A Gentzen-style Calculus for Substructural Logics with Axioms W and S *

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1 Introduction

Substructural logics are logics obtained from sequent calculus **LK** and **LJ** as a result of removing structural rules such as weakening, contraction, and exchange rules, and sometimes adding further axioms. As an important subclass of substructural logics, Lambek calculus, BCK logic, relevant logics, many-valued logics, fuzzy logics, and linear logic exist. Basic literature on substructural logics include books by Bimbó [1], Cintula and Noguera [3], Galatos et al. [4], Ono [6], Paoli [7], Restall [8], and Schroder-Heister and Došen [10].

Formulas $[\alpha \to (\alpha \to \beta)] \to (\alpha \to \beta)$ and $[\alpha \to (\beta \to \delta)] \to [(\alpha \to \beta) \to (\alpha \to \delta)]$ are sometimes called W and S, respectively, and they are related to contraction rule in sequent calculus **LK** and **LJ**. In substructural logics without exchange, two types of implications, denoted by $\backslash$ and $/$ in accordance with [4], must be considered. Therefore, several variations of W and S exist in which each $\to$ is replaced by $\backslash$ or $/$ in such logics.

In this paper, we discuss a Gentzen-style calculus for substructural logics with variations of W and S without assuming both associativity and exchange. Related studies are as follows: *non-associative* substructural logics were partially discussed in [1, 3, 7, 8, 10]. Galatos and Ono [5] introduced Gentzen and Hilbert-style calculi as a non-associative version of the full Lambek calculus **FL** and discussed algebraizable logic. Seki [9] introduced a slightly different Hilbert-style calculus and discussed metacompleteness. In these studies, variations of W and S were not fully considered.

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This paper is structured as follows: In Section 2, we present the Hilbert-style formulation of the basic logic **BSL** in which we do not assume all structural rules including associativity, and variations of W and S. Roughly, **BSL**, which originates from [9], is obtained from **SL** in both [3] and [5] by removing left unitness on 1. In Section 3, we provide a Gentzen-style calculus for **BSL**, and variations of W and S using consecution calculus, and discuss their cut elimination in Section 4.

2 Logics and Variations of W and S

The language of (propositional) substructural logic consists of (i) individual variables; (ii) binary logical connectives $\wedge$ (*and*), $\vee$ (*or*), $\cdot$ (*fusion*), $\backslash$ (*left division*), and $/$ (*right division*); and (iii) constant 1. Formulas are defined in the usual manner and denoted, for example, by $\alpha, \beta, \gamma, \delta$. We adopt the convention of writing $\alpha\beta$ for $\alpha \cdot \beta$, and in the absence of parentheses, we assume that $\cdot$ is performed first. Hence, for example, $\alpha\beta\backslash\delta$ is a simplification of $(\alpha \cdot \beta)\backslash\delta$.

The logic **BSL** is defined as follows:

(a) Axioms

(A1) $\alpha\backslash\alpha$

(A2) $(\alpha \wedge \beta)\backslash\alpha$

(A3) $(\alpha \wedge \beta)\backslash\beta$

(A4) $[(\alpha\backslash\beta) \wedge (\alpha\backslash\delta)]\backslash[\alpha\backslash(\beta \wedge \delta)]$

(A5) $\alpha\backslash(\alpha \vee \beta)$

(A6) $\beta\backslash(\alpha \vee \beta)$

(A7) $[(\alpha\backslash\delta) \wedge (\beta\backslash\delta)]\backslash[(\alpha \vee \beta)\backslash\delta]$

(A8) 1

(b) Rules of inference

$$\text{(R1)}\ \frac{\alpha \quad \alpha\backslash\beta}{\beta} \qquad \text{(R2)}\ \frac{\alpha \quad \beta}{\alpha \wedge \beta} \qquad \text{(R3)}\ \frac{\alpha\backslash\beta}{(\delta\backslash\alpha)\backslash(\delta\backslash\beta)}$$

$$\text{(R4)}\ \frac{\alpha\beta\backslash\delta}{\beta\backslash(\alpha\backslash\delta)} \qquad \text{(R5)}\ \frac{\beta\backslash(\alpha\backslash\delta)}{\alpha\backslash(\delta/\beta)} \qquad \text{(R6)}\ \frac{\alpha\backslash(\delta/\beta)}{\alpha\beta\backslash\delta} \qquad \text{(R7)}\ \frac{\alpha}{1\backslash\alpha}.$$

We remark that **BSL** is the propositional fragment of **BSL** in [9].

The following (1) and (2) are proved in Proposition 2.1 in [9].

Proposition 1 (1) The following formulas are theorems of **BSL**:

$$\text{(T1)}\ \alpha\backslash[(\beta/\alpha)\backslash\beta] \qquad \text{(T2)}\ \beta\backslash(\alpha\backslash\alpha\beta) \qquad \text{(T3)}\ \alpha 1\backslash\alpha \qquad \text{(T4)}\ \alpha\backslash\alpha 1.$$

(2) The following rules of inference are derivable in **BSL**:

$$\text{(Q1)}\ \frac{\alpha\backslash\beta \quad \beta\backslash\delta}{\alpha\backslash\delta} \qquad \text{(Q2)}\ \frac{\beta\backslash(\alpha\backslash\delta)}{\alpha\beta\backslash\delta} \qquad \text{(Q3)}\ \frac{\alpha\backslash(\delta/\beta)}{\beta\backslash(\alpha\backslash\delta)}$$

$$\text{(Q4)}\ \frac{\alpha\beta\backslash\delta}{\alpha\backslash(\delta/\beta)} \qquad \text{(Q5)}\ \frac{\alpha\backslash\beta}{\alpha\delta\backslash\beta\delta} \qquad \text{(Q6)}\ \frac{\alpha\backslash\beta}{\delta\alpha\backslash\delta\beta}$$

$$\text{(Q7)}\ \frac{\alpha\backslash\beta}{(\beta\backslash\delta)\backslash(\alpha\backslash\delta)} \qquad \text{(Q8)}\ \frac{\alpha\backslash\beta}{(\alpha/\delta)\backslash(\beta/\delta)} \qquad \text{(Q9)}\ \frac{\alpha\backslash\beta}{(\delta/\beta)\backslash(\delta/\alpha)}.$$

We consider extensions of **BSL** by adding the following variations (W1)–(W8) of W and (S1)–(S32) of S as axioms.

(W1)	$[\alpha\backslash(\alpha\backslash\beta)]\backslash(\alpha\backslash\beta)$	(W2)	$[\alpha\backslash(\alpha\backslash\beta)]\backslash(\beta/\alpha)$
(W3)	$[\alpha\backslash(\beta/\alpha)]\backslash(\alpha\backslash\beta)$	(W4)	$[\alpha\backslash(\beta/\alpha)]\backslash(\beta/\alpha)$
(W5)	$[(\alpha\backslash\beta)/\alpha]\backslash(\alpha\backslash\beta)$	(W6)	$[(\alpha\backslash\beta)/\alpha]\backslash(\beta/\alpha)$
(W7)	$[(\beta/\alpha)/\alpha]\backslash(\alpha\backslash\beta)$	(W8)	$[(\beta/\alpha)/\alpha]\backslash(\beta/\alpha)$
(S1)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma)]$	(S2)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\alpha\backslash\gamma)/(\alpha\backslash\beta)]$
(S3)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\alpha\backslash\beta)\backslash(\gamma/\alpha)]$	(S4)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\gamma/\alpha)/(\alpha\backslash\beta)]$
(S5)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\beta/\alpha)\backslash(\alpha\backslash\gamma)]$	(S6)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\alpha\backslash\gamma)/(\beta/\alpha)]$
(S7)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\beta/\alpha)\backslash(\gamma/\alpha)]$	(S8)	$[\alpha\backslash(\beta\backslash\gamma)]\backslash[(\gamma/\alpha)/(\beta/\alpha)]$
(S9)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\alpha\backslash\gamma)/(\beta/\alpha)]$	(S10)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\beta/\alpha)\backslash(\alpha\backslash\gamma)]$
(S11)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\gamma/\alpha)/(\beta/\alpha)]$	(S12)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\beta/\alpha)\backslash(\gamma/\alpha)]$
(S13)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\beta/\alpha)\backslash(\alpha\backslash\gamma)]$	(S14)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\alpha\backslash\gamma)/(\beta/\alpha)]$
(S15)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\beta/\alpha)\backslash(\gamma/\alpha)]$	(S16)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\gamma/\alpha)/(\beta/\alpha)]$
(S17)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\alpha\backslash\gamma)/(\alpha\backslash\beta)]$	(S18)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma)]$
(S19)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\gamma/\alpha)/(\alpha\backslash\beta)]$	(S20)	$[\alpha\backslash(\gamma/\beta)]\backslash[(\alpha\backslash\beta)\backslash(\gamma/\alpha)]$
(S21)	$[(\gamma/\beta)/\alpha]\backslash[(\alpha\backslash\gamma)/(\alpha\backslash\beta)]$	(S22)	$[(\gamma/\beta)/\alpha]\backslash[(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma)]$
(S23)	$[(\gamma/\beta)/\alpha]\backslash[(\gamma/\alpha)/(\alpha\backslash\beta)]$	(S24)	$[(\gamma/\beta)/\alpha]\backslash[(\alpha\backslash\beta)\backslash(\gamma/\alpha)]$
(S25)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma)]$	(S26)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\alpha\backslash\gamma)/(\alpha\backslash\beta)]$
(S27)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\alpha\backslash\beta)\backslash(\gamma/\alpha)]$	(S28)	$[(\beta\backslash\gamma)/\alpha]\backslash[(\gamma/\alpha)/(\alpha\backslash\beta)]$
(S29)	$[(\gamma/\beta)/\alpha]\backslash[(\alpha\backslash\gamma)/(\beta/\alpha)]$	(S30)	$[(\gamma/\beta)/\alpha]\backslash[(\beta/\alpha)\backslash(\alpha\backslash\gamma)]$
(S31)	$[(\gamma/\beta)/\alpha]\backslash[(\gamma/\alpha)/(\beta/\alpha)]$	(S32)	$[(\gamma/\beta)/\alpha]\backslash[(\beta/\alpha)\backslash(\gamma/\alpha)]$

Let X be a subset of $\{(\mathrm{W1}), \cdots, (\mathrm{W8}), (\mathrm{S1}), \cdots, (\mathrm{S32})\}$. Then $\mathbf{BSL} + X$ denotes the logic we obtain from **BSL** by adding all elements of X as axioms. In particular, if X is a singleton set, we write $\mathbf{BSL} + (\mathrm{W1})$ instead of $\mathbf{BSL} + \{(\mathrm{W1})\}$, for example.

3 Consecution Calculus

Gentzen-style calculus for non-associative substructural logics that uses the usual sequent calculus such as **LJ** and **FL** is impossible, but consecution calculus makes it possible. For general information on consecution calculus, see [1]. This paper uses the notation in [2].

To introduce consecution calculus, we prepare some notions and notation. *Structures* are defined as follows:

(i) A formula α is a structure.

(ii) If A and B are structures, then $(A; B)$ is also a structure.

We remark that empty structures are *not* allowed. For a structure A and formula α, an expression of the form '$A \Rightarrow \alpha$' is called a *consecution*.

We use Greek capital letters Γ and Δ for strings of symbols formed by truncating a structure. These strings of symbols can be empty, a well-formed structure or a part of a structure which is not well-formed, but would become well-formed if further symbols are added to it.

The consecution calculus **LBSL1** is defined as follows, where A and B denote structures:

(a) Axiom $\quad \alpha \Rightarrow \alpha$

(b) Cut rule

$$\frac{A \Rightarrow \gamma \quad \Gamma\gamma\Delta \Rightarrow \delta}{\Gamma A \Delta \Rightarrow \delta}(\mathrm{cut})$$

(c) Rules for logical connectives and the constant

$$\frac{\Gamma\alpha\Delta \Rightarrow \delta}{\Gamma\alpha \wedge \beta\Delta \Rightarrow \delta}(\wedge \Rightarrow) \quad \frac{\Gamma\beta\Delta \Rightarrow \delta}{\Gamma\alpha \wedge \beta\Delta \Rightarrow \delta}(\wedge \Rightarrow) \quad \frac{A \Rightarrow \alpha \quad A \Rightarrow \beta}{A \Rightarrow \alpha \wedge \beta}(\Rightarrow \wedge)$$

$$\frac{\Gamma\alpha\Delta \Rightarrow \delta \quad \Gamma\beta\Delta \Rightarrow \delta}{\Gamma\alpha \vee \beta\Delta \Rightarrow \delta}(\vee \Rightarrow) \quad \frac{A \Rightarrow \alpha}{A \Rightarrow \alpha \vee \beta}(\Rightarrow \vee) \quad \frac{A \Rightarrow \beta}{A \Rightarrow \alpha \vee \beta}(\Rightarrow \vee)$$

$$\frac{A \Rightarrow \alpha \quad \Gamma\beta\Delta \Rightarrow \delta}{\Gamma(A; \alpha\backslash\beta)\Delta \Rightarrow \delta}(\backslash \Rightarrow) \quad \frac{(\alpha; A) \Rightarrow \beta}{A \Rightarrow \alpha\backslash\beta}(\Rightarrow \backslash)$$

$$\frac{A \Rightarrow \alpha \quad \Gamma\beta\Delta \Rightarrow \delta}{\Gamma(\beta/\alpha; A)\Delta \Rightarrow \delta}(/ \Rightarrow) \quad \frac{(A; \alpha) \Rightarrow \beta}{A \Rightarrow \beta/\alpha}(\Rightarrow /)$$

$$\frac{\Gamma(\alpha; \beta)\Delta \Rightarrow \delta}{\Gamma\alpha \cdot \beta\Delta \Rightarrow \delta}(\cdot \Rightarrow) \quad \frac{A \Rightarrow \alpha \quad B \Rightarrow \beta}{(A; B) \Rightarrow \alpha \cdot \beta}(\Rightarrow \cdot)$$

$$\frac{\Gamma A\Delta \Rightarrow \delta}{\Gamma(A;1)\Delta \Rightarrow \delta}(1\mathrm{I}) \quad \frac{\Gamma(A;1)\Delta \Rightarrow \delta}{\Gamma A\Delta \Rightarrow \delta}(1\mathrm{E})$$

Lemma 2 If a formula α is a theorem of **BSL**, then a consecution $1 \Rightarrow \alpha$ is provable in **LBSL1**.

Proof. We prove the lemma by induction on the length of the proof in **BSL**. As an example, we present the case (R6).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1 \Rightarrow \alpha\backslash(\delta/\beta) \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\delta \Rightarrow \delta \qquad \beta \Rightarrow \beta}{(\delta/\beta;\beta) \Rightarrow \delta}(/\Rightarrow)}{((\alpha;\alpha\backslash(\delta/\beta));\beta) \Rightarrow \delta}(\backslash\Rightarrow)}{((\alpha;1);\beta) \Rightarrow \delta}(\mathrm{cut})}{(\alpha;\beta) \Rightarrow \delta}(1\mathrm{E})}{\alpha\beta \Rightarrow \delta}(\cdot\Rightarrow)}{(\alpha\beta;1) \Rightarrow \delta}(1\mathrm{I})$$
$$\dfrac{(\alpha\beta;1) \Rightarrow \delta}{1 \Rightarrow \alpha\beta\backslash\delta}(\Rightarrow\backslash)$$
■

To relate consecutions to formulas, we provide *interpretations* I for consecutions in terms of formulas as follows:

(i) For formulas α, $I(\alpha) = \alpha$.
(ii) For structures $(A;B)$, $I((A;B)) = I(A) \cdot I(B)$.
(iii) For structures A and formulas α, $I(A \Rightarrow \alpha) = I(A)\backslash I(\alpha)$.

Then we can prove the following by induction on the length of the proof in **LBSL1**.

Lemma 3 If a consecution $A \Rightarrow \alpha$ is provable in **LBSL1**, then $I(A \Rightarrow \alpha)$ is a theorem of **BSL**.

Thus, we have the following.

Theorem 4 A formula α is a theorem of **BSL** if and only if a consecution $1 \Rightarrow \alpha$ is provable in **LBSL1**.

We consider corresponding rules for (W1)–(W8) and (S1)–(S32). Let R be a set of rules $\{(\mathrm{r}_1), \cdots, (\mathrm{r}_n)\}$. Then **LBSL1** $+ R$ denotes the calculus obtained from **LBSL1** by adding all rules that belong to R. In particular, if R is a singleton $\{(\mathrm{r})\}$, we write **LBSL1**+(r) instead of **LBSL1** $+ \{(\mathrm{r})\}$.

Rules (w1)–(w8) are defined as follows:

$$\frac{\Gamma(A;(A;B))\Delta \Rightarrow \delta}{\Gamma(A;B)\Delta \Rightarrow \delta}\,(\mathrm{w1}) \qquad \frac{\Gamma(A;(A;B))\Delta \Rightarrow \delta}{\Gamma(B;A)\Delta \Rightarrow \delta}\,(\mathrm{w2})$$

$$\frac{\Gamma((A;B);A)\Delta \Rightarrow \delta}{\Gamma(A;B)\Delta \Rightarrow \delta}\,(\mathrm{w3}) \qquad \frac{\Gamma((A;B);A)\Delta \Rightarrow \delta}{\Gamma(B;A)\Delta \Rightarrow \delta}\,(\mathrm{w4})$$

$$\frac{\Gamma(A;(B;A))\Delta \Rightarrow \delta}{\Gamma(A;B)\Delta \Rightarrow \delta}\,(\mathrm{w5}) \qquad \frac{\Gamma(A;(B;A))\Delta \Rightarrow \delta}{\Gamma(B;A)\Delta \Rightarrow \delta}\,(\mathrm{w6})$$

$$\frac{\Gamma((B;A);A)\Delta \Rightarrow \delta}{\Gamma(A;B)\Delta \Rightarrow \delta}\,(\mathrm{w7}) \qquad \frac{\Gamma((B;A);A)\Delta \Rightarrow \delta}{\Gamma(B;A)\Delta \Rightarrow \delta}\,(\mathrm{w8}).$$

Theorem 5 For $i = 1, \cdots, 8$, a formula α is a theorem of $\mathbf{BSL} + (\mathrm{W}i)$ if and only if a consecution $1 \Rightarrow \alpha$ is provable in $\mathbf{LBSL1} + (\mathrm{w}i)$.

Proof. We prove the case $i = 1$. For 'if' part:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{(\alpha;\alpha\backslash\beta) \Rightarrow \beta}\,(\backslash\Rightarrow)}{(\alpha;(\alpha;\alpha\backslash(\alpha\backslash\beta))) \Rightarrow \beta}\,(\backslash\Rightarrow)}{(\alpha;\alpha\backslash(\alpha\backslash\beta)) \Rightarrow \beta}\,(\mathrm{w1})}{\alpha\backslash(\alpha\backslash\beta) \Rightarrow \alpha\backslash\beta}\,(\Rightarrow\backslash)}{(\alpha\backslash(\alpha\backslash\beta);1) \Rightarrow \alpha\backslash\beta}\,(1\mathrm{I})}{1 \Rightarrow [\alpha\backslash(\alpha\backslash\beta)]\backslash(\alpha\backslash\beta)}\,(\Rightarrow\backslash)$$

For 'only if' part, we show only the simplest case, where α, β, δ denote $I(A), I(B), I(\delta)$, respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha(\alpha\beta)\backslash\delta}{\alpha\beta\backslash(\alpha\backslash\delta)}\,(\mathrm{R4})}{\beta\backslash[\alpha\backslash(\alpha\backslash\delta)]}\,(\mathrm{R4}) \qquad \dfrac{(\mathrm{W1})}{[\alpha\backslash(\alpha\backslash\delta)]\backslash(\alpha\backslash\delta)}}{\beta\backslash(\alpha\backslash\delta)}\,(\mathrm{Q1}).}{\alpha\beta\backslash\delta}\,(\mathrm{Q2}).$$

■

Rules (s1)–(s16) are defined as follows.

$$\frac{\Gamma((A;B);(A;C))\Delta \Rightarrow \delta}{\Gamma(A;(B;C))\Delta \Rightarrow \delta}\,(\mathrm{s1}) \qquad \frac{\Gamma((A;B);(A;C))\Delta \Rightarrow \delta}{\Gamma(A;(C;B))\Delta \Rightarrow \delta}\,(\mathrm{s2})$$

$$\frac{\Gamma((A;B);(A;C))\Delta \Rightarrow \delta}{\Gamma((B;C);A)\Delta \Rightarrow \delta}\,(\mathrm{s3}) \qquad \frac{\Gamma((A;B);(A;C))\Delta \Rightarrow \delta}{\Gamma((C;B);A)\Delta \Rightarrow \delta}\,(\mathrm{s4})$$

$$\frac{\Gamma((B;A);(A;C))\Delta \Rightarrow \delta}{\Gamma(A;(B;C))\Delta \Rightarrow \delta}\ (\text{s5}) \qquad \frac{\Gamma((B;A);(A;C))\Delta \Rightarrow \delta}{\Gamma(A;(C;B))\Delta \Rightarrow \delta}\ (\text{s6})$$

$$\frac{\Gamma((B;A);(A;C))\Delta \Rightarrow \delta}{\Gamma((B;C);A)\Delta \Rightarrow \delta}\ (\text{s7}) \qquad \frac{\Gamma((B;A);(A;C))\Delta \Rightarrow \delta}{\Gamma((C;B);A)\Delta \Rightarrow \delta}\ (\text{s8})$$

$$\frac{\Gamma((A;B);(C;A))\Delta \Rightarrow \delta}{\Gamma(A;(B;C))\Delta \Rightarrow \delta}\ (\text{s9}) \qquad \frac{\Gamma((A;B);(C;A))\Delta \Rightarrow \delta}{\Gamma(A;(C;B))\Delta \Rightarrow \delta}\ (\text{s10})$$

$$\frac{\Gamma((A;B);(C;A))\Delta \Rightarrow \delta}{\Gamma((B;C);A)\Delta \Rightarrow \delta}\ (\text{s11}) \qquad \frac{\Gamma((A;B);(C;A))\Delta \Rightarrow \delta}{\Gamma((C;B);A)\Delta \Rightarrow \delta}\ (\text{s12})$$

$$\frac{\Gamma((B;A);(C;A))\Delta \Rightarrow \delta}{\Gamma(A;(B;C))\Delta \Rightarrow \delta}\ (\text{s13}) \qquad \frac{\Gamma((B;A);(C;A))\Delta \Rightarrow \delta}{\Gamma(A;(C;B))\Delta \Rightarrow \delta}\ (\text{s14})$$

$$\frac{\Gamma((B;A);(C;A))\Delta \Rightarrow \delta}{\Gamma((B;C);A)\Delta \Rightarrow \delta}\ (\text{s15}) \qquad \frac{\Gamma((B;A);(C;A))\Delta \Rightarrow \delta}{\Gamma((C;B);A)\Delta \Rightarrow \delta}\ (\text{s16}).$$

Theorem 6 For $i = 1, \cdots, 16$, the following are equivalent:

(i) A formula α is a theorem of $\mathbf{BSL} + (\mathrm{S}i)$.

(ii) A formula α is a theorem of $\mathbf{BSL} + (\mathrm{S}j)$, for $j = i + 16$.

(iii) A consecution $1 \Rightarrow \alpha$ is provable in $\mathbf{LBSL1} + (\mathrm{s}i)$.

Proof. We prove the case $i = 1$. From (i) to (iii):

$$\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\beta \Rightarrow \beta \qquad \gamma \Rightarrow \gamma}{(\beta;\beta\backslash\gamma) \Rightarrow \gamma}\,(\backslash \Rightarrow)}{(\beta;(\alpha;\alpha\backslash(\beta\backslash\gamma))) \Rightarrow \gamma}\,(\backslash \Rightarrow)}{\dfrac{((\alpha;\alpha\backslash\beta);(\alpha;\alpha\backslash(\beta\backslash\gamma))) \Rightarrow \gamma}{\dfrac{(\alpha;(\alpha\backslash\beta;\alpha\backslash(\beta\backslash\gamma))) \Rightarrow \gamma}{\dfrac{(\alpha\backslash\beta;\alpha\backslash(\beta\backslash\gamma)) \Rightarrow \alpha\backslash\gamma}{\dfrac{\alpha\backslash(\beta\backslash\gamma) \Rightarrow (\alpha\backslash\beta)\backslash(\alpha\backslash\gamma)}{\dfrac{(\alpha\backslash(\beta\backslash\gamma);1) \Rightarrow (\alpha\backslash\beta)\backslash(\alpha\backslash\gamma)}{1 \Rightarrow [\alpha\backslash(\beta\backslash\gamma)]\backslash[(\alpha\backslash\beta)\backslash(\alpha\backslash\gamma)]}\,(\Rightarrow \backslash).}\,(1\mathrm{I})}\,(\Rightarrow \backslash)}\,(\Rightarrow \backslash)}\,(\text{s1})}\,(\backslash \Rightarrow)$$

From (ii) to (iii) is similar.

From (iii) to (i), we show only the simplest case, where $\alpha, \beta, \gamma, \delta$ denote $I(A), I(B), I(C), I(\delta)$, respectively:

$$\dfrac{\dfrac{\dfrac{(\alpha\beta)(\alpha\gamma)\backslash\delta}{\alpha\gamma\backslash(\alpha\beta\backslash\delta)}\,(\mathrm{R4})}{\gamma\backslash[\alpha\backslash(\alpha\beta\backslash\delta)]}\,(\mathrm{R4}) \qquad \dfrac{\overset{(\mathrm{S1})}{[\alpha\backslash(\alpha\beta\backslash\delta)]\backslash[(\alpha\backslash\alpha\beta)\backslash(\alpha\backslash\delta)]} \quad \dfrac{\overset{(\mathrm{T2})}{\beta\backslash(\alpha\backslash\alpha\beta)}}{[(\alpha\backslash\alpha\beta)\backslash(\alpha\backslash\delta)]\backslash[\beta\backslash(\alpha\backslash\delta)]}\,(\mathrm{Q8})}{[\alpha\backslash(\alpha\beta\backslash\delta)]\backslash[(\alpha\backslash\alpha\beta)\backslash(\alpha\backslash\delta)]}\,(\mathrm{Q1})}{\dfrac{\dfrac{\gamma\backslash[\beta\backslash(\alpha\backslash\delta)]}{\beta\gamma\backslash(\alpha\backslash\delta)}\,(\mathrm{Q3})}{\alpha(\beta\gamma)\backslash\delta}\,(\mathrm{Q3}).}\,(\mathrm{Q1})$$

From (iii) to (ii) is similar. ■

4 Cut Elimination Theorem

We consider the cut elimination theorem for the calculi **LBSL1** with at least one of (w1)–(w8) and (s1)–(s16). Basically, we can prove it by following the method described in [1, 4].

The cut elimination theorem for **LBSL1**, in which neither (w1)–(w8) nor (s1)–(s16) contains, is proved in a well-known manner, namely by double induction on the degree and rank; however, it is not necessary to introduce alternative rules, such as the mix rule. Roughly speaking, we eliminate cut, which is one of the uppermost applications of cut in the proof, by either pushing the cut up (reducing the rank) or replacing the cut formula with a simpler formula (reducing the degree).

We prove the cut elimination theorem for the calculus containing (w1)–(w8) or (s1)–(s16) by eliminating the following *multiple cut* rule, abbreviated by *m-cut*, which is essentially the same as the cut rule:

$$\dfrac{A \Rightarrow \gamma \quad B\langle\gamma\rangle \Rightarrow \delta}{B\langle A\rangle \Rightarrow \delta}\,(\text{m-cut}).$$

The notation $\langle\rangle$ indicates *at least one*, but possibly *several* occurrences of the formula γ being selected in the structure B, each of which is replaced by A in the lower consecution.

To prove m-cut elimination, to enable the tracking of each occurrence of the formulas in a consecution in the proof, we use *parametric ancestors* as in [1]. The detailed definition is as follows:

An occurrence of α in the upper consecution of a rule is the *immediate ancestor* of an occurrence of α in the lower consecution of a rule according to one of (1)–(7), as appropriate.

(1) In the rules $(\wedge \Rightarrow)$, $(\vee \Rightarrow)$ and $(\cdot \Rightarrow)$, the elements of Γ, Δ and the δ in the upper consecutions are immediate ancestors of the matching formulas in the lower consecution;

(2) In the rules $(\Rightarrow \wedge)$, $(\Rightarrow \vee)$, $(\Rightarrow \backslash)$, and $(\Rightarrow /)$, the elements of A in the upper consecutions are immediate ancestors of the matching formulas in the lower consecution;

(3) In the rules $(\backslash \Rightarrow)$, $(/ \Rightarrow)$ and (1I), the elements of A, Γ, Δ and the δ in the upper consecutions are immediate ancestors of the matching formulas in the lower consecution;

(4) In the rule $(\Rightarrow \cdot)$, the elements of A and B in the upper consecutions are immediate ancestors of the matching formulas in the lower consecution;

(5) In the rule (1E), the elements of Γ, A, Δ and the 1 and δ in the upper consecutions are immediate ancestors of the matching formulas in the lower consecution;

(6) In the rules (w1)–(w8), the elements of Γ, Δ, two A's, B and the δ in the upper consecutions are immediate ancestors of the matching elements of Γ, Δ, A, B in the lower consecution;

(7) In the rules (s1)–(s16), the elements of Γ, Δ, two A's, B, C and the δ in the upper consecutions are immediate ancestors of the matching elements of Γ, Δ, A, B, C in the lower consecution.

The *parametric ancestor* relation is the transitive closure of the immediate ancestor relation.

Thus, parametric ancestors of an occurrence of α in a consecution are intuitively all matching occurrences of α in its upper consecutions in the proof. Therefore, the principal formula has no parametric ancestors.

We slightly modify the definition of the rank of the m-cut. In the above m-cut, (1) the *left rank* is the length of the path for consecutions with γ as the succedent of the consecution in the proof ending $A \Rightarrow \gamma$, (2) the *right rank* is the maximal length of any paths containing parametric ancestors of γ in the proof ending $B\langle\gamma\rangle \Rightarrow \delta$, and (3) the *rank* is the sum of the left rank and right rank.

Theorem 7 For any subset (including emptyset) X of $\{(\mathrm{w1}), \cdots, (\mathrm{w8}), (\mathrm{s1}), \cdots, (\mathrm{s16})\}$, the cut elimination theorem holds in $\mathbf{LBSL1} + X$.

Proof. We prove the theorem by double induction on the degree and rank. The case in which the rank is equal to 2 is proved in the usual manner. For the case in which the left rank is greater than 1, the m-cut is moved upward. Finally, we consider the case in which the right rank is greater than 1. We only show the following cases:

Case 1.

$$\dfrac{D \Rightarrow \gamma \qquad \dfrac{\Gamma A \Delta\langle\gamma\rangle \Rightarrow \delta}{\Gamma(A;1)\Delta\langle\gamma\rangle \Rightarrow \delta}\,(1\mathrm{I})}{\Gamma(A;1)\Delta\langle D\rangle \Rightarrow \delta}\,(\text{m-cut})$$

If we can write the end consecution $\Gamma(A;1)\Delta\langle D\rangle \Rightarrow \delta$ as $\Gamma(A;D)\Delta\langle D\rangle \Rightarrow \delta$, then γ must be 1 and the transform is as follows, where the ranks of both m-cuts are smaller than that of original m-cut; in particular, the rank of the second m-cut is equal to the left rank plus 1 because the second m-cut formula γ, which is 1, is the principal formula of (1I):

$$\dfrac{D \Rightarrow \gamma \qquad \dfrac{\dfrac{D \Rightarrow \gamma \qquad \Gamma A \Delta\langle\gamma\rangle \Rightarrow \delta}{\Gamma A \Delta\langle D\rangle \Rightarrow \delta}\,(\text{m-cut})}{\Gamma(A;1)\Delta\langle D\rangle \Rightarrow \delta}\,(1\mathrm{I})}{\Gamma(A;D)\Delta\langle D\rangle \Rightarrow \delta}\,(\text{m-cut});$$

otherwise, the transform is as follows, where the rank of the m-cut is smaller than that of orginal m-cut:

$$\dfrac{\dfrac{D \Rightarrow \gamma \qquad \Gamma A \Delta\langle\gamma\rangle \Rightarrow \delta}{\Gamma A \Delta\langle D\rangle \Rightarrow \delta}\,(\text{m-cut})}{\Gamma(A;1)\Delta\langle D\rangle \Rightarrow \delta}\,(1\mathrm{I}).$$

Case 2.

$$\dfrac{D \Rightarrow \gamma \qquad \dfrac{\Gamma(A;1)\Delta\langle\gamma\rangle \Rightarrow \delta}{\Gamma A \Delta\langle\gamma\rangle \Rightarrow \delta}\,(1\mathrm{E})}{\Gamma A \Delta\langle D\rangle \Rightarrow \delta}\,(\text{m-cut})$$

Then the transform is as follows, where the rank of the m-cut is smaller than that of original m-cut. We remark that the consecution $\Gamma(A;1)\Delta\langle D\rangle \Rightarrow \delta$ must not be written as $\Gamma(A;D)\Delta\langle D\rangle \Rightarrow \delta$:

$$\dfrac{\dfrac{D \Rightarrow \gamma \qquad \Gamma(A;1)\Delta\langle\gamma\rangle \Rightarrow \delta}{\Gamma(A;1)\Delta\langle D\rangle \Rightarrow \delta}\,(\text{m-cut})}{\Gamma A \Delta\langle D\rangle \Rightarrow \delta}\,(1\mathrm{E}).$$

Case 3. Let X contain (s1).

$$\dfrac{D \Rightarrow \gamma \qquad \dfrac{\Gamma((A;B);(A;C))\Delta\langle\gamma\rangle \Rightarrow \delta}{\Gamma(A;(B;C))\Delta\langle\gamma\rangle \Rightarrow \delta}\,(\mathrm{s1})}{\Gamma(A;(B;C))\Delta\langle D\rangle \Rightarrow \delta}\,(\text{m-cut})$$

Then the transform is as follows, where the rank of the m-cut is smaller than that of original m-cut.

$$\dfrac{\dfrac{D \Rightarrow \gamma \qquad \Gamma((A;B);(A;C))\Delta\langle\gamma\rangle \Rightarrow \delta}{\Gamma((A;B);(A;C))\Delta\langle D\rangle \Rightarrow \delta}\ (\text{m-cut})}{\Gamma(A;(B;C))\Delta\langle D\rangle \Rightarrow \delta}\ (\text{s1})$$

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