L^2 HODGE THEORY FOR COMPLEX HYPERBOLIC SPACE FORM WITH FINITE VOLUME

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Abstract. The goal of this article is to review a version of L^2-Hodge theory in [6], for symmetric powers of holomorphic cotangent bundle over complex hyperbolic space forms with finite volume.

Let (X, E) be a hermitian manifold with a smooth hermitian vector bundle E over X. Let $L_2^{r,s}(X, E)$ be the L^2 space of measurable E-valued (r, s) forms for the induced metric from X and E. The Hilbert adjoint of densely defined operator $\bar{\partial} : L_2^{r,s}(X, E) \to L_2^{r,s+1}(X, E)$ is denoted by $\bar{\partial}^*$. If the dimension of L^2 Dolbeault cohomology group $H^{r,s}_{L^2,\bar{\partial}}(X, E)$ is finite, then the following L^2 Hodge decomposition holds.

Theorem 0.1. *(L^2 Hodge decomposition) Let (X, E) be a hermitian manifold with a smooth hermitian vector bundle E over X. If the dimension of $H^{r,s}_{L^2,\bar{\partial}}(X, E)$ is finite, then*

(1) $L_2^{r,s}(X, E)$ has an orthogonal decomposition:

$$L_2^{r,s}(X, E) = Im\bar{\partial} \oplus Im\bar{\partial}^* \oplus \mathcal{H}_2^{r,s}(X, E)$$

where $\Delta_{\bar{\partial}} := \partial \circ \bar{\partial}^* + \bar{\partial}^* \circ \bar{\partial}$ and $\mathcal{H}_2^{r,s}(X, E) := \ker \Delta_{\bar{\partial}}$ is the space of L^2-harmonic (r, s) forms. This decomposition implies that the following isomorphism

$$H^{r,s}_{L^2,\bar{\partial}}(X, E) \cong \mathcal{H}_2^{r,s}(X, E)$$

(2) Let H be the projection operator from $L_2^{r,s}(X, E)$ to $\mathcal{H}_2^{r,s}(X, E)$. Then, there is a bounded linear operator G satisfying

$$\Delta_{\bar{\partial}} \circ G = G \circ \Delta_{\bar{\partial}} = I - H, \quad H \circ G = G \circ H = 0.$$

The operator G is called the Green operator.

If X is compact, then the dimension of L^2 Dolbeault cohomology group is finite by the Hodge theory. However, in non-compact manifolds, the L^2 Dolbeault cohomology group may have infinite dimension (for example, see [2]). Therefore, to guarantee the finite dimensionality of the cohomology group, we need to impose some geometric conditions to (X, E).

If X is an n-dimensional complete Kähler manifold and E is Nakano-positive, then $H^{n,s}_{L^2,\bar{\partial}}(X, E)$ vanishes due to the Bochner-Kodaira-Nakano inequality. However, if E is Nakano semi-positive, additional assumptions on (X, E) are needed to guarantee the finiteness of $H^{n,s}_{L^2,\bar{\partial}}(X, E)$: suppose that X has a complete Kähler metric. The associated Kähler form is denoted by ω. Assume that there is a smooth real-valued function φ on X such that $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$ on the complement of a compact proper subset K of X. If the supremum of the induced norm $|d\varphi|^2_\omega$ of φ from ω is finite, then the finiteness of $H^{n,s}_{L^2,\bar{\partial}}(X, E)$ follows when E is Nakano semi-positive.

Unfortunately, these results cannot be used for non-Nakano semi-positive bundle: Let $\mathbb{B}^n$ be a complex unit ball in $\mathbb{C}^n$ and Γ be a torsion-free lattice in $\mathrm{Aut}(\mathbb{B}^n)$. Consider a smooth finite

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volume quotient $\Sigma = \mathbb{B}^n/\Gamma$, that is, a complex hyperbolic space form with finite volume for the induced Bergman metric of $\mathbb{B}^n$. Note that its holomorphic sectional curvature is of negative constant. We denote the holomorphic cotangent bundle of Σ by T^*_Σ. Let $S^mT^*_\Sigma$ be the m-th symmetric power of T^*_Σ and K_Σ be the canonical line bundle. If m is not sufficiently large, then $S^mT^*_\Sigma \otimes K_\Sigma^{-1}$ is not Nakano semi-positive.

Nevertheless, Σ has a toroidal compactification $\overline{\Sigma}$ up to a finite cover. The toroidal compactification $\overline{\Sigma}$ is a smooth compactification of Σ whose boundary divisor consists of the disjoint union of smooth Abelian varieties, each with a normal bundle (for details, see [7, 8]). Since several isomorphism theorems between a sheaf cohomology over a compact Kähler manifold X with a simple normal crossing divisor D and a L^2-Dolbeault cohomology on $X - D$ with a certain type of complete Kähler metric have been established (for instance, see [4]), we can expect that a similar type of theorem for $S^mT^*_\Sigma$ exists. If it is true, then the finite dimensionality of $H^{r,s}_{L^2,\bar\partial}(\Sigma, S^mT^*_\Sigma)$ directly follows from the compactness of $\overline{\Sigma}$.

Theorem 0.2. *(Main Theorem of [6]) Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice that has only unipotent parabolic automorphisms. If the volume of $\Sigma = \mathbb{B}^n/\Gamma$ for the induced Bergman metric from $\mathbb{B}^n$ is finite, then for every $m, r, s \in \mathbb{N} \cup \{0\}$, there exists a holomorphic vector bundle $E_{r,m}$ over the smooth toroidal compactification $\overline{\Sigma}$ such that if $r = 0, n$ or $m \geq n-1$, then the following isomorphism holds:*

$$H^{r,s}_{L^2,\bar\partial}(\Sigma, S^mT^*_\Sigma) \cong H^s(\overline{\Sigma}, E_{r,m}). \tag{0.1}$$

Remark 0.3. Any torsion-free lattice has a finite index subgroup that only has unipotent parabolic automorphisms. Therefore, any complex hyperbolic space form with finite volume has a smooth toroidal compactification up to a finite cover.

Sketch of the proof of Theorem 0.2. Let $\mathcal{O}(E_{r,m})$ be the sheaf of germs of holomorphic sections of $E_{r,m}$. Let $L^{r,*}_{2,loc}(U, S^mT^*_\Sigma)$ be the space of measurable $S^mT^*_\Sigma$-valued $(r,*)$ forms which are locally integrable on U with respect to the induced Bergman metric of Σ and the hermitian metric of $S^mT^*_\Sigma$. Define a sheaf $L^{r,*}_{2,S^mT^*_\Sigma}$ by $L^{r,*}_{2,S^mT^*_\Sigma}(U) := \{\nu \in L^{r,*}_{2,\mathrm{loc}}(U, S^mT^*_\Sigma) : \bar\partial\nu \in L^{r,*+1}_{2,\mathrm{loc}}(U, S^mT^*_\Sigma)\}$ for any open set U in $\overline{\Sigma}$. Note that $L^{r,*}_{2,S^mT^*_\Sigma}$ is a fine sheaf by the completeness of ω.

Since $L^{r,*}_{2,S^mT^*_\Sigma}$ is a fine resolution of $\mathcal{O}(E_{r,m})$, if the resolution is exact, then a standard homological algebra argument (for example, see Lemma 4.1 [6]) induces the isomorphism (0.1). Therefore, to establish (0.1), we need to verify the exactness of

$$0 \to \mathcal{O}(E_{r,m}) \to L^{r,0}_{2,S^mT^*_\Sigma} \xrightarrow{\bar\partial} \ker \bar\partial_{(r,1),S^mT^*_\Sigma} \to 0 \tag{0.2}$$

and

$$0 \to \ker \bar\partial_{(r,s),S^mT^*_\Sigma} \to L^{r,s}_{2,S^mT^*_\Sigma} \xrightarrow{\bar\partial} \ker \bar\partial_{(r,s+1),S^mT^*_\Sigma} \to 0 \tag{0.3}$$

for all $1 \leq s \leq n$.

Let D_j, $j = 1, \cdots, k$ be the boundary divisor corresponding to the cusp b_j in the Satake-Baily-Borel compactification of Σ, where k is the number of cusps of Σ. To verify (0.2), take any point p in $D_j \subset \Sigma \subset \overline{\Sigma}$. Since $\Lambda^{r,s}T^*_\Sigma \otimes S^mT^*_\Sigma$ can be trivialized near p, we can consider the Laurent expansion of a holomorphic section of $S^mT^*_\Sigma$ near p. Now, the exactness of (0.2) follows by some integral calculations because the induced Bergman metric on Σ is locally quasi-isometric to a ball quotient type metric near p (see section 3 in [6]).

Now, the remaining part is to verify (0.3). For this, we need to establish a version of L^2 Dolbeault-Grothendieck type Lemma for $S^mT^*_\Sigma$. To do this, take a point $p \in D_j$ and a sufficiently small open polydisc U_p near p. Then, the induced Bergman metric on Σ is quasi-isometric to a ball quotient type metric in U_p. Hence, we can control the curvature term of $\Lambda^{r,0}T^*_{U_p-D_j} \otimes$

$S^mT^*_\Sigma \otimes K^{-1}_{U_p-D_j}$ in the Bochner-Kodaira-Nakano inequality, using a suitable weight function near p. Therefore, we can prove a local vanishing of a L^2-Dolbeault cohomology for $\Lambda^{r,0}T^*_{U_p-D_j} \otimes S^mT^*_\Sigma \otimes K^{-1}_{U_p-D_j}$ on $U_p - D_j$ if $s \geq 2$.

However, if $s = 1$, this approach does not work due to the geometry of the ball quotient type metric. When $r = 0$ or n, Lee and Seo define a local coordinate system using the local description of ball quotient type metric near D_j to overcome these difficulties. Using this, they change the local $\bar\partial$-solvability problem for $\Lambda^{r,0}T^*_\Sigma \otimes S^mT^*_\Sigma$-valued $(0,1)$ forms to $\Lambda^{r,0}T^*_\Sigma \otimes S^mT^*_\Sigma$-valued $(n,1)$ forms. If $r = 0$ or n, then $\Lambda^{r,0}T^*_\Sigma \otimes S^mT^*_\Sigma$ is Nakano-positive, and so, in this case, the local $\bar\partial$-solvability issue for $(0,1)$ form can be solved.

Also, for the case where $r \neq 0, n$, and $m \geq n-1$, the authors apply the argument of Chen [3] to induce a Donnelly-Fefferman type estimate, which is useful to establish a local vanishing result of some L^2-Dolbeault cohomology near p (for details, See Lemma 4.4 and Proposition 4.5 in [6]).

□

The result follows directly by the compactness of $\overline{\Sigma}$.

Corollary 0.4. *Suppose that $r = 0, n$ or $m \geq n-1$, then $H^{r,s}_{L^2,\bar\partial}(\Sigma, S^mT^*_\Sigma) < \infty$.*

A holomorphic ball bundle Ω over Σ is given by the quotient of $\mathbb{B}^n \times \mathbb{B}^n$ under the diagonal action

$$(z,w) \in \mathbb{B}^n \times \mathbb{B}^n \mapsto (\gamma z, \gamma w) \in \mathbb{B}^n \times \mathbb{B}^n, \quad \forall \gamma \in \Gamma.$$

Since any automorphism of $\mathbb{B}^n$ is extended as an automorphism of n-dimensional complex projective space $\mathbb{CP}^n$, Ω is a relatively compact smooth domain in a certain $\mathbb{CP}^n$ bundle over Σ.

Let $K_{\mathbb{B}^n}$ be the Bergman kernel of $\mathbb{B}^n$. A weighted Bergman space $A^2_\alpha(\Omega)$ of order $\alpha > -1$ is defined by a Kähler form

$$\omega_\Omega = \frac{\sqrt{-1}}{n+1}\partial\bar\partial K_{\mathbb{B}^n}(z,z) + \frac{\sqrt{-1}}{n+1}\partial\bar\partial K_{\mathbb{B}^n}(w,w)$$

and $\delta := 1 - |T_z w|^2$, where $T_z w$ is an involution automorphism in $\mathbb{B}^n$ (for details, see [5, 6]).

Theorem 0.5. *(Theorem 1.5 in [6]) Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice that has only unipotent parabolic automorphisms. If the volume of $\Sigma = \mathbb{B}^n/\Gamma$ is finite for the induced Bergman metric of $\mathbb{B}^n$, then there exists a linear injective map*

$$\Phi : \bigoplus_{m=0}^{\infty} H^{0,0}_{L^2,\bar\partial}(\Sigma, S^mT^*_\Sigma) \to \bigcap_{\alpha>-1} A^2_\alpha(\Omega)$$

and the image of Φ is dense in $\mathcal{O}(\Omega)$ equipped with the compact open topology.

Corollary 0.6. *(Corollary 1.6 in [6]) Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice. If the volume of $\Sigma = \mathbb{B}^n/\Gamma$ for the induced Bergman metric of $\mathbb{B}^n$ is finite, then there are no non-constant bounded holomorphic functions on Ω.*

Sketch of the proof of Theorem 0.5. We identify $S^0T^*_\Sigma$ with the trivial line bundle $\Sigma \times \mathbb{C}$. Now, for a $\psi \in H^{0,0}_{L^2,\bar\partial}(\Sigma, S^0T^*_\Sigma)$, we define $\Phi(\psi)(z,w) := \tilde\psi(z)$, where $\tilde\psi$ is given by the quotient map $\mathbb{B}^n \to \Sigma$. Since the volume of Ω is finite for the induced volume form by ω_Ω, $\Phi(\psi)$ is well-defined.

If $N \geq 1$, for any given $\psi \in H^{0,0}_{L^2,\bar\partial}(\Sigma, S^NT^*_\Sigma)$, we define a sequence $\{\varphi_k\}$ by

$$\begin{cases} \varphi_k = 0 & \text{if } k < m_0 \\ \varphi_N = \psi \end{cases}$$

and for $s \geq 1$, φ_{N+s} is the L^2-minimal solution of the recursive formula

$$\bar\partial\varphi_{N+s} = -(N+s-1)\mathcal{R}_G\varphi_{N+s-1} \tag{0.4}$$

on Σ. For the definition of $\mathcal{R}_G$, see [5, 6].

If the Green operator G exists in $L_2^{0,0}(\Sigma, S^m T^*_\Sigma)$, then the minimal solution of (0.4) exists, and so we can explicitly calculate the L^2 norm of φ_k. If Σ is compact, then the existence of the Green operator G is due to the Hodge theory. However, if Σ is not compact, the existence of G is not apparent.

To establish the existence of the Green operator G, Lee and Seo apply Theorem 0.1 and Corollary 0.4. As a result, the authors calculate the explicit L^2 norm of $\{\varphi_k\}$. Using these L^2 norm data, they construct an associated holomorphic function $\Phi(\varphi_N)$ from φ_N. Also, they show that the L^2 norm of $\Phi(\varphi_N)$ is finite and $\bar{\partial}\Phi(\varphi_N) = 0$ in the sense of distribution by following the argument in [1, 5]. Therefore, $\Phi(\varphi_N)$ is well-defined. Now, Φ is defined by extending linearly.

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References

[1] Adachi Masanori, *On weighted Bergman spaces of a domain with Levi-flat boundary.* Trans. Amer. Math. Soc. 374 (2021), no. 10, 7499–7524.

[2] Chakrabarti Debraj; Laurent-Thiébaut Christine; Shaw Mei-Chi, *On the L^2-Dolbeault cohomology of annuli.* Indiana Univ. Math. J. 67, No. 2, 831-857 (2018).

[3] Chen Bo-Yong, *A simple proof of the Ohsawa-Takegoshi extension theorem.* arXiv:1105.2430v1.

[4] Chunle Huang, Kefeng Liu, Xueyuan Wan, Xiaokui Yang, *Vanishing Theorems for Sheaves of Logarithmic Differential Forms on Compact Kähler Manifolds.* Int. Math. Res. Not. ranc204, 1–23 (2022), https://doi.org/10.1093/imrn/rnac204.

[5] Lee Seungjae; Seo Aeryeong, *Symmetric differentials and Jets extension of L^2 holomorphic functions*, to appear in Indiana University Mathematics Journal.

[6] Lee Seungjae; Seo Aeryeong, *Cohomology isomorphism of symmetric power of cotangent bundle of ball quotient and its toroidal compactification*, arXiv: arXiv:2311.01931.

[7] Mok Ngaiming, *Projective-Algebraicity of Minimal Compactifications of Complex-Hyperbolic Space forms of Finite Volume*, Itenberg, Ilia (ed.) et al., (eds) Perspectives in analysis, geometry, and topology, Progr. Math. 296, pp. 331–354. Birkhäuser/Springer, New York (2012).

[8] Wong Kwok-Kin, *On effective existence of symmetric differentials of complex hyperbolic space forms*, Math. Z. 290 (2018), no. 3–4, 711–733.

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