ON CERTAIN SEXTIC ALGEBRAIC NUMBER FIELDS

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Abstract. In this paper, we give an overview of the papers [8] and [9] which deal with the computation of discriminant, integral basis and testing of monogenity of certain classes of sextic number fields.

1. Introduction and statements of results

The problem of computation of discriminant as well as integral basis of algebraic number fields has attracted the attention of several mathematicians. Let K be an algebraic number field and p be a prime number. Let $\mathbf{Z}_{(p)}$ denote the localisation of $\mathbf{Z}$ at the prime ideal $p\mathbf{Z}$ and $I_{(p)}$ the integral closure of the ring $\mathbf{Z}_{(p)}$ in K. Then $I_{(p)} = \{\frac{\alpha}{a} |\ \alpha \in A_K,\ a \in \mathbf{Z} \setminus p\mathbf{Z}\}$ is a free $\mathbf{Z}_{(p)}$-module of rank equal to the degree of K. A basis of $I_{(p)}$ as a $\mathbf{Z}_{(p)}$-module is called a p-integral basis of K.

Clearly an integral basis of an algebraic number field K is a p-integral basis of K for each prime number p. For an algebraic number field $K = \mathbf{Q}(\theta)$ with θ in the ring A_K of algebraic integers of K, if a prime p does not divide $\operatorname{ind}\theta$, then by Lagrange's theorem for finite groups, $A_K \subseteq \mathbf{Z}_{(p)}[\theta]$ and hence $I_{(p)} = \mathbf{Z}_{(p)}[\theta]$, i.e., $\{1,\theta,\cdots,\theta^{n-1}\}$ is a p-integral basis of K, n being the degree of K. In what follows, for a prime number p, v_p will stand for the p-adic valuation of $\mathbf{Q}$ defined for any non-zero integer m to be the highest power of p dividing m and m_p for the integer $m/p^{v_p(m)}$.

In [9], we find a p-integral basis and $v_p(d_K)$ for all primes p when $K = \mathbf{Q}(\theta)$ is a sextic field with θ a root of an irreducible trinomial of the type $x^6 + ax + b$ belonging to $\mathbf{Z}[x]$. If a prime p is such that p^5 divides a and p^6 divides b, then θ/p is a root of the polynomial $x^6 + (a/p^5)x + (b/p^6)$ having integer coefficients. So we may assume that for each prime p

$$\text{either } v_p(a) < 5 \ \text{ or } v_p(b) < 6. \tag{1.1}$$

It is well-known that the discriminant D of $f(x)$ is given by (cf. [5, Exercise 4.5.4])

$$D = 5^5a^6 - 6^6b^5. \tag{1.2}$$

Assuming (1.1), in [9], we have proved:

Theorem 1.1. Let $K = \mathbf{Q}(\theta)$ be an algebraic number field with θ a root of an irreducible trinomial $f(x) = x^6 + ax + b$ belonging to $\mathbf{Z}[x]$. Let D, d_K denote respectively the discriminants of $f(x)$ and K. Then a p-integral basis together with the values $v_p(D)$, $v_p(d_K)$ are given in Tables $1 - 4$ according as p equals 2 or 3 or 5 or $p > 5$.

Table 1. 2-integral basis of sextic field K and $v_2(d_K)$

Case	Conditions	$v_2(D)$	$v_2(d_K)$	2-integral basis
E1	$v_2(a)=0$	0	0	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
E2	$v_2(b)=1, v_2(a)=1$	6	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
E3	$v_2(b)=1, v_2(a)\geq 2$	11	11	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
E4	$v_2(b)\geq 2, v_2(a)=1$	6	4	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5/2\}$
E5	$v_2(b)\geq 3, v_2(a)=2$	12	4	$\{1,\theta,\theta^2,\theta^3/2,\theta^4/2,\theta^5/2^2\}$
E6	$v_2(b)=3, v_2(a)=3$	18	6	$\{1,\theta,\theta^2/2,\theta^3/2,\theta^4/2^2,\theta^5/2^2\}$
E7	$v_2(b)=3, v_2(a)\geq 4$	21	9	$\{1,\theta,\theta^2/2,\theta^3/2,\theta^4/2^2,\theta^5/2^2\}$
E8	$v_2(b)\geq 4, v_2(a)=3$	18	4	$\{1,\theta,\theta^2/2,\theta^3/2,\theta^4/2^2,\theta^5/2^3\}$
E9	$v_2(b)\geq 5, v_2(a)=4$	24	4	$\{1,\theta,\theta^2/2,\theta^3/2^2,\theta^4/2^3,\theta^5/2^4\}$
E10	$v_2(b)=5, v_2(a)=5$	30	10	$\{1,\theta,\theta^2/2,\theta^3/2^2,\theta^4/2^3,\theta^5/2^4\}$
E11	$v_2(b)=5, v_2(a)\geq 6$	31	11	$\{1,\theta,\theta^2/2,\theta^3/2^2,\theta^4/2^3,\theta^5/2^4\}$
E12	$v_2(a)=1$ $b\equiv 3 (\mathrm{mod}\ 4)$	7	7	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
E13	$v_2(a)=1$ $b\equiv 1 (\mathrm{mod}\ 4)$ $v_2(D)$ odd	$v_2(D)\geq 9$ odd	7	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(-5x_0^5+x_0^4\theta+x_0^3\theta^2+x_0^2\theta^3+x_0\theta^4+\theta^5)/2^{\frac{v_2(D)-7}{2}}\}$, $5(\frac{a}{2})x_0+3b\equiv 0(\mathrm{mod}\ 2^{\frac{v_2(D)-7}{2}})$
E14	$v_2(a)=1$ $b\equiv 1 (\mathrm{mod}\ 4)$, $v_2(D)$ even, $D_2\equiv 3 (\mathrm{mod}\ 4)$	$v_2(D)\geq 8$ even	4	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(-5x_1^5+x_1^4\theta+x_1^3\theta^2+x_1^2\theta^3+x_1\theta^4+\theta^5)/2^{\frac{v_2(D)-4}{2}}\}$, $5(\frac{a}{2})x_1-2^u+3b\equiv 0(\mathrm{mod}\ 2^{\frac{v_2(D)-4}{2}})$, $u=\frac{v_2(D)-6}{2}$
E15	$v_2(a)=1$ $b\equiv 1 (\mathrm{mod}\ 4)$, $v_2(D)$ even, $D_2\equiv 1 (\mathrm{mod}\ 4)$	$v_2(D)\geq 8$ even	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(-5x_2^5+x_2^4\theta+x_2^3\theta^2+x_2^2\theta^3+x_2\theta^4+\theta^5)/2^{\frac{v_2(D)-6}{2}}\}$, $5(\frac{a}{2})x_2+3b\equiv 0(\mathrm{mod}\ 2^{\frac{v_2(D)-6}{2}})$
E16	$v_2(a)\geq 2$ $b\equiv 1 (\mathrm{mod}\ 4)$	6	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
E17	$v_2(a)\geq 2$ $b\equiv 3 (\mathrm{mod}\ 4)$	6	0	$\{1,\theta,\theta^2,(1+\theta^3)/2,(\theta+\theta^4)/2,(\theta^2+\theta^5)/2\}$
E18	$v_2(b)=2, v_2(a)=2$	12	6	$\{1,\theta,\theta^2,\theta^3/2,\theta^4/2,\theta^5/2\}$
E19	$v_2(b)=2, v_2(a)=3$ $\frac{b}{4}\equiv 3 (\mathrm{mod}\ 4)$	16	6	$\{1,\theta,\theta^2,\theta^3/2,(2\theta+\theta^4)/2^2,(2\theta^2+\theta^5)/2^2\}$
E20	$v_2(b)=2, v_2(a)\geq 4$ $\frac{b}{4}\equiv 3 (\mathrm{mod}\ 4)$	16	4	$\{1,\theta,\theta^2,(2+\theta^3)/2^2,(2\theta+\theta^4)/2^2,(2\theta^2+\theta^5)/2^2\}$
E21	$v_2(b)=2, v_2(a)\geq 3$ $\frac{b}{4}\equiv 1 (\mathrm{mod}\ 4)$	16	8	$\{1,\theta,\theta^2,\theta^3/2,\theta^4/2,(2\theta^2+\theta^5)/2^2\}$
E22	$v_2(b)=4, v_2(a)=4$ $\frac{b}{16}\equiv 1 (\mathrm{mod}\ 4)$	24	4	$\{1,\theta,\theta^2/2,\theta^3/2^2,(4\theta+\theta^4)/2^3,$ $(8\theta+4\theta^2+\theta^5)/2^4\}$
E23	$v_2(b)=4, v_2(a)=4$ $\frac{b}{16}\equiv 3 (\mathrm{mod}\ 4)$	24	6	$\{1,\theta,\theta^2/2,\theta^3/2^2,(4\theta+\theta^4)/2^3,(4\theta^2+\theta^5)/2^3\}$
E24	$v_2(b)=4, v_2(a)=5$ $\frac{b}{16}\equiv 3 (\mathrm{mod}\ 4)$	26	6	$\{1,\theta,\theta^2/2,\theta^3/2^2,(4\theta+\theta^4)/2^3,(4\theta^2+\theta^5)/2^4\}$
E25	$v_2(b)=4, v_2(a)\geq 6$ $\frac{b}{16}\equiv 3 (\mathrm{mod}\ 4)$	26	4	$\{1,\theta,\theta^2/2,(4+\theta^3)/2^3,(4\theta+\theta^4)/2^3,(4\theta^2+\theta^5)/2^4\}$ 2
E26	$v_2(b)=4, v_2(a)\geq 5$ $\frac{b}{16}\equiv 1 (\mathrm{mod}\ 4)$	26	8	$\{1,\theta,\theta^2/2,\theta^3/2^2,(4\theta+\theta^4)/2^3,\theta^5/2^3\}$

Table 2. 3-integral basis of sextic field K and $v_3(d_K)$

Case	Conditions	$v_3(D)$	$v_3(d_K)$	3-integral basis
F1	$v_3(a)=0$	0	0	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
F2	$v_3(b)=1, v_3(a)=1$	6	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
F3	$v_3(b)=1, v_3(a)\geq 2$	11	11	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
F4	$v_3(b)\geq 2, v_3(a)=1$	6	4	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5/3\}$
F5	$v_3(b)=2, v_3(a)=2$	12	6	$\{1,\theta,\theta^2,\theta^3/3,\theta^4/3,\theta^5/3\}$
F6	$v_3(b)=2, v_3(a)\geq 3$	16	10	$\{1,\theta,\theta^2,\theta^3/3,\theta^4/3,\theta^5/3\}$
F7	$v_3(b)\geq 3, v_3(a)=2$	12	4	$\{1,\theta,\theta^2,\theta^3/3,\theta^4/3,\theta^5/3^2\}$
F8	$v_3(b)\geq 4, v_3(a)=3$	18	4	$\{1,\theta,\theta^2/3,\theta^3/3,\theta^4/3^2,\theta^5/3^3\}$
F9	$v_3(b)=4, v_3(a)=4$	24	8	$\{1,\theta,\theta^2/3,\theta^3/3^2,\theta^4/3^2,\theta^5/3^3\}$
F10	$v_3(b)=4, v_3(a)\geq 5$	26	10	$\{1,\theta,\theta^2/3,\theta^3/3^2,\theta^4/3^2,\theta^5/3^3\}$
F11	$v_3(b)\geq 5, v_3(a)=4$	24	4	$\{1,\theta,\theta^2/3,\theta^3/3^2,\theta^4/3^3,\theta^5/3^4\}$
F12	$v_3(b)=5, v_3(a)=5$	30	10	$\{1,\theta,\theta^2/3,\theta^3/3^2,\theta^4/3^3,\theta^5/3^4\}$
F13	$v_3(b)=5, v_3(a)\geq 6$	31	11	$\{1,\theta,\theta^2/3,\theta^3/3^2,\theta^4/3^3,\theta^5/3^4\}$
F14	$v_3(a)=1$ $b\equiv 1 \pmod 3$	6	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
F15	$v_3(a)\geq 2$ $b\equiv 4$ *or* $7 \pmod 9$	6	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
F16	$v_3(a)\geq 2$ $b\equiv 1 \pmod 9$	6	2	$\{1,\theta,\theta^2,\theta^3,(1-\theta^2+\theta^4)/3,$ $(\theta-\theta^3+\theta^5)/3\}$
F17	$v_3(a)\geq 2$ $b\equiv 2$ *or* $5 \pmod 9$	6	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
F18	$v_3(a)\geq 2$ $b\equiv -1 \pmod 9$	6	2	$\{1,\theta,\theta^2,\theta^3,(1+\theta^2+\theta^4)/3,$ $(\theta+\theta^3+\theta^5)/3\}$
F19	$b\equiv 2 \pmod 9$ $a\equiv \pm 3 \pmod 9$	7	5	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(\epsilon+\theta+\epsilon\theta^2+\theta^3+\epsilon\theta^4+\theta^5)/3\}$, ϵ is -1 or 1 according as $a\equiv 3$ or $-3 \pmod 9$
F20	$b\equiv -1 \pmod 9$ $a\equiv \pm 3 \pmod 9$	7	7	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
F21	$b\equiv 5 \pmod 9$ $a\equiv \pm 3 \pmod 9$ $v_3(D)=8$	8	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(\epsilon+\theta+\epsilon\theta^2+\theta^3+\epsilon\theta^4+\theta^5)/3\}$, ϵ is -1 or 1 according as $a\equiv -3$ or $3 \pmod 9$
F22	$b\equiv 5 \pmod 9$ $a\equiv \pm 3 \pmod 9$ $v_3(D)=9$	9	3	$\{1,\theta,\theta^2,\theta^3,(-1-\theta+\theta^3+\theta^4)/3,$ $(-5x_1^5+x_1^4\theta+x_1^3\theta^2+x_1^2\theta^3+x_1\theta^4+\theta^5)/3^2\}$, $5(\frac{a}{3})x_1+2b\equiv 0 \pmod 9$
F23	$b\equiv 5 \pmod 9$ $a\equiv \pm 3 \pmod 9$ $v_3(D)\geq 10$ and even	$v_3(D)\geq 10$ even	4	$\{1,\theta,\theta^2,\theta^3,(-1-\theta+\theta^3+\theta^4)/3,$ $(-5x_2^5+x_2^4\theta+x_2^3\theta^2+x_2^2\theta^3+x_2\theta^4+\theta^5)/3^{\frac{v_3(D)-6}{2}}\}$, $5(\frac{a}{3})x_2+2b\equiv 0 (\bmod 3^{\frac{v_3(D)-6}{2}})$
F24	$b\equiv 5 \pmod 9$ $a\equiv \pm 3 \pmod 9$ $v_3(D)\geq 11$ and odd	$v_3(D)\geq 11$ odd	3	$\{1,\theta,\theta^2,\theta^3,(-1-\theta+\theta^3+\theta^4)/3,$ $(-5x_3^5+x_3^4\theta+x_3^3\theta^2+x_3^2\theta^3+x_3\theta^4+\theta^5)/3^{\frac{v_3(D)-5}{2}}\}$, $5(\frac{a}{3})x_3+2b\equiv 0 (\bmod 3^{\frac{v_3(D)-5}{2}})$
F25	$v_3(b)=3, v_3(a)=3$	18	6	$\{1,\theta,\theta^2/3,\theta^3/3,\theta^4/3^2,\theta^5/3^2\}$
F26	$v_3(b)=3, v_3(a)\geq 4$ $B=\frac{b}{3^3}$, $v_3(B^3-B)=1$	21	7	$\{1,\theta,\theta^2/3,\theta^3/3,\theta^4/3^2,$ $(\theta^2+3B)^2\theta/3^3\}$
F27	$v_3(b)=3, v_3(a)\geq 4$ $B=\frac{b}{3^3}$, $v_3(B^3-B)\geq 2$	21	3	$\{1,\theta,\theta^2/3,(\theta^2+3B)\theta/3^2,$ $(\theta^2+3B)^2/3^3,(\theta^2+3B)^2\theta/3^3\}$

Table 3. 5-integral basis of sextic field K and $v_5(d_K)$

Case	Conditions	$v_5(D)$	$v_5(d_K)$	5-integral basis
G1	$v_5(b)=0$	0	0	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
G2	$v_5(b)=1, v_5(a)=0$, $a^4\not\equiv 21(\text{mod } 25)$, $b\not\equiv a^2-a^6(\text{mod } 25)$, $a^2\not\equiv (b/5)(\text{mod } 5)$	5	5	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
G3	$v_5(b)=1, v_5(a)=0$, $a^4\not\equiv 21(\text{mod } 25)$, $b\not\equiv a^2-a^6(\text{mod } 25)$, $a^2\equiv (b/5)(\text{mod } 5)$	6	6	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
G4	$v_5(b)=1, v_5(a)=0$, $a^4\not\equiv 21(\text{mod } 25)$, $b\equiv a^2-a^6(\text{mod } 25)$	5	3	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(\theta-a^3\theta^2+a^2\theta^3-a\theta^4+\theta^5)/5\}$
G5	$v_5(b)=1, v_5(a)=0$, $a^4\equiv 21(\text{mod } 25)$, $b\not\equiv a^2-a^6(\text{mod } 25)$	5	5	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
G6	$v_5(b)=1, v_5(a)=0$, $a^4\equiv 21(\text{mod } 25)$, $b\equiv a^2-a^6(\text{mod } 25)$, $v_5(D)$ is odd	$v_5(D)\geq 7$	3	$\{1,\theta,\theta^2,\theta^3,$ $(-4a^3\theta+3a^2\theta^2-2a\theta^3+\theta^4)/5,$ $(-5x_0^5+x_0^4\theta+x_0^3\theta^2+x_0^2\theta^3+x_0\theta^4+\theta^5)/5^{\frac{v_5(D)-5}{2}}\}$, $ax_0+6(\frac{b}{5})\equiv 0(\text{mod } 5^{\frac{v_5(D)-5}{2}})$
G7	$v_5(b)=1, v_5(a)=0$, $a^4\equiv 21(\text{mod } 25)$, $b\equiv a^2-a^6(\text{mod } 25)$, $v_5(D)$ is even	$v_5(D)\geq 8$	2	$\{1,\theta,\theta^2,\theta^3,$ $(-4a^3\theta+3a^2\theta^2-2a\theta^3+\theta^4)/5,$ $(-5x_1^5+x_1^4\theta+x_1^3\theta^2+x_1^2\theta^3+x_1\theta^4+\theta^5)/5^{\frac{v_5(D)-4}{2}}\}$, $ax_1+6(\frac{b}{5})\equiv 0(\text{mod } 5^{\frac{v_5(D)-4}{2}})$
G8	$v_5(b)=1, v_5(a)\geq 1$	5	5	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
G9	$v_5(b)\geq 2, v_5(a)=0$ $a^4\not\equiv 1(\text{mod } 25)$	5	5	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
G10	$v_5(b)\geq 2, v_5(a)=0$, $a^4\equiv 1(\text{mod } 25)$	5	3	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(\theta-a^3\theta^2+a^2\theta^3-a\theta^4+\theta^5)/5\}$
G11	$v_5(b)=2, v_5(a)=1$	10	8	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5/5\}$
G12	$v_5(b)=2, v_5(a)\geq 2$	10	4	$\{1,\theta,\theta^2,\theta^3/5,\theta^4/5,\theta^5/5\}$
G13	$v_5(b)\geq 3, v_5(a)=1$	11	9	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5/5\}$
G14	$v_5(b)=3, v_5(a)=2$	15	7	$\{1,\theta,\theta^2,\theta^3/5,\theta^4/5,\theta^5/5^2\}$
G15	$v_5(b)=3, v_5(a)\geq 3$	15	3	$\{1,\theta,\theta^2/5,\theta^3/5,\theta^4/5^2,\theta^5/5^2\}$
G16	$v_5(b)\geq 4, v_5(a)=2$	17	9	$\{1,\theta,\theta^2,\theta^3/5,\theta^4/5,\theta^5/5^2\}$
G17	$v_5(b)=4, v_5(a)=3$	20	6	$\{1,\theta,\theta^2/5,\theta^3/5,\theta^4/5^2,\theta^5/5^3\}$
G18	$v_5(b)=4, v_5(a)\geq 4$	20	4	$\{1,\theta,\theta^2/5,\theta^3/5^2,\theta^4/5^2,\theta^5/5^3\}$
G19	$v_5(b)\geq 5, v_5(a)=3$	23	9	$\{1,\theta,\theta^2/5,\theta^3/5,\theta^4/5^2,\theta^5/5^3\}$
G20	$v_5(b)=5, v_5(a)=4$	25	5	$\{1,\theta,\theta^2/5,\theta^3/5^2,\theta^4/5^3,\theta^5/5^4\}$
G21	$v_5(b)=5, v_5(a)\geq 5$	25	5	$\{1,\theta,\theta^2/5,\theta^3/5^2,\theta^4/5^3,\theta^5/5^4\}$
G22	$v_5(b)\geq 6, v_5(a)=4$	29	9	$\{1,\theta,\theta^2/5,\theta^3/5^2,\theta^4/5^3,\theta^5/5^4\}$

Table 4. p-integral basis of sextic field K and $v_p(d_K)$ for $p > 5$

Case	Conditions	$v_p(D)$	$v_p(d_K)$	p-integral basis
H1	$v_p(b) = 0, v_p(a) \geq 1$ or $v_p(a) = 0, v_p(b) \geq 1$	0	0	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
H2	$v_p(b) = 1, v_p(a) \geq 1$	5	5	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5\}$
H3	$v_p(a) = 1, v_p(b) \geq 2$	6	4	$\{1,\theta,\theta^2,\theta^3,\theta^4,\theta^5/p\}$
H4	$v_p(b) = 2, v_p(a) \geq 2$	10	4	$\{1,\theta,\theta^2,\theta^3/p,\theta^4/p,\theta^5/p\}$
H5	$v_p(a) = 2, v_p(b) \geq 3$	12	4	$\{1,\theta,\theta^2,\theta^3/p,\theta^4/p,\theta^5/p^2\}$
H6	$v_p(b) = 3, v_p(a) \geq 3$	15	3	$\{1,\theta,\theta^2/p,\theta^3/p,\theta^4/p^2,\theta^5/p^2\}$
H7	$v_p(a) = 3, v_p(b) \geq 4$	18	4	$\{1,\theta,\theta^2/p,\theta^3/p,\theta^4/p^2,\theta^5/p^3\}$
H8	$v_p(b) = 4, v_p(a) \geq 4$	20	4	$\{1,\theta,\theta^2/p,\theta^3/p^2,\theta^4/p^2,\theta^5/p^3\}$
H9	$v_p(a) = 4, v_p(b) \geq 5$	24	4	$\{1,\theta,\theta^2/p,\theta^3/p^2,\theta^4/p^3,\theta^5/p^4\}$
H10	$v_p(b) = 5, v_p(a) \geq 5$	25	5	$\{1,\theta,\theta^2/p,\theta^3/p^2,\theta^4/p^3,\theta^5/p^4\}$
H11	$v_p(ab) = 0$, $v_p(D)$ is even	$v_p(D)$	0	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(x + y\theta + z\theta^2 + v\theta^3 + w\theta^4 + \theta^5)/p^m\}$, $m = v_p(D)/2$, $6x \equiv 5a (\text{mod } p^m)$, $(5a)^4 y \equiv (6b)^4 (\text{mod } p^m)$, $(5a)^3 z \equiv -(6b)^3 (\text{mod } p^m)$, $(5a)^2 v \equiv (6b)^2 (\text{mod } p^m)$, $5aw \equiv -6b (\text{mod } p^m)$
H12	$v_p(ab) = 0$, $v_p(D)$ is odd	$v_p(D)$	1	$\{1,\theta,\theta^2,\theta^3,\theta^4,$ $(x + y\theta + z\theta^2 + v\theta^3 + w\theta^4 + \theta^5)/p^m\}$, $m = (v_p(D) - 1)/2$, $6x \equiv 5a (\text{mod } p^m)$, $(5a)^4 y \equiv (6b)^4 (\text{mod } p^m)$, $(5a)^3 z \equiv -(6b)^3 (\text{mod } p^m)$, $(5a)^2 v \equiv (6b)^2 (\text{mod } p^m)$, $5aw \equiv -6b (\text{mod } p^m)$

It can be easily shown that for any prime p, an algebraic number field K has a p-integral basis of the type $\mathcal{C} = \{1, \gamma_1, \cdots, \gamma_{n-1}\}$, where γ_i's are of the form

$$\gamma_i = \frac{c_{i0} + c_{i1}\theta + \cdots + c_{i(i-1)}\theta^{i-1} + \theta^i}{p^{k_i}}$$

with $c_{ij} \in \mathbf{Z}$ and k_i's are integers with $0 \leq k_i \leq k_{i+1}$ for $1 \leq i \leq n-1$. The p-integral basis constructed in the above theorem is of this type. We wish to point out that these type of p-integral bases of any algebraic number field K quickly lead to an integral basis of K in view of the following theorem which has been proved in [9].

Theorem 1.2. Let $L = \mathbf{Q}(\xi)$ be an algebraic number field of degree n with ξ an algebraic integer. Let $p_1, \cdots, p_s$ are the primes dividing $ind\ \xi$ and $\{\alpha_{r0}, \alpha_{r1}, \cdots, \alpha_{r(n-1)}\}$ be a p_r-integral basis of L, $1 \leq r \leq s$ with $\alpha_{r0} = 1$, $\alpha_{ri} = \frac{c_{i0}^{(r)} + c_{i1}^{(r)}\xi + \cdots + c_{i(i-1)}^{(r)}\xi^{i-1} + \xi^i}{p_r^{k_{i,r}}}$, $1 \leq i \leq n-1$, where $c_{ij}^{(r)}$ and $0 \leq k_{i,r} \leq k_{i+1,r}$ are integers. If $c_{ij} \in \mathbf{Z}$ are such that $c_{ij} \equiv c_{ij}^{(r)} (mod\ p_r^{k_{i,r}})$ for $1 \leq r \leq s$ and if t_i stands for $\prod_{r=1}^{s} p_r^{k_{i,r}}$, then $\{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$ is an integral basis of L where $\alpha_0 = 1$, $\alpha_i = \frac{c_{i0} + c_{i1}\xi + \cdots + c_{i(i-1)}\xi^{i-1} + \xi^i}{t_i}$ for $1 \leq i \leq n-1$.

Definition 1.5 An algebraic number field K of degree n is said to be monogenic if there exists an algebraic integer α generating the extension K over $\mathbf{Q}$ such that $A_K = \mathbf{Z}[\alpha]$. In this case $\{1, \alpha, \cdots, \alpha^{n-1}\}$ will be an integral basis of K; such an integral basis of K is called a power basis of K. An algebraic number field which is not monogenic is said to be non-monogenic.

The problem of existence and construction of power bases in algebraic number fields has been intensively studied. It may be pointed out that not all algebraic number fields are monogenic. In 1878, Dedekind was the first to give an example of a non-monogenic algebraic number field; in fact he showed that the cubic field $\mathbf{Q}(\theta)$ is non-monogenic when θ is a root of the irreducible polynomial $x^3 - x^2 - 2x - 8$ over $\mathbf{Q}$. Hasse raised the problem of characterizing those algebraic number fields which are monogenic. Since then this problem has been tackeled by several mathematicians (cf. [1], [2], [6], [7], [12], [13]). In [2], Ahmad, Nakahara and Husnine proved that the sextic number field generated by $m^{\frac{1}{6}}$ is monogenic if $m \equiv 2$ or $3 \mod 4$ and $m \not\equiv \mp 1 \mod 9$. In 2017, Gaál and Remete [7] studied monogenity of algebraic number fields of the type $\mathbf{Q}(a^{1/n})$ where $3 \leq n \leq 9$ and a is squarefree.

In [8], we have given some non-monogenic classes of algebraic number fields $K = \mathbf{Q}(\theta)$, where θ satisfies an irreducible trinomial $f(x) = x^6 + ax^m + b$ belonging to $\mathbf{Z}[x]$ with $1 \leq m \leq 5$. Precisely stated, we have proved:

Theorem 1.3. Let $K = \mathbf{Q}(\theta)$ be an algebraic number field with θ a root of an irreducible trinomial $f(x) = x^6 + ax^m + b$ belonging to $\mathbf{Z}[x]$ with $1 \leq m \leq 5$. Then K is non-monogenic if a and $b+1$ both are divisible by either 8 or 9.

2. Preliminary Results

The following elementary lemma, which is an immediate consequence of Dedekind's theorem on splitting of primes [11, Chapter 1, Proposition 8.1], will play a significant role in the proof of Theorems 1.3.

Lemma 2.1. Let K be an algebraic number field and p be a rational prime. For every natural number f, let P_f denote the number of distinct prime ideals of O_K lying above p having residual degree f and N_f denote the number of irreducible polynomials over $\mathbb{F}_p$ of degree f. If $P_f > N_f$ for some f, then for every algebraic integer γ generating the field extension $K/\mathbf{Q}$, the prime p divides ind γ.

To find the number of distinct prime ideals of O_K lying above a rational prime p, we will use a weaker version of the Classical Theorem of Ore. Before stating that theorem, we will first introduce the notion of Gauss valuation, ϕ-Newton polygon, ϕ-index of a polynomial where $\phi(x)$ belonging to $\mathbf{Z}_p[x]$ is a monic polynomial with $\overline{\phi}(x)$ irreducible over $\mathbb{F}_p$.
We shall denote by v_p^x the Gauss valuation of the field $\mathbf{Q}_p(x)$ of rational functions in an indeterminate x which extends the valuation v_p of $\mathbf{Q}_p$ and is defined on $\mathbf{Q}_p[x]$ by

$$v_p^x(\sum_i c_i x^i) = \min_i \{v_p(c_i)\}, c_i \in \mathbf{Q}_p.$$

Definition 2.2. Let p be a prime number. Let $\phi(x) \in \mathbf{Z}_p[x]$ be a monic polynomial which is irreducible modulo p and $f(x) \in \mathbf{Z}_p[x]$ be a monic polynomial not divisible by $\phi(x)$. Let $\sum_{i=0}^{n} a_i(x)\phi(x)^i$ with $\deg a_i(x) < \deg \phi(x)$, $a_n(x) \neq 0$ be the $\phi(x)$-expansion of $f(x)$ obtained on dividing it by successive powers of $\phi(x)$. Let P_i stand for the point in the plane having coordinates $(i, v_p^x(a_{n-i}(x)))$ when $a_{n-i}(x) \neq 0$, $0 \leq i \leq n$. Let μ_{ij} denote the slope of the line joining the point P_i with P_j if $a_{n-i}(x)a_{n-j}(x) \neq 0$. Let i_1 be the largest positive index not exceeding n such that

$$\mu_{0i_1} = \min\{\ \mu_{0j} \mid 0 < j \leq n,\ a_{n-j}(x) \neq 0\}.$$

If $i_1 < n$, let i_2 be the largest index such that $i_1 < i_2 \leq n$ with

$$\mu_{i_1 i_2} = \min\{\ \mu_{i_1 j} \mid i_1 < j \leq n,\ a_{n-j}(x) \neq 0\}$$

and so on. The ϕ-Newton polygon of $f(x)$ with respect to p is the polygonal path having segments $P_0P_{i_1}, P_{i_1}P_{i_2}, \ldots, P_{i_{k-1}}P_{i_k}$ with $i_k = n$. These segments are called the edges of the ϕ-Newton polygon and their slopes form a strictly increasing sequence; these slopes are non-negative as $f(x)$ is a monic polynomial with coefficients in $\mathbf{Z}_p$.

Example 2.3. Consider $\phi(x) = x^2 + 1$. We determine the ϕ-Newton polygon of the polynomial $f(x) = (x^2+1)^3 + (9x+27)(x^2+1)^2 + (12x+27)(x^2+1) + 27x + 81$ with respect to 3. The ϕ-Newton polygon of $f(x)$ with respect to 3 being the lower convex hull of the points $(0,0)$, $(1,2)$, $(2,1)$ and $(3,3)$ has two edges; the first edge is the line segment joining the point $(0,0)$ with $(2,1)$ and the second edge is the line segment joining $(2,1)$ with $(3,3)$.

Definition 2.4. Let $\phi(x) \in \mathbf{Z}_p[x]$ be a monic polynomial which is irreducible modulo a rational prime p having a root α in $\widetilde{\mathbf{Q}}_p$. Let $f(x) \in \mathbf{Z}_p[x]$ be a monic polynomial not divisible by $\phi(x)$ with $\phi(x)$-expansion $\phi(x)^n + a_{n-1}(x)\phi(x)^{n-1} + \cdots + a_0(x)$ such that $\overline{f}(x)$ is a power of $\overline{\phi}(x)$. Suppose that the ϕ-Newton polygon of $f(x)$ consists of a single edge, say S having positive slope denoted by $\frac{d}{e}$ with d, e coprime, i.e.,

$$\min\left\{\frac{v_p^x(a_{n-i}(x))}{i} \;\middle|\; 1 \le i \le n\right\} = \frac{v_p^x(a_0(x))}{n} = \frac{d}{e}$$

so that n is divisible by e, say $n = et$ and $v_p^x(a_{n-ej}(x)) \ge dj$ for $1 \le j \le t$. Thus the polynomial $b_j(x) := \frac{a_{n-ej}(x)}{p^{dj}}$ has coefficients in $\mathbf{Z}_p$ and hence $b_j(\alpha) \in \mathbf{Z}_p[\alpha]$ for $1 \le j \le t$. The polynomial $T(Y)$ in an indeterminate Y defined by $T(Y) = Y^t + \sum_{j=1}^{t} \overline{b_j}(\overline{\alpha})Y^{t-j}$ having coefficients in $\mathbb{F}_p[\overline{\alpha}] \cong \frac{\mathbb{F}_p[x]}{\langle \phi(x) \rangle}$ is said to be the polynomial associated to $f(x)$ with respect to (ϕ, S).

The following definition gives the notion of associated polynomial when $f(x)$ is more general.

Definition 2.5. Let $\phi(x), \alpha$ be as in Definition 2.4. Let $g(x) \in \mathbf{Z}_p[x]$ be a monic polynomial not divisible by $\phi(x)$ such that $\overline{g}(x)$ is a power of $\overline{\phi}(x)$. Let $\lambda_1 < \cdots < \lambda_k$ be the slopes of the edges of the ϕ-Newton polygon of $g(x)$ and S_i denote the edge with slope λ_i. In view of a classical result proved by Ore (cf. [4, Theorem 1.5], we can write $g(x) = g_1(x) \cdots g_k(x)$, where the ϕ-Newton polygon of $g_i(x) \in \mathbf{Z}_p[x]$ has a single edge, say S_i', which is a translate of S_i. Let $T_i(Y)$ belonging to $\mathbb{F}_p[\overline{\alpha}][Y]$ denote the polynomial associated to $g_i(x)$ with respect to $(\phi,\ S_i')$ described as in Definition 2.4. For convenience, the polynomial $T_i(Y)$ will be referred to as the polynomial associated to $g(x)$ with respect to (ϕ, S_i). The polynomial $g(x)$ is said to be p-regular with respect to ϕ if none of the polynomials $T_i(Y)$ has a repeated root in the algebraic closure of $\mathbb{F}_p$, $1 \le i \le k$. In general, if $F(x)$ belonging to $\mathbf{Z}_p[x]$ is a monic polynomial and $\overline{f}(x) = \overline{\phi}_1(x)^{e_1} \cdots \overline{\phi}_r(x)^{e_r}$ is its factorization modulo p into irreducible polynomials with each $\phi_i(x)$ belonging to $\mathbf{Z}_p[x]$ monic and $e_i > 0$, then by Hensel's Lemma there exist monic polynomials $f_1(x), \cdots, f_r(x)$ belonging to $\mathbf{Z}_p[x]$ such that $f(x) = f_1(x) \cdots f_r(x)$ and $\overline{f}_i(x) = \overline{\phi}_i(x)^{e_i}$ for each i. The polynomial $f(x)$ is said to be p-regular (with respect to $\phi_1, \cdots, \phi_r$) if each $f_i(x)$ is p-regular with respect to ϕ_i.

The following theorem is a weaker version of Theorem 1.2 of [10].

Theorem 2.6. Let $L = \mathbf{Q}(\xi)$ be an algebraic number field with ξ satisfying an irreducible polynomial $g(x) \in \mathbf{Z}[x]$ and p be a rational prime. Let $\overline{\phi}_1(x)^{e_1} \cdots \overline{\phi}_r(x)^{e_r}$ be the factorization of $g(x)$ modulo p into powers of distinct irreducible polynomials over $\mathbb{F}_p$ with each $\phi_i(x) \neq g(x)$ belonging to $\mathbf{Z}[x]$ monic. Suppose that the ϕ_i-Newton polygon of $g(x)$ has k_i edges, say S_{ij} having slopes $\lambda_{ij} = \frac{d_{ij}}{e_{ij}}$ with gcd $(d_{ij},\ e_{ij}) = 1$ for $1 \leq j \leq k_i$. If $g(x)$ is p-regular with respect to $\phi_1, \cdots, \phi_r$ and the associated polynomials $T_{ij}[Y]$ with respect to $(\phi_i,\ S_{ij})$ are irreducible for $1 \leq j \leq k_i$, then

$$pO_L = \prod_{i=1}^{r}\prod_{j=1}^{k_i} \mathfrak{p}_{ij}^{e_{ij}},$$

where $\mathfrak{p}_{ij}$ are distinct prime ideals of O_L having residual degree $\deg \phi_i(x) \times \deg T_{ij}[Y]$.

3. Proof of Theorem 1.2

Recall that $\mathbf{Z}_{(p)}$ stands for the localisation of $\mathbf{Z}$ at $p\mathbf{Z}$. Clearly α_i is integral over $\mathbf{Z}_{(p)}$ for all primes p not belonging to the set $\{p_1, \cdots, p_s\}$ for each i. Also in view of the choice of c_{ij} we see that each α_i is integral over $\mathbf{Z}_{(p)}$ for p belonging to the set $\{p_1, p_2, \cdots, p_s\}$. So each α_i is integral over $\mathbf{Z}$. Therefore if Γ denotes the subgroup of $\mathbb{C}$ defined by $\Gamma = \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1 + \cdots + \mathbf{Z}\alpha_{n-1}$, then $\mathbf{Z}[\xi] \subseteq \Gamma \subseteq A_L$. Further by virtue of a basic result (cf. [3, Chapter 2, Section 2, Theorem 2]) the index of the subgroup $\mathbf{Z}[\xi]$ in Γ being the absolute value of the determinant of the transition matrix from $\{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$ to $\{1, \xi, \cdots, \xi^{n-1}\}$ equals $\prod_{i=1}^{n-1} t_i$. Since $v_{p_r}(ind\ \xi) = k_{1,r} + \cdots + k_{n-1,r}$ for $1 \leq r \leq s$. By hypothesis $p_1, \cdots, p_s$ are the only primes dividing $ind\ \xi$. Therefore

$$ind\ \xi = \prod_{r=1}^{s} p_r^{k_{1,r}+\cdots+k_{n-1,r}} = \prod_{i=1}^{n-1}(\prod_{r=1}^{s} p_r^{k_{i,r}}) = \prod_{i=1}^{n-1} t_i.$$

Since $\Gamma \subseteq A_L$ and $[\Gamma : \mathbf{Z}[\xi]] = [A_L : \mathbf{Z}[\xi]]$, it follows that $A_L = \Gamma$ and hence $\{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}$ is an integral basis of L.

4. Proof of Theorem 1.3

Firstly we will consider the case when a and $b+1$ both are divisible by 8. In this case $f(x) \equiv (x^2+x+1)^2(x+1)^2 \mod 2$. Set $\phi_1(x) = x^2+x+1$ and $\phi_2(x) = x+1$. One can easily see that the ϕ_1-exapansion of $f(x)$ is given by the following equations corresponding to each m.

$$m = 1 : \phi_1^3 - 3x\phi_1^2 + (2x-2)\phi_1 + ax + b + 1. \tag{4.1}$$

$$m = 2 : \phi_1^3 - 3x\phi_1^2 + (2x+a-2)\phi_1 + (-ax + b + 1 - a). \tag{4.2}$$

$$m=3:\phi_1^3-3x\phi_1^2+((a+2)x-(a+2))\phi_1+a+b+1. \tag{4.3}$$

$$m=4:\phi_1^3-(3x-a)\phi_1^2+(2(1-a)x-(a+2))\phi_1+(ax+b+1). \tag{4.4}$$

$$m=5:\phi_1^3-((3-a)x+2a)\phi_1^2+((a+2)x+3a-2)\phi_1+(-ax+b-a+1). \tag{4.5}$$

Keeping in mind the definition of Gauss valuation and the fact that $v_2(a)\geq 3,\ v_2(b+1)\geq 3$, it can be easily seen that the ϕ_1-Newton polygon of $f(x)$ for $1\leq m\leq 5$ has two edges, say S_1, and S_2 of positive slopes. The polynomials attached to $f(x)$ with respect to $(\phi_1,\ S_1)$ and $(\phi_1,\ S_2)$ are linear. Now we will write the ϕ_2-expansion of $f(x)$. For each m, $1\leq m\leq 5$, let c_m denote the integer $(-1)^m a+b+1$. It can be easily verified that the ϕ_2-exapansion of $f(x)$ is given by the following equations corresponding to each m.

$$m=1:\phi_2^6-6\phi_2^5+15\phi_2^4-20\phi_2^3+15\phi_2^2+(a-6)\phi_2+c_1. \tag{4.6}$$

$$m=2:\phi_2^6-6\phi_2^5+15\phi_2^4-20\phi_2^3+(a+15)\phi_2^2-2(a+3)\phi_2+c_2. \tag{4.7}$$

$$m=3:\phi_2^6-6\phi_2^5+15\phi_2^4+(a-20)\phi_2^3+3(5-a)\phi_2^2+3(a-2)\phi_2+c_3. \tag{4.8}$$

$$m=4:\phi_2^6-6\phi_2^5+(a+15)\phi_2^4-4(a+5)\phi_2^3+3(2a+5)\phi_2^2-2(2a+3)\phi_2+c_4. \tag{4.9}$$

$$m=5:\phi_2^6+(a-6)\phi_2^5-5(a-3)\phi_2^4+10(a-2)\phi_2^3+5(3-2a)\phi_2^2+(5a-6)\phi_2+c_5. \tag{4.10}$$

It is easy to check that the ϕ_2-Newton polygon of $f(x)$ for $1\leq m\leq 5$ has two edges, say S_1', and S_2' of positive slope. The polynomial associated to $f(x)$ with respect to $(\phi_2,\ S_i')$ is linear for $i=1,\ 2$. Thus $f(x)$ is 2-regular with respect to $\phi_1,\ \phi_2$. Hence, Theorem 2.6 is applicable. Using this theorem, we see that there are two prime ideals of O_K with residual degree two that lies above 2. But there is only one irreducible polynomial over $\mathbb{F}_2$ of degree 2. So by Lemma 2.1, 2 will divide ind γ for every generator $\gamma\in O_K$. Thus K is non-monogenic.

Now we will consider the case when both a and $b+1$ are divisible by 9. In this situation, $f(x)\equiv(x-1)^3(x+1)^3 \mod 3$. Set $\phi_1(x)=x-1$ and $\phi_2(x)=x+1$. We will write the ϕ_1-expansion of the polynomial $f(x)$. Let c denote the integer $a+b+1$. It is easy to check that the ϕ_1-expansion of $f(x)$ is given by the following equations corresponding to each m.

$$m=1:\phi_1^6+6\phi_1^5+15\phi_1^4+20\phi_1^3+15\phi_1^2+(a+6)\phi_1+c. \tag{4.11}$$

$$m=2:\phi_1^6+6\phi_1^5+15\phi_1^4+20\phi_1^3+(a+15)\phi_1^2+2(a+3)\phi_1+c. \tag{4.12}$$

$$m=3:\phi_1^6+6\phi_1^5+15\phi_1^4+(a+20)\phi_1^3+3(5+a)\phi_1^2+3(a+2)\phi_1+c. \tag{4.13}$$

$$m=4:\phi_1^6+6\phi_1^5+(a+15)\phi_1^4+4(a+5)\phi_1^3+3(2a+5)\phi_1^2+2(2a+3)\phi_1+c. \tag{4.14}$$

$$m=5:\phi_1^6+(a+6)\phi_1^5+5(a+3)\phi_1^4+10(a+2)\phi_1^3+5(3+2a)\phi_1^2+(5a+6)\phi_1+c. \tag{4.15}$$

By virtue of the fact that $v_3(a)\geq 2,\ v_3(b+1)\geq 2$, it can be easily seen that the ϕ_1-Newton polygon of $f(x)$ for $1\leq m\leq 5$ has two edges, say S_1, and S_2 of positive slope. The associated polynomials to $f(x)$ with respect to both $(\phi_1,\ S_1)$ and $(\phi_1,\ S_2)$ are linear. Recall that for each m, $1\leq m\leq 5$, if c_m denote the integer $(-1)^m a+b+1$, then the ϕ_2-exapansion of $f(x)$ is given by the Equations (4.6)–(4.10) corresponding to each m. One can check that ϕ_2-Newton polygon of $f(x)$ for $1\leq m\leq 5$ has two edges of positive slopes. The polynomials associated to $f(x)$ with respect to ϕ_2 corresponding to these two edges are linear. Hence $f(x)$ is 3-regular with respect to $\phi_1,\ \phi_2$. So by Theorem 2.6, we see that there exist four distinct prime ideals of O_K lying above 3 with residual degree one each. Since there are only three irreducible polynomials of degree 1 over $\mathbb{F}_3$, by Lemma 2.1, K is non-monogenic.

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