Remarks on the paper "On the infinite dimensionality of the middle L^2 cohomology of complex domains"

Kazuhisa Miyazawa

1.Preliminaies

In this note, we will give an outline of the proof of Theorem in [Oh] and remark on the paper. Then we will describe conditions and some properties for infinite dimensionality of the middle L^2 cohomology.

Let $\mathrm{D} \subset \mathbf{C}^n$ be a domain with the smooth boundary in the n-dimensional comlplex Euclidean space. Then $\partial \mathrm{D}$ denotes the boundary of D and $\overline{\mathrm{D}}$ denotes the topological closure of D.

Let $\varphi: \mathbf{C}^n \longrightarrow \mathbf{R}$ be a defining function of D satisfying $\mathrm{D} = \{z \in \boldsymbol{C}^n | \varphi(z) < 0\}$. For a subset $\mathrm{A} \subset \mathbf{C}^n$, let $C^{p,q}(A)$ be the restriction on A for the space of smooth (p,q)- forms on the complex Euclidean space. For $x \in \partial\mathrm{D}$, let $\{\mathrm{U}, (\zeta_1, \dots, \zeta_n) \mid \mathrm{U} \subset \mathbf{C}^n, (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n\}$ be a local coordinate of x. We assume that $d\varphi \neq 0$ holds on $\overline{U}$. Let $\{\tau_l\}_{l=1\sim n}$ be a set of $C^{1,0}(\overline{\mathrm{D}} \cap U)$ such that $\{\tau_k(y)\}_{k=1\sim n}$ are an ortho-normal basis with respect to the complex Euclidean metric at $y \in \overline{\mathrm{D}} \cap U$ with $\tau_1 = \partial\varphi / |\partial\varphi|_E$. Here $|\cdot|_E$ denotes the pointwise norm with respect to the complex Euclidean metric. For a differential form θ, let $e(\theta)$ be the exterior product $e(\theta): u \mapsto \theta \wedge u$ and $e^*(\theta)$ be the adjoint of $e(\theta)$ with respect to the complex Euclidean metric.

For a smooth (p,q)-form $u \in C^{p,q}(\overline{\mathrm{D}} \cap U)$, there are unique four differential forms $\{u_k\}_{k=1\sim 4}$ satisfying

$$u = u_1 + \partial\varphi \wedge u_2 + \overline{\partial}\varphi \wedge u_3 + \partial\varphi \wedge \overline{\partial}\varphi \wedge u_4,$$
$$e^*(\tau_1)u_k = e^*(\overline{\tau_1})u_k = 0 \quad (k = 1\sim 4).$$

Moreover, the following holds.

Proposition 1 (cf : [We])

For $p + q = n$, we assume that Levi form $\sqrt{-1}\,\partial\overline{\partial}\varphi$ is not non-degenerated on U. Then we have the following.

(1) For $w \in C^{p,q}(\overline{\mathrm{D}} \cap U)$, there are unique four forms on $(\overline{\mathrm{D}} \cap U)$ such that $v_1 \in C^{p-1,q-1}(\overline{\mathrm{D}} \cap U)$, $v_2 \in C^{p-1,q}(\overline{\mathrm{D}} \cap U)$, $v_3 \in C^{p,q-1}(\overline{\mathrm{D}} \cap U)$, $v_4 \in C^{p-1,q-1}(\overline{\mathrm{D}} \cap U)$ satisfies that

(i) $w = \partial\overline{\partial}\varphi \wedge v_1 + \partial\varphi \wedge v_2 + \overline{\partial}\varphi \wedge v_3 + \partial\varphi \wedge \overline{\partial}\varphi \wedge v_4$,

(ii) $e^*(\tau_1)v_k = e^*(\overline{\tau_1})v_k = 0$ $(k = 1\sim 4)$ hold.

(2) For $v_f \in C^{0,1}(\overline{\mathrm{D}} \cap U)$ and $u_1 \in C^{p,q-1}(\overline{\mathrm{D}} \cap U)$, we put $w = v_f \wedge u_1 \in C^{p,q}(\overline{\mathrm{D}} \cap U)$. assume that $e^*(\tau_1)w = e^*(\overline{\tau_1})w = 0$ hold. Then there exist $u_{11} \in C^{p-1,q-1}(\overline{\mathrm{D}} \cap U)$, $u_{12} \in C^{p-1,q-1}(\overline{\mathrm{D}} \cap U)$

This work was supported by the Research Institute for Mathematical Sciences, an international Joint Usage/Research Center located in Kyoto University.
E-mail address: kz-miyazawa@ezweb.ne.jp

uniquely satisfying the following:

(i) $w = \partial\bar{\partial}\varphi \wedge u_{11} - \partial\varphi \wedge \bar{\partial}\varphi \wedge u_{12}$,

(ii) $e^*(\tau_1)u_{11} = e^*(\bar{\tau_1})u_{11} = 0,\ e^*(\tau_1)u_{12} = e^*(\bar{\tau_1})u_{12} = 0.$

Especially, the form u_{11} may have a divisor v_f in general.

2. Theorem of Ohsawa

We will explain Theorem of Ohsawa and describe an outline of the proof [Oh] by using the argument of Ohsawa in view of Proposition 1.

Let ds^2 be a complete hermitian metric on a domain D with the smooth boundary and $L^{p,q}(D)$ be the space of square integrable (p, q)-forms.

From now on, we assume the following:

(i) There exist positive constants C_0 and a, b satisfying

$$C_0^{-1}ds^2 < \left(-\frac{1}{\varphi}\right)^a \sum_{l=2}^{n} d\zeta_l \wedge d\bar{\zeta_l} + \left(-\frac{1}{\varphi}\right)^b \partial\varphi \wedge \bar{\partial}\varphi < C_0\ ds^2 \text{ on } D \cap U,$$

(ii) $1 \leq a, 1 \leq b, a < b + 1$ hold,

(iii) $\partial\varphi,\ d\zeta_2, \dots, d\zeta_n$ are linearly independent on $\bar{D} \cap U$.

Then we have the following theorem about unreduced L^2cohomology.

Theorem([Oh])

For $p + q = n$, we have that

$$dim_C \frac{\{u \in L^{p,q}(D) \mid \bar{\partial}u = 0\}}{\{\bar{\partial}u \in L^{p,q}(D) \mid u \in L^{p,q-1}(D)\}} = +\infty$$

holds.

Proof of the theorem is as follows.

For $p + q = n$, we set $u = f\, d\bar{\zeta_2} \wedge \cdots \wedge d\bar{\zeta_q} \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{p+1}$ which is a smooth(p, q)- form on $\bar{D} \cap U$.

Then there exist four forms $\{v_k\}_{k=1\sim4}$ uniquely from Proposition 1 satisfying

$$\bar{\partial}u = \partial\bar{\partial}\varphi \wedge v_1 + \partial\varphi \wedge v_2 + \bar{\partial}\varphi \wedge v_3 + \partial\varphi \wedge \bar{\partial}\varphi \wedge v_4.$$

We put

$$w = w(u) = u - \partial\varphi \wedge v_1. \qquad (*)$$

Then we have $\bar{\partial}w = \partial\varphi \wedge (v_2 - \bar{\partial}v_1) + \bar{\partial}\varphi \wedge v_3 + \partial\varphi \wedge \bar{\partial}\varphi \wedge v_4$. By calculating, the following hold.

Proposition 2

(1) $\partial\varphi \wedge v_2 \in L^{p,q}(D)$

(2) $\bar{\partial}\varphi \wedge v_3 \in L^{p,q}(D)$

(3) $\partial\varphi \wedge \bar{\partial}\varphi \wedge v_4 \in L^{p,q}(D)\ for\ b - a > -1$

(4) $\partial\varphi \wedge \bar{\partial}v_1 \in L^{p,q}(D)\ for\ b - a > -1$

Then the following proposition holds from Proposition 2.

Proposition 3 For $\mathrm{w} \in C^{p,q-1}(D)$ in (∗), $\bar{\partial}\mathrm{w} \in L^{p,q}(D)$ holds.

Next step is to find conditions such that $\bar{\partial}\mathrm{w}$ is L^2-cohomologous to 0. We assume that there exists $w_{p,q-1} \in L^{p,q-1}(D)$ satisfying the following:

(i)
$$w_{p,q-1} = \sum_{|I|=p,|J|=q-1} w_{1I\,J} d\zeta_I \wedge d\overline{\zeta_J}$$
$$+ \partial\varphi \wedge \sum_{|K|=p-1,|L|=q-1} w_{2K\,L} d\zeta_K \wedge d\overline{\zeta_L}$$
$$+ \bar{\partial}\varphi \wedge \left(\partial\varphi \wedge \sum_{|M|=p-1,|N|=q-2} w_{3M\,N} d\zeta_M \wedge d\overline{\zeta_N} \right)$$
$$+ \bar{\partial}\varphi \wedge \left(\sum_{|P|=p,|Q|=q-1} w_{4P\,Q} d\zeta_P \wedge d\overline{\zeta_Q} \right)$$

(ii) $\bar{\partial}\mathrm{w} = \bar{\partial} w_{p,q-1}$

For a defining function φ of the domain D and $\mathrm{t} < 0$, we put $D_t := \{ \varphi < \mathrm{t} \}$ and $\mathrm{d}S_t$ denotes the volume element of D_t with respect to the complex Euclidean metric.

Then we have the following by Fubini's theorem.

Proposition 4

(1) $\liminf_{t \nearrow 0} \int_{D_t \cap U} | w_{1IJ}|^2 \, \mathrm{d}S_t = 0 \; for \; \mathrm{b} \geq 1$

(2) $\liminf_{t \nearrow 0} \int_{D_t \cap U} | w_{2K\,L}|^2 \, \mathrm{d}S_t = 0 \; for \; \mathrm{a} \geq 1$

By comparing $\bar{\partial}\mathrm{w}$ with $\bar{\partial} w_{p,q-1}$, we have the following from Proposition 4.

Proposition 5

We assume that $\bar{\partial}\mathrm{w}$ is L^2-cohomologous to 0 with respect to the complete metric $\mathrm{d}s^2$. Then we have the following:

(1) $\bar{\partial}\varphi \wedge (v_2 - \bar{\partial} v_1) = 0$ holds on $\partial \mathrm{D} \cap U$.

(2) $\bar{\partial}\varphi \wedge \bar{\partial}\mathrm{w} = 0$ holds on $\partial \mathrm{D} \cap U$.

Here we will prove Theorem of Ohsawa by using previous propositions.

Proof of Theorem of Ohsawa

Let ρ be a real-valued smooth function which has the support in a neighborhood U of $x \in \partial \mathrm{D}$ and

$\rho \equiv 1$ on a neighborhood $V \subset U$. We put

$$u_k := \rho\, \overline{\zeta_2^k}\, d\overline{\zeta_2} \wedge \cdots \wedge d\overline{\zeta_q} \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{p+1} \text{ for } k \in \mathbf{N}$$

and $w_k :=$ w (u_k) in accordance with (∗). Then we see that $\bar\partial w_k - \bar\partial w_l$ is not to L^2-cohomologous to 0 if $k \neq l$ from Proposition 5. Hence our claim holds. □

3. Remarks

For $p + q = n$, we put $u = f\, d\overline{\zeta_2} \wedge \cdots \wedge d\overline{\zeta_q} \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{p+1}$ which is a smooth $(p, q-1)$-form with the support in $\overline{D} \cap U$. Then we see that $\bar\partial u = \bar\partial f \wedge d\overline{\zeta_2} \wedge \cdots \wedge d\overline{\zeta_q} \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{p+1}$ is the smooth (p, q)- form.

On the other hand, by using four forms $\{u_k\}$ $(k = 1 \sim 4)$ with $e^*(\tau_1)u_k = e^*(\overline{\tau_1})u_k = 0$, we can uniquely describe

$$d\overline{\zeta_2} \wedge \cdots \wedge d\overline{\zeta_q} \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{p+1} = u_1 + \partial\varphi \wedge u_2 + \bar\partial\varphi \wedge u_3 + \partial\varphi \wedge \bar\partial\varphi \wedge u_4$$

from Proposition 1 (1). From now on, we will describe

$$\bar\partial f = f_{\bar\partial\varphi}\, \bar\partial\varphi + \nu_f$$

for any smooth function f by using a $(0,1)$- form ν_f with $e^*(\tau_1)\nu_f = e^*(\overline{\tau_1})\nu_f = 0$. The we have

$$\bar\partial u = \nu_f \wedge u_1 + \partial\varphi \wedge (-\nu_f \wedge u_2) + \bar\partial\varphi \wedge f_{\bar\partial\varphi} u_1 + \partial\varphi \wedge \bar\partial\varphi \wedge \left(f_{\bar\partial\varphi} u_2 + \nu_f \wedge u_4\right)$$

On the other hand, there exist $u_{11} \in C^{p-1,q-1}(\overline{D} \cap U)$ and $u_{12} \in C^{p-1,q-1}(\overline{D} \cap U)$ uniquely such that

$$\nu_f \wedge u_1 = \partial\bar\partial\varphi \wedge u_{11} - \partial\varphi \wedge \bar\partial\varphi \wedge u_{12},$$

satisfying $e^*(\tau_1)u_{11} = e^*(\overline{\tau_1})u_{11} = 0$ and $e^*(\tau_1)u_{12} = e^*(\overline{\tau_1})u_{12} = 0$ from Proposition 1 (2). Especially, u_{11} may have a divisor ν_f.

Then we have

$$\bar\partial u = \partial\bar\partial\varphi \wedge u_{11} + \partial\varphi \wedge (-\nu_f \wedge u_2) + \bar\partial\varphi \wedge f_{\bar\partial\varphi} u_1 + \partial\varphi \wedge \bar\partial\varphi \wedge \left(f_{\bar\partial\varphi} u_2 + \nu_f \wedge u_4 - u_{12}\right).$$

We put

$$v_1 = u_{11},\ v_2 = -\nu_f \wedge u_2,\ v_3 = f_{\bar\partial\varphi} u_1,\ v_4 = f_{\bar\partial\varphi} u_2 + \nu_f \wedge u_4 - u_{12}.$$

Then we have

$$\bar\partial u = \partial\bar\partial\varphi \wedge v_1 + \partial\varphi \wedge v_2 + \bar\partial\varphi \wedge v_3 + \partial\varphi \wedge \bar\partial\varphi \wedge v_4.$$

Remark 6

In our situation, both v_1 and v_2 may have a divisor ν_f. Hene we should claim that

$$\bar\partial\varphi \wedge \left(v_2 - \bar\partial v_1\right) = 0$$

holds if $\bar\partial w$ is L^2-cohomologous to 0 from Proposition 5 (1).

Remark 7

In [Oh], assumptions about the claim for complete metrics is the following: " $1 \leq a \leq b < a + 3$". The condition $b < a + 3$ is needed for L^2-integrability of φv_4 in [Oh] p.107. In our argument, this term does not appear.

4. Infinite dimensionality of reduced L^2 cohomology

Let X be an n-dimensional non-compact complex manifold and (X, ds^2) be a complete hermitian manifolds. Let $(\cdot, \cdot)$ ($resp.$ $\|\cdot\|$) be the inner product (resp. norm) of (p, q)-forms on X. Let $|\cdot|$ be the pointwise norm of (p, q)-forms on X. Let $C_0^{p,q}(X)$ be the space of (p, q)-forms on X with compact supports and $L^{p,q}(X)$ be the space of L^2-integrable (p, q)-forms on X with respect to ds^2. Let $\bar{\partial}: L^{p,q}(X) \to L^{p,q+1}(X)$ be the $\bar{\partial}$ operator and $\bar{\partial}^*: L^{p,q+1}(X) \to L^{p,q}(X)$ be the adjoint operator. Then

$$\mathcal{H}_{(2)}^{p,q}(X) := \{\varphi \in L^{p,q}(X) \mid \bar{\partial}\varphi = 0, \bar{\partial}^*\varphi = 0\}$$

denotes the L^2harmonic space with degree (p, q). We put

$$N_{\bar{\partial}}^{p,q}(X) := \{u \in L^{p,q}(X) | \bar{\partial}u = 0\}, R_{\bar{\partial}}^{p,q}(X) := \{\bar{\partial}u \in L^{p,q}(X) | \ u \in L^{p,q-1}(X)\}$$

and

$$H^{p,q}(X) := N_{\bar{\partial}}^{p,q}(X)/\overline{R_{\bar{\partial}}^{p,q}(X)}$$

, where $\overline{R_{\bar{\partial}}^{p,q}(X)}$ denotes the topological closure of $R_{\bar{\partial}}^{p,q}(X)$ in $L^{p,q}(X)$. Let $H^{p,q}(X)$ denotes the reduced L^2cohomology with degree (p, q). These are isomorphic to each other.

In general, it is well known that the following holds for the closedness of $R_{\bar{\partial}}^{p,q}(X)$.

Proposition 8 ([H]) For any $\varphi \in C_0^{p,q}(X)$, there is a constant $C > 0$ such that the following holds:

$$\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 > C\|\varphi\|^2.$$

Then $R_{\bar{\partial}}^{p,q} \subset L^{p,q}(X)$ and $R_{\bar{\partial}}^{p,q+1} \subset L^{p,q+1}(X)$ are closed subsets.

Infinite dimensionality of the middle L^2cohomoloy has been investigated by many articles. It is expected that this property will play important roles in various situations. If an n-dimensional complex manifold satisfies Kähler hyperbolicity([Gr]) or strictly Kähler convexity([Mc]), it is well known that Proposition 8 holds for $p + q \neq n$.

In 2006, B. Y. Chen had claimed that infinite dimensionality of the middle L^2cohomoloy holds for any non-compact Kähler hyperbolic manifolds([Ch]). However, unfortunately, the claim has counterexamples in the case of finite volumes and bounded curvatures([Ye]).

5. Localization

For nonvanishing L^2cohomoloy, it is known that the following holds.

Proposition 9 (cf : [G-T] Proposition 8.4)

Let Xe a complete hermitian manifold. Let α be an element of $N_{\bar{\partial}}^{p,q}(X)$. We assume that there exist a $\gamma \in N_{\bar{\partial}}^{n-p,n-q}(X)$ such that $\int_X \alpha \wedge \gamma \neq 0$ holds. Then $\alpha \notin N_{\bar{\partial}}^{p,q}(X)$ holds. Especially, the L^2 cohomoloy

$H^{p,q}$(X) is nonvanishing.

For the unit ball $\mathrm{B} := \{(z_1, z_2) \in \mathbf{C}^2 \mid \|z\| < 1\}$, we will induce a complete hermitian metric ds_φ^2 satisfying the following conditions by using a differential automorphism φ:

(i) For $p = (1,0) \in \partial\mathrm{B}$ and a sufficiently small neighborhood $p \in \mathrm{U} \subset \mathbf{C}^2$, we put $ds_\varphi^2 = ds_{hyp}^2$ on U, where ds_{hyp}^2 denotes the Poincaré metric of B. This metric satisfies conditions in Theorem of Ohsawa.

(ii) We put $ds_\varphi^2 = ds_E^2$ on the hemisphere $\mathrm{B}_- := \{z \in \mathrm{B} \mid Re\, z_1 < 0\}$, where ds_E^2 denotes the complex Euclidean metric of $\mathbf{C}^2$.

For the complex Euclidean metric $(\mathbf{C}^2, ds_E^2)$, it is well-known that $H^{p,q}(\mathbf{C}^n, ds_E^2) = \{0\}$ holds for $0 \leq \mathrm{p} \leq \mathrm{n},\ 0 \leq \mathrm{q} \leq \mathrm{n}$ ([Lo], p345). Therefore $H^{1,1}(\mathrm{B}, ds_\varphi^2) = \{0\}$ holds. Then the following claim holds by using Proposition 9. It is a Serre duality-like proposition.

Proposition 10

Let $\alpha \in N_{\bar\partial}^{1,1}(B, ds_{hyp}^2)$ be a representative of non-zero L^2 cohomology with $supp\ \alpha \subset \bar{U}$ and γ be an element of $N_{\bar\partial}^{1,1}(B, ds_{hy}^2\ )$ such that $\int_X \alpha \wedge \gamma \neq 0$. Then $supp\ \gamma := \overline{\{z \in \bar{\mathrm{B}} \mid \gamma(z) \neq 0\}} \not\subset \bar{U}$ holds.

6. Examples

We will give examples which have infinite dimensionality of the middle $\boldsymbol{L}^2$ cohomology by Theorem of Ohsawa.

Example 11 Let $D = \{ z \in \mathbf{C}^n \mid |z| < 1\}$ be the unit ball and ds_D^2 be the Poincaré metric. Then (D, ds_D^2) satisfies conditions for Proposition 8 and any point of ∂D has a neighborhood satisfying conditions for Theorem of Ohsawa. Therefore $dim_C H^{p,q}(\mathrm{D}) = +\infty$ holds.

Example 12 Let $D = \left\{ Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \middle| I_2 - Z\ {}^t\bar{Z} > 0 \right\} \subset \mathbf{C}^3$ be the bounded symmetric domain and ds_D^2 be the Bergman metric of D. Then(D, ds_D^2) satisfies conditions for Proposition 8 ([Do]) and almost everywhere point of ∂D has a neighborhood satisfying conditions for Theorem of Ohsawa. Therefore $dim_C H^{p,q}(\mathrm{D}) = +\infty$ holds.

References

[Ch] B. Y. Chen, Infinite Dimensionality of the Middle L^2–cohomology on Non-compact K¨ahler Hyperbolic Manifolds, Publ. RIMS, Kyoto Univ., 42 (2006), 683–689.

[Do] H. Donnelly, L_2cohomology of the Bergman metric for weakly pseudoconvex domains, Illinois J. Math., 41 (1997), 151-160.

[Gr] M. Gromov, Kähler hyperbolicity and L^2-Hodge theory, J. Differential Geom. 33 (1991), 263-292.

[G-T] V. Gol'dshtein, and M. Troyanov, Sobolev Inequalities for Differential Forms and $L_{q,p}$-Cohomology, The Journal of Geometric Analysis Volume 16, Number 4, 2006.

[H] L.Hörmander, An introduction to complex analysis in several variables, North Holland1973.

[Lo] J Lott, The zero-in-the -spectrum question, 42 (1996), 341-376.

[Mc] JD. McNeal, L^2 harmonic forms on some complete Kähler manifolds."Mathematische Annalen 323.2 (2002), 319-34

[Oh] T. Ohsawa, On the infinite dimensionality of the middle L^2 cohomology of complex domains, Publ. RIMS, Kyoto Univ., 25 (1989), 499-502.

[Ye] N. Yeganefar, L^2-cohomology of negatively curved Kaehler manifolds of finite volume, Geometric and Functional Analysis GAFA, Volume 15, pages 1128-1143, (2005)

[We] A. Weil, Introduction à l'étude des variétés kählériennes, Hermann, 1958.