pi,\chi)=2^{l-1}.$$

If H is the even special orthogonal group, we let H_0 be another even special orthogonal group as above. We use $m_0(\pi,\chi)$ to denote the multiplicity for the model (G,H_0,χ). Then we have

$$\sum_{\pi\in\Pi_\phi(G)} m(\pi,\chi)+m_0(\pi,\chi)=2^{l-1}.$$

Remark 3.5. *By Theorem 1 of* [2]*, the above two theorems also hold if we replace H (and H_0 if we are in the even orthogonal group case) by the non quasi-split classical group.*

We also proved similar results for the unitary Shalika model. We refer the reader to Theorem 8.3, 8.4 and 8.7 of [4] for details. The idea of the proof is to compare the local trace formula for the pair (G,H) with the local twisted trace formula for the pair (G',H'), and then apply the theory of twisted endoscopy.

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