Transfinite iterations of Σ^1_1 inductive definitions –Reconsideration of Nemoto's Conjecture –

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Abstract

In prior research, we have examined Σ^1_1-ID_0, an axiom system of second-order arithmetic, along with its variations, to measure the determinacy strength of Σ^0_2 sets and their differences. However, the study by Tanaka [12] on the collapse phenomena of multiple inductive definitions has prompted a need for a more meticulous discussion. Therefore, we first reexamined the following conjecture proposed by Nemoto: a transfinite iteration of Σ^1_1-ID can be deduced from the determinacy of $\mathbf{\Sigma}^0_2$ (Conjecture 3.11, [6]). Nemoto [6] showed that Σ^1_1-IDTR, a full form of transfinite iterations of Σ^1_1-ID, implies the determinacy of $(\mathbf{\Sigma}^0_2)_2 \cap \neg(\mathbf{\Sigma}^0_2)_2$. While this stronger conjecture was proposed, the reversal has not been established until [2], [17], [18]. In this paper, we demonstrate that Σ^1_1-IDTR(or $\mathbf{\Delta}((\mathbf{\Sigma}^0_2)_2)$-$\mathsf{Det}$) is strictly stronger than Σ^1_1-ID (or $\mathbf{\Sigma}^0_2$-Det). Note that we primarily consider boldface assertions with set parameters.

1 Introduction

Our study concerns the determinacy of games and the axiom systems within the context of reverse mathematics. The first study on the determinacy of games in reverse mathematics was conducted by J. Steel [11], who proved the equivalence between the determinacy of open (Σ^0_1) games and ATR, an axiom system of second order arithmetic. This result is also known as one of the earliest findings in reverse mathematics.

Subsequently, research on stronger determinacy than that of open games and the corresponding axiom systems of second order arithmetic was initiated by Tanaka

[13], [15], [16]. While it is impossible to cover all such studies, notable examples include the work by MedSalem and Tanaka [4], which characterized the determinacy of Δ^0_3 games; the research by Welch [20] and Hachtman [1] on characterizing Σ^0_3 games; the study by Montalbán and Shore [5], which clarified the "limit" concerning the determinacy of games in second-order arithmetic; the work by Pacheco and Yokoyama [9] on characterizing the determinacy of arbitrary finite defferences of Σ^0_3 sets; and the analyses of determinacy in Cantor space conducted by Nemoto, Tanaka, and MedSalem [7], [8]. For more information, see also [19]

In the 2009 RIMS Kôkyûroku [6], Nemoto demonstrated that the determinacy of $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$ in the Baire space can be derived from the transfinite iteration of Σ^1_1 inductive definitions. In the 2010 RIMS Kôkyûroku [2], we reported on the equivalence of Σ^1_1-IDTR and $\Delta((\Sigma^0_2)_2)$-Det. The following year, at the Computability in Europe, we [18] presented results on determincay of the finite differences of Σ^0_2 sets and variations of Σ^1_1 inductive definitions. After a considerable period, we also contributed detailed proofs of these results in [17]. However, in 2023 [12], it was demonstrated that the hierarchy concerning multiple inductive definitions introduced in [2], and used in [18], and [17] collapses. This finding indicated the necessity for a more meticulous discussion of the inductive definitions and the corresponding hierarchy of determinacy. Therefore, we began by reconsidering the transfinite iterations of inductive definitions and the conjecture proposed by Nemoto in [6]: a transfinite iteration of Σ^1_1-ID can be deduced from the determinacy of $\boldsymbol{\Sigma}^0_2$. In considering this conjecture, we speculated that Σ^0_2 determinacy might imply $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$ determinacy (as presented in our Proof Theory Workshop 2023 at RIMS oral presentation). However, recognizing the error in this line of thought, we have detailed these issues in Appendix II.

In this paper, we observe the separation of Σ^1_1-ID_0 and Σ^1_1-IDTR_0 in Theorem 4.6. Before proving it in Chapter 4, we review the relevant definitions and results in Chapters 2 and 3. Some proofs closely related to our main theorem, which help in better understanding, are detailed in Appendix I.

2 Preliminaries

In this section, we recall some basic definitions and facts about second order arithmetic. The language $\mathcal{L}_2$ of second-order arithmetic is a two-sorted language consisting of constant symbols $0, 1, +, \cdot, =, <$ with number variables $x, y, z, \ldots$ and unary function variables $f, g, h, \ldots$. We also use set variables $X, Y, Z, \ldots$, intending to range over the $\{0,1\}$-valued functions, that is, the characteristic functions of sets.

The formulas can be classified as follows:

- φ is *bounded* (Π^0_0) if it is built up from atomic formulas by using propositional connectives and bounded number quantifiers $(\forall x < t), (\exists x < t)$, where t does not contain x.
- φ is Π^1_0 if it does not contain any function quantifier. Π^1_0-formulas are called *arithmetical* formulas.
- $\neg\varphi$ is Σ^i_n if φ is a Π^i_n-formula ($i \in \{0,1\}, n \in \omega$).
- $\forall x_1 \cdots \forall x_k \varphi$ is Π^0_{n+1} if φ is a Σ^0_n-formula ($n \in \omega$),
- $\forall f_1 \cdots \forall f_k \varphi$ is Π^1_{n+1} if φ is a Σ^1_n-formula ($n \in \omega$).

We loosely say that a formula is Σ^i_n (resp. Π^i_n) if it is equivalent over a base theory (such as ACA_0) to a $\psi \in \Sigma^i_n$ (resp. Π^i_n).

We now define some popular axiom schemata of second order arithmetic.

Definition 2.1 *Let $\mathcal{C}$ be a set of $\mathcal{L}_2$-formulas.*

(1) $\mathcal{C}$-IND*:* $(\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1))) \to \forall x \varphi(x)$,
where $\varphi(x)$ belongs to $\mathcal{C}$.

(2) $\mathcal{C}$-CA *:* $\exists X \forall x(x \in X \leftrightarrow \varphi(x))$,
where $\varphi(x)$ belongs to $\mathcal{C}$ and X does not occur freely in $\varphi(x)$.

(3) $\Delta(\mathcal{C})$-CA : $\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x))$,
where $\varphi(x)$ and $\neg\psi(x)$ belong to $\mathcal{C}$ and X does not occur freely in $\varphi(x)$.

(4) $\mathcal{C}$-AC : $\forall x \exists X \varphi(x, X) \to \exists X \forall x \varphi(x, X_x)$,
where $\varphi(x, X)$ belongs to $\mathcal{C}$ and $X_x = \{y : (x, y) \in X\}$.

The system ACA_0 consists of the ordered semiring axioms for $(\omega, +, \cdot, 0, 1, <)$, Σ^0_1-CA and Σ^0_1-IND. For a set Λ of sentences, $\mathsf{\Lambda}_0$ denotes the system consisting of ACA_0 plus Λ.

By Δ^i_n-CA, we denote $\Delta(\Sigma^i_n)$-CA. We can easily show that for any $k \geq 0$,

$$\Delta^1_k\text{-}\mathsf{CA}_0 \subset \Sigma^1_k\text{-}\mathsf{AC}_0.$$

Moreover, if $k = 2$, the above two axioms are known to be equivalent to each other.

Finally, we introduce an axiom of determinacy. For a formula φ with a distinct variable f ranging over $\mathbb{N}^\mathbb{N}$, we associate a two-person *game* G_φ (or simply denote φ) as follows: player I and player II alternately choose a natural number (starting with player I) to form an infinite sequence $f \in \mathbb{N}^\mathbb{N}$ and player I (resp. II) wins iff $\varphi(f)$ (resp. $\neg\varphi(f)$). We say that φ is *determinate* if one of the players has a *winning strategy* $\sigma : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ in the game φ. For a class $\mathcal{C}$ of formulas, $\mathcal{C}$-Det is the axiom which states that any game in $\mathcal{C}$ is determinate.

3 Difference Hierarchy

To separate the axiom system Σ^1_1-IDTR_0 from other axioms, we review some definitions and related results of the effective version of the Hausdorff-Kuratowski hierarchy, which is introduced to characterize the determinacy of Δ^0_2 games in [13]. While the proof of Theorem 3.6 is also found in [17], it is included in Appendix I for better understanding of Theorem 4.6 in this paper. See also [3], [13], [14], [17], and [18] for further works.

Let $\mathcal{C}$ be a class of formulas (or inductive operators) which closed under arithmetical quantifiers (e.g., Σ^1_n under assuming AC).

Definition 3.1 *Let $\mathcal{C}$ and $\mathcal{C}'$ be classes of formulas. By $\mathcal{C} \wedge \mathcal{C}'$, $\mathcal{C} \vee \mathcal{C}'$ and $\neg\mathcal{C}$, we denote the classes of formulas in the form $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\neg\varphi$, respectively ($\varphi \in \mathcal{C}, \psi \in \mathcal{C}'$).*

Definition 3.2 *For $k \geq 1$, let $(\mathcal{C})_k$ be the class of formulas inductively defined as follows:*

$$(\mathcal{C})_1 = \mathcal{C}, \quad (\mathcal{C})_k = \mathcal{C} \wedge \neg(\mathcal{C})_{k-1} \text{ for } k > 1.$$

Then, we easily calculate

- $(\mathcal{C})_2 = \mathcal{C} \wedge \neg\mathcal{C}, \quad \neg(\mathcal{C})_2 = \mathcal{C} \vee \neg\mathcal{C}$,
- $(\mathcal{C})_3 = \mathcal{C} \wedge \neg(\mathcal{C})_2 = \mathcal{C} \wedge (\mathcal{C} \vee \neg\mathcal{C}) = (\mathcal{C} \wedge \mathcal{C}) \vee (\mathcal{C} \wedge \neg\mathcal{C}) = \mathcal{C} \vee (\mathcal{C})_2$,
- $\neg(\mathcal{C})_3 = \neg\mathcal{C} \vee (\mathcal{C})_2$, etc.

In general, we can easily show the following.

Lemma 3.3 *For any $k \geq 1$, the following hold.*

- $(\mathcal{C})_{2k} = (\mathcal{C})_2 \vee (\mathcal{C})_2 \vee \cdots \vee (\mathcal{C})_2$ (k times),
- $(\mathcal{C})_{2k+1} = \mathcal{C} \vee (\mathcal{C})_{2k}, \quad \neg(\mathcal{C})_{2k+1} = \neg\mathcal{C} \vee (\mathcal{C})_{2k}$.

Definition 3.4 *Let $\mathcal{C}$ be a class of formulas. A $\mathcal{C}$-formula φ is also called a $\Delta(\mathcal{C})$-formula if there exists a $\neg\mathcal{C}$-formula φ' such that φ and φ' are equivalent over an appropriate system, e.g., ACA_0. In particular, we write Δ^i_n for $\Delta(\Sigma^i_n)$.*

Definition 3.5 *Let $\mathcal{C}, \mathcal{C}'$ be classes of formulas. For $\psi \in \Delta(\mathcal{C})$ and $\eta, \eta' \in \mathcal{C}'$, the following formula $\varphi(f)$ is called a $\Delta(\mathcal{C})$-separated co-difference of $\mathcal{C}'$, denoted as* $\mathsf{Sep}(\Delta(\mathcal{C}), \mathcal{C}')$*,*

$$(\psi(f) \wedge \eta(f)) \vee (\neg\psi(f) \wedge \neg\eta'(f)).$$

Next, we see the following theorem 3.6. This theorem is proved through the following Lemma 3.7, Lemma 3.8, and Lemmma 3.9. For the proofs of these lemmas, see [17], or Appendix I.

Theorem 3.6 *Over* ACA_0*, a $\Delta((\Sigma^0_n)_{k+1})$-formula is equivalent to a* $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_k)$*-formula for $n, k \geq 1$, and vice versa.*

Lemma 3.7 *For $1 \leq k < \omega$,* $\mathsf{Sep}(\Delta(\mathcal{C}), (\mathcal{C})_k) \subseteq \Delta((\mathcal{C})_{k+1})$.

Lemma 3.8 *Suppose $k, n \geq 1$. Two disjoint $\neg(\Sigma^0_n)_k$-formulas φ_0, φ_1 are separated by a* $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_{k-1})$*-formula δ, i.e., $\varphi_0 \to \delta \to \neg\varphi_1$ holds.*

Lemma 3.9 *Suppose $k, n \geq 1$. For any $\Delta((\Sigma^0_n)_k)$-formula $\zeta(f)$, there exist a Δ^0_n-formula $\psi(f)$, a $(\Sigma^0_n)_k$-formula $\eta(f)$ and a $(\Pi^0_n)_k$-formula $\eta'(f)$ such that*

$$\forall f(\zeta(f) \leftrightarrow ((\psi(f) \wedge \eta(f) \vee (\neg\psi(f) \wedge \eta'(f)))).$$

Finally, we recall an effective (or ACA_0) version of the Hausdorff-Kuratowski theorem, which states that a Δ^0_n set can be splitted into a transfinite difference of Π^0_{n-1} sets. By using this result, a $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_k)$ set can be treated as a certain combination of $(\Sigma^0_n)_k$ sets, which enables us to prove $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_k)$-determinacy by transfinite recursion.

Let $\prec$ be a recursive well-ordering on $\mathbb{N}$. We define a recursive well-ordering $\prec^*$ on $\mathbb{N} \times \{0, 1\}$ as follows:

$$(x, i) \prec^* (y, j) \quad \text{iff} \quad x \prec y \vee (x = y \wedge i < j).$$

We say that a formula $\varphi(n, i, f)$ is *decreasing along* $\prec^*$ if and only if

$$\forall f \in \mathbb{N}^{\mathbb{N}}\ \forall n \forall i \forall m \forall j\ (((m, j) \prec^* (n, i) \wedge \varphi(n, i, f)) \to \varphi(m, j, f)).$$

Definition 3.10 *For $n \geq 1$, $\mathrm{D}^0_{\mathrm{n+1}}$ is defined to be the class of all sets (or formally formulas) A such that*

$$A(f) \equiv \exists x(\neg\varphi(x, 1, f) \wedge \varphi(x, 0, f)),$$

where $\varphi(x, i, f)$ $(i = 0, 1)$ is a decreasing Π^0_n-formula along some recursive well-ordering relation $\prec^*$.

The following theorem shows that the class D^0_n and the class Δ^0_n coincide for each $n \geq 2$.

Theorem 3.11 (Tanaka [13], [14]. See also [3]) *Over* ACA_0*, we have* $\mathrm{D}^0_\mathrm{n} = \Delta^0_n$ $(n \geq 2)$*. Strictly speaking, for each formula* $A \in \mathrm{D}_\mathrm{n}$*, there exists a formula* $B \in \Delta^0_n$ *such that* $\mathsf{ACA}_0 \vdash A \leftrightarrow B$*, and vice versa.*

4 Σ^1_1 Inductive Definitions and their Transfinite Recursion

We begin with formalizing inductive definitions in second order arithmetic. An operator $\Gamma : P(\mathbb{N}) \to P(\mathbb{N})$ belongs to a class $\mathcal{C}$ of formulas if its graph $\{(x, X) : x \in \Gamma(X)\}$ is defined by a formula in $\mathcal{C}$.

A relation W is a *pre-ordering* if it is reflexive, connected and transitive. W is a *pre-well-ordering* if it is a well-founded pre-ordering. The *field* of W is the set $F = \{x : \exists y\ (x, y) \in W \vee \exists y\ (y, x) \in W\}$. An axiom of inductive definition by an operator Γ asserts the existence of a pre-well-ordering constructed by iterative applications of operator Γ. This can be stipulated as the following definition.

Definition 4.1 *Let* $\mathcal{C}$ *be a set of* $\mathcal{L}_2$ *formulas.* $\mathcal{C}$-ID *asserts that for any operator* $\Gamma \in \mathcal{C}$*, there exists a set* $W \subset \mathbb{N} \times \mathbb{N}$ *such that*

1. W *is a pre-well-ordering on its field* F*,*
2. $\forall x \in F \quad W_x = \Gamma(W_{<x}) \cup W_{<x}$*,*
3. $\Gamma(F) \subset F$*,*

where $W_x = \{y \in F : (y, x) \in W\}$ *and* $W_{<x} = \{y \in F : (y, x) \in W \text{ and } (x, y) \notin W\}$.

Theorem 4.2 (Tanaka [16] , MedSalem-Tanaka [3]) *Over* ACA_0*, the following are equivalent.*

(1) $\boldsymbol{\Sigma}^0_2$-Det.

(2) $\mathsf{Sep}(\boldsymbol{\Delta}^0_1, \boldsymbol{\Sigma}^0_2)$-$\mathsf{Det}$.

(3) $\boldsymbol{\Sigma}^1_1$-ID.

Remark that while Tanaka ([16] and others) studied relations between lightface statements without parameters, we here concentrate on boldface statements, since the use of parameters makes arguments much easier (and hence the base system weaker). Also note that with a minor change, an operator Γ can also define a pre-wellordering W rather than its field F. Formally, instead of W, it suffices to define a pre-well-ordering $\bar{W} = \{((w,x),(y,z)) : (w,x),(y,z),(x,z)$ all belong to $W\}$ inductively. So, we may say that Γ has a fixed point W.

In [16] and [4], we have shown that Σ^1_1-ID without free parameters, denoted as Σ^{1-}_1-ID, is equivalent to lightface Σ^0_2-Det, denoted as Σ^{0-}_2-Det, over ATR_0. Note that the both assertions are Σ^1_2, and they become Π^1_3 if parameters are allowed. Also, remark that all Σ^1_1 operators and all Σ^0_2 games are expressed by a single universal Σ^1_1 operator and a single universal Σ^0_2 game, respectively.

In the following, we finally see the definition of $\mathcal{C}$-IDTR.

Definition 4.3 *The formal system $\mathcal{C}$-IDTR_0 consists of ACA_0 and the following axiom scheme ($\mathcal{C}$-IDTR): for any well-ordering $\preceq$ and $\mathcal{C}$-operator Γ, there exists a transfinite sequence $\langle V^r : r \in \mathrm{field}(\preceq)\rangle$ such that for each $r \in \mathrm{field}(\preceq)$,*

1. *V^r is a pre-well-ordering on its field $F^r = \mathrm{field}(V^r)$.*
2. *$\forall x \in F^r (V^r_x = \Gamma^{F^{\prec r}}(V^r_{<x}) \cup V^r_{<x})$.*
3. *$\Gamma^{F^{\prec r}}(F^r) \subset F^r$.*

where $V^r_x = \{y \in F^r : y \leq_{V^r} x\}$, $V^r_{<x} = \{y \in F^r : y <_{V^r} x\}$, $F^{\prec r} = \bigcup\{F^{r'} : r' \prec r\}$.

As stated above, the definition of $\mathcal{C}$-IDTR was introduced in [6]. Intuitively, by ($\mathcal{C}$-IDTR), inductive definitions by a $\mathcal{C}$-operator Γ are iterated transfinitely along $\preceq$ in the following way. First apply inductive operator $\Gamma^{\emptyset}$ with the empty parameter to obtain a fixed point F^{r_0}, where r_0 is the $\preceq$-least element. Then, apply $\Gamma^{F^{r_0}}$ with parameter F^{r_0} to obtain a fixed point F^{r_1} with the second $\preceq$-least r_1. Then, apply $\Gamma^{F^{r_0} \cup F^{r_1}}$ to obtain F^{r_2}. We iterate this procedure transfinitely along well-ordering $\preceq$, and then we obtain the sequence of pre-well-orderings $\langle V^r : r \in \mathrm{field}(\preceq)\rangle$.

We here remark that the oracle parameter $F^{\prec r}$ of operator $\Gamma^{F^{\prec r}}$ can be replaced with any set $G^{\prec r}$ obtained from $F^{\prec r}$ arithmetically, i.e., $G^r = \Gamma_1(F^r) \cup G^{\prec r}$ with arithmetical Γ_1. This is because the description of Γ_1 may be inserted into the description of Γ, which is still Σ^1_1. Or we may modify Γ to produce a pair (F^r, G^r).

Theorem 4.4 (Yoshii-Tanaka [18], Tanaka-Yoshii [17]) *Over RCA_0, the following are equivalent.*

(1) $\Delta((\mathbf{\Sigma}^0_2)_2)$-Det.

(2) $\mathsf{Sep}(\mathbf{\Delta}^0_2, \mathbf{\Sigma}^0_2)$-Det.

(3) Σ^1_1-IDTR.

For a proof of (3)⇒(2) and a related issue, see the appendix II.

Now, we will show that Σ^1_1-IDTR is properly stronger than Σ^1_1-ID. Let $\varphi_1(e, m, X, Y)$ be a universal Σ^1_1 formula with only the displayed free variables. Let $\Gamma^X_e(Y)$ be a Σ^1_1 operator defined as $m \in \Gamma^X_e(Y) \leftrightarrow \varphi_1(e, m, X, Y)$. Moreover, we define a universal Σ^1_1 operator $\Gamma^X(Y)$ by $(e, m) \in \Gamma^X(Y) \leftrightarrow m \in \Gamma^X_e(Y_e)$. Then a fixed point of $\Gamma^X_e(Y)$ is also obtained as $F^X_e = \{m : (e, m) \in F^X\}$ if F^X is a fixed point of $\Gamma^X(Y)$. Thus, Σ^1_1-ID is equivalent to Γ-ID with a single universal Σ^1_1 operator Γ.

Consider the iteration of Γ-ID with a universal Σ^1_1 operator Γ along the standard order type of ω, which can be carried out in Σ^1_1-IDTR obviously. Intuitively, we first use the inductive operator $\Gamma^\emptyset$ (with the empty parameter) to obtain a fixed point F^0. Then, by Γ^{F^0} with parameter F^0, we obtain a fixed point F^1. Next, by $\Gamma^{F^0 \uplus F^1}$ to obtain F^2. We iterate this procedure and finally we obtain $\langle F^n : n \in \omega \rangle$.

Let us recall the definition of β-models in RCA_0. A set M represents a c.c. (abbr. for countable coded) ω-model $(\mathbb{N}, \{M_n : n \in \mathbb{N}\})$ of second order arithmetic. M is called a c.c. β-model if for all $e, m \in \mathbb{N}$ and $X, Y \in \{M_n\}$, $\varphi_1(e, m, X, Y)$ iff $M \models \varphi_1(e, m, X, Y)$ (Definition VII.2.3, Simpson [10]).

Lemma 4.5 (Lemma VII.2.9, Simpson [10]) *In* RCA_0*, for all* $X \subset \mathbb{N}$*, the hyperjump of* X *exists iff there exists a c.c.* β*-model* M *such that* $X \in M$*, i.e.,* $X = M_n$ *for some* n*.*

Proof. For the necessary condition, the key idea of the proof is the Kleene basis theorem that the sets recursive in $\mathcal{O}$ form a basis for Σ^1_1. □

Theorem 4.6 Σ^{1-}_1-IDTR_0 *proves there exists a c.c.* β*-model of* Σ^1_1-ID*.*

Proof. Let $\Gamma^X(Y)$ be a universal Σ^1_1 operator. By iterating the inductive definition Γ along ω, we obtain $\langle F^n : n \in \omega \rangle$. Let M be $\langle (F^n)_e : n, e \in \omega \rangle$. Then for any $X \in M$, the fixed point of Γ^X belongs to M, and the hyperjump of X is also included in M. Therefore, M is a c.c. β-model of Σ^1_1-ID. □

Corollary 4.7 Σ^1_1-IDTR_0 *is properly stronger than* Σ^1_1-ID_0*. Hence,* $\Delta((\mathbf{\Sigma}^0_2))$-Det *is properly stronger than* $\mathbf{\Sigma}^0_2$-Det *over* RCA_0*.*

5 Appendix I

Here, the proofs of the lemmas from Chapter 3 are recorded.

Lemma 3.7 For $1 \leq k < \omega$, $\mathsf{Sep}(\Delta(\mathcal{C}), (\mathcal{C})_k) \subseteq \Delta((\mathcal{C})_{k+1})$.

Proof. Take a $\Delta(\mathcal{C})$-formula ψ. We show by induction on k that for $\eta \in (\mathcal{C})_k$ and $\eta' \in \neg(\mathcal{C})_k$, $\theta \equiv (\psi \wedge \eta) \vee (\neg\psi \wedge \eta')$ is $\Delta((\mathcal{C})_{k+1})$.

First suppose $k = 1$. Then obviously, $(\psi \wedge \eta) \vee (\neg\psi \wedge \eta')$ is $\mathcal{C} \vee \neg\mathcal{C}$, i.e., $\neg(\mathcal{C})_2$. Its negation $\neg\theta$ can be written as $(\psi \wedge \neg\eta) \vee (\neg\psi \wedge \neg\eta')$, which is also $\neg(\mathcal{C})_2$, and thus θ is $\Delta((\mathcal{C})_2)$.

Take a $(\mathcal{C})_{k+1}$ formula $\eta \equiv \sigma \wedge \eta_1$ with $\sigma \in \mathcal{C}$, $\eta_1 \in \neg(\mathcal{C})_k$, and a $\neg(\mathcal{C})_{k+1}$ formula $\eta' \equiv \pi \vee \eta'_1$ with $\pi \in \neg\mathcal{C}$, $\eta'_1 \in (\mathcal{C})_k$. By the induction hypothesis, $\theta_1 \equiv (\psi \wedge \eta_1) \vee (\neg\psi \wedge \eta'_1)$ is $\Delta((\mathcal{C})_k)$. We easily observe that

$$\theta \equiv (\psi \wedge (\sigma \wedge \eta_1)) \vee (\neg\psi \wedge (\pi \vee \eta'_1))$$

$$\leftrightarrow (((\psi \wedge \sigma) \vee \neg\psi) \wedge \theta_1) \vee (\neg\psi \wedge \pi).$$

Since $((\psi \wedge \sigma) \vee \neg\psi) \wedge \theta_1$ is $\mathcal{C} \wedge (\mathcal{C})_k$ and so $(\mathcal{C})_k$, and $\neg\psi \wedge \pi$ is $\neg\mathcal{C}$, θ is $\neg(\mathcal{C})_{k+1}$. Similarly, we can show that $\neg\theta$ is also $(\mathcal{C})_{k+1}$. Hence, θ is $\Delta((\mathcal{C})_{k+1})$, which completes the proof. $\square$

To prove $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_k) \supseteq \Delta((\Sigma^0_n)_{k+1})$, we first need the following lemma.

Lemma 3.8 Suppose $k, n \geq 1$. Two disjoint $\neg(\Sigma^0_n)_k$-formulas φ_0, φ_1 are separated by a $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_{k-1})$-formula δ, i.e., $\varphi_0 \to \delta \to \neg\varphi_1$ holds.

We abbreviate $(\alpha_1 \to \alpha_2) \wedge (\alpha_2 \to \alpha_3) \wedge \cdots \wedge (\alpha_{i-1} \to \alpha_i)$ as $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_i$.

Proof. Suppose $k = 1$. Let $\varphi_i \equiv \forall x \theta_i$ with $\theta_i \in \Sigma^0_{n-1}$. Assume that φ_0 and φ_1 are disjoint. So, $\neg\varphi_0 \vee \neg\varphi_1$ always holds.

Now, let $\delta \equiv \exists x(\neg\theta_1 \wedge \forall y < x\theta_0)$. Then, it is easy to see

$$\neg\delta \leftrightarrow \exists x(\neg\theta_0 \wedge \forall y \leq x\theta_1).$$

Thus, δ is Δ^0_n. Also, it is clear that $\delta \to \neg\varphi_1$ and $\neg\delta \to \neg\varphi_0$. Hence, δ separates φ_0 and φ_1.

Suppose $k > 1$. Let $\varphi_i \equiv \pi_i \vee \psi_i$ with $\pi_i \in \Pi^0_n$ and $\psi_i \in (\Sigma^0_n)_{k-1}$. Assume φ_0 and φ_1 are disjoint. Then π_0 and π_1 are also disjoint, and so by the above argument, there exists $\delta \in \Delta^0_n$ such that $\pi_0 \to \delta \to \neg\pi_1$ holds. Now let $\overline{\delta} \equiv (\delta \wedge \neg\psi_1) \vee \psi_0$, i.e., $(\delta \wedge \neg\psi_1) \vee (\neg\delta \wedge \psi_0)$. So, $\overline{\delta} \in \mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_{k-1})$. Then, noticing π_i and ψ_{1-i} are disjoint, we have $\pi_0 \to \neg\psi_1$ and so $\varphi_0 \equiv (\pi_0 \vee \psi_0) \to (\delta \wedge \neg\psi_1) \vee \psi_0 [\equiv \overline{\delta}] \to (\neg\varphi_1 \vee \psi_0) \to \neg\varphi_1$, since φ_1 and ψ_0 are also disjoint. Thus $\overline{\delta}$ separates φ_0, φ_1. $\square$

Now, $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_k) \supseteq \Delta((\Sigma^0_n)_{k+1})$ is straightforward.

Lemma 3.9 Suppose $k, n \geq 1$. For any $\Delta((\Sigma^0_n)_k)$-formula $\zeta(f)$, there exist a Δ^0_n-formula $\psi(f)$, a $(\Sigma^0_n)_k$-formula $\eta(f)$ and a $(\Pi^0_n)_k$-formula $\eta'(f)$ such that

$$\forall f(\zeta(f) \leftrightarrow ((\psi(f) \wedge \eta(f) \vee (\neg\psi(f) \wedge \eta'(f)))).$$

Proof. If ζ is $\Delta((\Sigma^0_n)_k)$, then ζ and $\neg\zeta$ are disjoint $(\Sigma^0_n)_k$. So, by the above lemma, there exists a $\mathsf{Sep}(\Delta^0_n, (\Sigma^0_n)_k)$-formula ψ such that $\zeta \to \psi \to \neg\neg\zeta$, i.e., $\zeta \leftrightarrow \psi$. □

6 Appendix II

We will first present a proof of $\mathsf{Sep}(\mathbf{\Delta}^0_2, \mathbf{\Sigma}^0_2)$ determinacy from Σ^1_1-IDTR in this appendix. Then, at first glance, $\mathsf{Sep}(\mathbf{\Delta}^0_2, \mathbf{\Sigma}^0_2)$ determinacy seems to be also derivable from Σ^0_2 determinacy (or Σ^1_1-ID). However, in the final remark, we will explain why it cannot be proved from Σ^1_1-ID.

A proof of (3) Σ^1_1-IDTR $\Rightarrow$ (2) $\mathsf{Sep}(\mathbf{\Delta}^0_2, \mathbf{\Sigma}^0_2)$-$\mathsf{Det}$ **of Theorem 4.4.**

Assume (3) and let $\varphi(f)$ be a $\mathsf{Sep}(\mathbf{\Delta}^0_2, \mathbf{\Sigma}^0_2)$-game. Then, there exist a Δ^0_2-formula ψ, a Π^0_2-formula η_0 and a Σ^0_2-formula η_1 (with parameters undisplayed) such that

$$\forall f(\varphi(f) \leftrightarrow (\psi(f) \wedge (\eta_0(f))) \vee (\neg\psi(f) \wedge (\eta_1(f))).$$

Since $\psi(f)$ is a Δ^0_2-formula, there are Δ^0_1-formulas $\theta_0(y), \theta_1(y)$ such that

$$\forall f((\psi(f) \leftrightarrow \forall n \exists m > n \theta_0(f[m])) \wedge (\neg\psi(f) \leftrightarrow \forall n \exists m > n \theta_1(f[m]))).$$

Without loss of generality, we may assume $\neg\exists s(\theta_0(s) \wedge \theta_1(s))$ and $\theta_0(\langle\rangle)$ holds. Note that $\langle\rangle$ is the empty sequence.

Now, we define a recursive tree T as follows:

$$T = \{s \in (\mathbb{N}^{<\mathbb{N}})^{<\mathbb{N}} : s(0) \subsetneq s(1) \subsetneq \cdots \subsetneq s(|s|-1) \wedge$$

$$\forall k < |s|(\text{ if } k \text{ is even, then } \theta_0(s(k)),\ \text{else } \theta_1(s(k))\}$$

Clearly, T does not have an infinite path. Let $\preceq$ be the Kleene-Blouwer ordering on T. Since T is a well-founded tree, for any $f \in \mathbb{N}^{\mathbb{N}}$ there exists the $\preceq$-least $x \in T$ such that $\cup x = x(|x|-1) \subset f$. So, we define a Π^0_1-formula $\xi(x, f)$ as follows:

$$\xi(x, f) \leftrightarrow x \in T \wedge \cup x \subset f \wedge \forall y(y \in T \wedge \cup y \subset f \to x \preceq y).$$

Then, it is easy to see

$$\psi(f) \leftrightarrow \exists x(|x| \text{ is odd} \wedge \xi(x, f)), \quad \neg\psi(f) \leftrightarrow \exists x(|x| \text{ is even} \wedge \xi(x, f)).$$

Now, we define $\eta'_0(x, f, Y)$ and $\eta'_1(x, f, Y)$ with parameter Y as follows:

$$\eta'_0(x, f, Y) \equiv (\xi(x, f) \wedge \eta_0(f)) \vee \exists n(f[n] \in Y),$$

$$\eta_1'(x,f,Y) \equiv (\xi(x,f) \wedge \eta_1(f)) \vee \exists n(f[n] \in Y).$$

Clearly, the formula η_0' is Π^0_2 and η_1' is Σ^0_2. Thus, their determinacy is deduced from Σ^1_1-ID. More precisely, the set of sure winning positions for player I (II) in a $\mathbf{\Sigma}^0_2$ ($\mathbf{\Pi}^0_2$) game can be defined by boldface Σ^1_1-ID from Theorem4.2.

By using Σ^1_1-IDTR, we inductively define the set of sure winning positions for player I in the game φ as follows: if $|x|$ is odd then

$$\overline{W}_x = \{s \in \mathbb{N}^{\mathbb{N}} : \cup x \subset s \text{ and II has a winning strategy in } \eta_0'(x,f,W_{\prec x}) \text{ starting at } s\},$$

and if $|x|$ is even then

$$W_x = \{s \in \mathbb{N}^{\mathbb{N}} : \cup x \subset s \text{ and I has a winning strategy in } \eta_1'(x,f,W_{\prec x}) \text{ starting at } s\},$$

where $W_{\prec x} = \bigcup\{W_y : y \prec x,\ y \text{ is even}\} \cup \bigcup\{\overline{W}^c_y : y \prec x,\ y \text{ is odd}\}$. We may identify W_y as $\overline{W}^c_y$.

We set $W = \bigcup\{W_x : x \in T\}$. Then, we can easily prove the following:

$\langle\rangle \in W \to$ I wins the game $\varphi(f)$, and

$\langle\rangle \notin W \to$ II wins the game $\varphi(f)$.

This completes (3)⇒(2).

Remark. At first glance, the above proof seems to be improved to a proof of Σ^1_1-ID ⇒ (2) in the following way. Since $\mathbf{\Sigma}^0_2$-Det is deducible from Σ^1_1-ID, we define the sets W_0, W_1 of I's winning positions for Π^0_2-game η_0 and Σ^0_2-game η_1, respectively, i.e., for $i = 0, 1$,

$$W_i = \{s \in \mathbb{N}^{\mathbb{N}} : \text{I has a winning strategy in } \eta_i(f) \text{ starting at } s\},$$

Then, we define the following game G.

$$G(f) \leftrightarrow \exists x(|x| \text{ is even} \wedge \xi(x,f) \wedge x \in W_0) \vee \exists x(|x| \text{ is odd} \wedge \xi(x,f) \wedge x \in W_1).$$

This is $\mathbf{\Sigma}^0_2$ with parameters W_0, W_1, and so it is determinate, too.

Now, suppose I has a winning strategy σ for G. Let f be a play consistent with σ. If $\psi(f)$ holds, then $\exists x(|x| \text{ is even} \wedge \xi(x,f))$ holds and so $\exists x(|x| \text{ is even} \wedge \xi(x,f) \wedge x \in W_0)$, since $G(f)$ holds. Hence at such a position x, I may switch to a winning strategy for $\eta_0(f)$ and then we may assume that f satisfies $\psi(f) \wedge \eta_0(f)$, that is, $\varphi(f)$. If $\neg\psi(f)$ holds, then $\exists x(|x| \text{ is odd} \wedge \xi(x,f) \wedge x \in W_1)$ must hold. At such a position x, I may switch to a winning strategy for $\eta_1(f)$ and then f satisfies $\neg\psi(f) \wedge \eta_1(f)$, that is, $\varphi(f)$. Suppose II has a winning strategy τ for G. Letting f be a play consistent with τ, we similarly show that $\neg\varphi(f)$.

A deficiency of this argument lies behind a fact that $\xi(x, f)$ and $\cup x \subset g$ does not imply $\xi(x, g)$, although for each f, there exists a unique x such that $\xi(x, f)$.

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References

[1] S. J. Hachtman, *Determinacy and monotone inductive definitions* Israel Journal of Mathematics, vol. 230 pp. 71-96 (2019)

[2] K. Mashiko, K.Tanaka, K. Yoshii, *Determinacy of Infinite Games and Inductive Definition in Second Order Arithmetic* RIMS Kokyuroku, vol. 1729, pp.167-177 (2010)

[3] M.O. MedSalem and K. Tanaka, Δ^0_3*-determinacy, comprehension and induction,* Journal of Symbolic Logic 72 (2007) 452-462.

[4] M.O. MedSalem and K. Tanaka, *Weak determinacy and iterations of inductive definitions,* in the Proc. of Computational Prospects of Infinity, World Scientific (2008), 333-353.

[5] M.Montalbán and R.A.Shore, *The Limits of determinacy in second order arithmetic,* Proc. London Math. Soc. (3) 104, no. 2 (2012) 223-252.

[6] T. Nemoto, *Determinacy of Wadge classes in the Baire space and simple iteration of inductive definition,* RIMS Kokyuroku1635 (2009) 121-133.

[7] T. Nemoto, M.O. MedSalem and K. Tanaka, *Infinite games in the Cantor space and subsystems of second order arithmetic,* Math. Log. Quart. 53, No,3, 226-236, (2007)

[8] T. Nemoto, *Determincay of Wadge classes and subsystems of second order arithmetic,* Math. Log. Quart. 55, No,2, 154-176, (2009)

[9] L. Pacheco, K. Yokoyama, *Determinacy and reflection principles in second-order arithmetic,* (arXive 2023)

[10] S.G. Simpson, *Subsystems of Second Order Arithmetic,* Springer (1999).

[11] J.R. Steel, *Determinateness and subsystems of analysis,* Ph.D. thesis, University of California, Berkeley (1977).

[12] K. Tanaka, *Collapsing the hierarchy of multiple inductive definitions?* BIMSA, 2023 (preprint).

[13] K. Tanaka, *Descriptive set theory and subsystems of analysis,* Ph.D. thesis, University of California, Berkeley (1986).

[14] K. Tanaka, *A recursion-theoretic proof of Hausdorff's theorem on ambiguous Borel sets,* Research Reports on Info. Sciences, Tokyo Inst. Tech. C 80 (1986), 1–9.

[15] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis I:* Δ^0_2-*games,* Zeitschrift für mathematische Logik und Grundlagen der Mathematik 36 (1990), 481-491.

[16] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis II:* Σ^0_2-*games,* Annals of Pure and Applied Logic 52 (1991), 181-193.

[17] K. Tanaka and K. Yoshii, *Infinite games, inductive definitions and transfinite recursion,* Computability Theory and Foundations of Mathematics 2019 (Wuhan), World-Scientific (2022), 47–78.

[18] K. Yoshii and K. Tanaka, *Infinite games and transfinite recursion of multiple inductive definitions,* How the world computes, Lecture Notes in Comput. Sci. 7318 (2012), 374–383.

[19] K. Yoshii, *A survey of determinacy of infinite games in second order arithmetic,* Annals of the Japan Association for Philosophy of Science Vol.25 (2017), 35-44.

[20] P.D. Welch, *Weak Systems of determinacy and arithmetical quasi-inductive definitions,* Journal of Symbolic Logic 76 (2011) 418-436.