ON THE $\mathfrak{sl}_2$-ALGEBRA OF FORMAL MULTIPLE EISENSTEIN SERIES

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Abstract. In this survey article we summarize the results of [BIM] in which the authors introduced the algebra of formal multiple Eisenstein. This algebra is motivated by the classical multiple Eisenstein series, introduced by Gangl–Kaneko–Zagier as a hybrid of classical Eisenstein series and multiple zeta values. This algebra is an $\mathfrak{sl}_2$-algebra by formalizing the usual derivations for quasimodular forms and extending them naturally to the whole algebra. A quotient of this algebra is isomorphic to the algebra of formal multiple zeta values. This gives a novel and purely formal approach to classical (quasi)modular forms and builds a new link between (formal) multiple zeta values and modular forms.

1. Introduction

The purpose of this note is to provide a summary of the work [BIM], where the authors introduced formal multiple Eisenstein and studied their derivations. These objects are formalization of multiple Eisenstein series, which are a hybrid of classical Eisenstein series and multiple zeta values. Multiple zeta values, which are defined for integers $r \geq 1$ and $k_1 \geq 2, k_2, \dots, k_r \geq 1$ by

$$\zeta(k_1, \dots, k_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \tag{1.1}$$

are subject to many relations. Denote the $\mathbb{Q}$-algebra of all multiple zeta values by $\mathcal{Z}$. Conjecturally, the *extended double shuffle relations* of multiple zeta values provide all algebraic relations among multiple zeta values [IKZ]. These relations are obtained (after possible regularization) from the two ways of expressing the product of multiple zeta values—the 'usual' (stuffle) product of real numbers, and a (shuffle) product from the iterated integral representation of multiple zeta values—which both can be interpreted as quasi-shuffle products [H]. In this note we use the standard algebraic setup as in [IKZ] (see the introduction of [BIM] for details) to describe these relations.

Multiple zeta values and (quasi)modular forms are connected in various ways. For example, in the case $r = 1$, they appear as the constant term of the Eisenstein series. The Eisenstein series of weight $k \geq 2$ is given for $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ by

$$\mathbb{G}_k(\tau) := \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{m,n \geq 1} n^{k-1} q^{mn} \qquad (q = e^{2\pi i \tau}).$$

For even $k \geq 2$ these series are (quasi)modular forms for the full modular group. In [GKZ] the authors defined double Eisenstein series, which have double zeta values ((1.1) in the case $r = 2$) as their constant terms, and which can be seen as a natural depth two version of Eisenstein

Date: April 8, 2024.

2020 *Mathematics Subject Classification.* Primary 11F11,11M32. Secondary 13N15,16T30 .

series. This construction was generalized by the first author in [Ba1]. Given a *depth* $r \geq 1$ and integers $k_1, \dots, k_r \geq 2$ the *multiple Eisenstein series* are defined[1] for $\tau \in \mathbb{H}$ by

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) := \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}},$$

where the order $\succ$ on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ is defined by $m_1\tau + n_1 \succ m_2\tau + n_2$ iff $m_1 > m_2$ or $m_1 = m_2 \wedge n_1 > n_2$. The multiple Eisenstein series are holomorphic functions, i.e. elements in $\mathcal{O}(\mathbb{H})$. Since $\mathbb{G}_{k_1,\dots,k_r}(\tau+1) = \mathbb{G}_{k_1,\dots,k_r}(\tau)$, which can be obtained directly by the above definition, the multiple Eisenstein series possess a Fourier expansion

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \zeta(k_1,\dots,k_r) + \sum_{n\geq 1} a_n\, q^n \qquad (a_n \in \mathcal{Z}[2\pi i]), \tag{1.2}$$

which was calculated in depth $r = 2$ in [GKZ] and for arbitrary depth by the first author in [Ba1].

Based on various conjectures[2] regarding the relations of the q-series appearing in this Fourier expansions, the authors of [BIM] came up with the following definition: the algebra of formal multiple Eisenstein series $\mathcal{G}^f$ is defined (Definition 2.4) to be the $\mathbb{Q}$-vector space spanned by symbols G^f, which are swap invariant and whose product is given by (a double indexed version of) the stuffle product. More precisely, for $r \geq 1, k_1, \dots, k_r \geq 1, d_1, \dots, d_r \geq 0$ we introduce formal variables $G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$ and impose the relations (swap invariance)

$$\mathfrak{G}_r\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \mathfrak{G}_r\binom{Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2} \qquad \text{(for all } r \geq 1), \tag{1.3}$$

where $\mathfrak{G}_r$ denotes the generating series

$$\mathfrak{G}_r\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} := \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 0}} G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r} X_1^{k_1-1}\cdots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!}\cdots\frac{Y_r^{d_r}}{d_r!}.$$

We then define the *algebra of formal multiple Eisenstein series* $\mathcal{G}^f$ as the $\mathbb{Q}$-vector space spanned by these formal symbols G^f equipped with the *stuffle product* (Definition 2.1), e.g.,

$$G^f\binom{k_1}{d_1}G^f\binom{k_2}{d_2} = G^f\binom{k_1,k_2}{d_1,d_2} + G^f\binom{k_2,k_1}{d_2,d_1} + G^f\binom{k_1+k_2}{d_1+d_2},$$

$$G^f\binom{k_1}{d_1}G^f\binom{k_2,k_3}{d_2,d_3} = G^f\binom{k_1,k_2,k_3}{d_1,d_2,d_3} + G^f\binom{k_2,k_1,k_3}{d_2,d_1,d_3} + G^f\binom{k_2,k_3,k_1}{d_2,d_3,d_1} + G^f\binom{k_1+k_2,k_3}{d_1+d_2,d_3} + G^f\binom{k_1,k_2+k_3}{d_1,d_2+d_3}.$$

For example, in the case $r = 1$ they satisfy the relation $G^f\binom{k}{d} = \frac{d!}{(k-1)!}G^f\binom{d+1}{k-1}$. The use of double indices is necessary to make sense of the swap invariance (1.3). But often we will be interested in the case when $d_1 = \dots = d_r = 0$ and set

$$G^f(k_1,\dots,k_r) := G^f\binom{k_1,\dots,k_r}{0,\dots,0}. \tag{1.4}$$

Then we expect that the symbols $G^f(k_1,\dots,k_r)$ should satisfy the same relations as the (stuffle regularized) multiple Eisenstein series. In fact, we will see that the $G^f(k)$ satisfy exactly the

[1]In the case $k_1 = 2$ one needs to use Eisenstein summation. See [Ba5] for details.

[2]See the introduction of [BIM] for a detailed explanation and motivation for these objects.

algebraic relations as the classical Eisenstein series (Theorem 2.7), which is quite surprising considering the simple family of relations that we impose on G^f. Even though it seems like the single indexed $G^f(k_1,\dots,k_r)$ in (1.4) just give a small portion of the whole space $\mathcal{G}^f$, we conjecture (Conjecture 2.5) it is already spanned by them. This justifies calling $\mathcal{G}^f$ the algebra of formal multiple Eisenstein series.

By the Fourier expansion (1.2) of multiple Eisenstein series we see that projecting onto the constant term gives a surjective algebra homomorphism from the space of multiple Eisenstein series to multiple zeta values. This raises the natural question of whether there is a formal analogue of this projection onto the algebra of formal multiple zeta values $\mathcal{Z}^f$. We give a positive answer to this question together with an explicit description of the kernel of this 'formal projection onto the constant term' as follows.

Theorem 1.1 ([BIM, Theorem 1.1])**.** *There exists a surjective algebra homomorphism*

$$\pi : \mathcal{G}^f \to \mathcal{Z}^f,$$

with $\pi(G^f(k_1,\dots,k_r)) = \zeta^f(k_1,\dots,k_r)$*. The kernel of* π *is the ideal generated by all formal multiple Eisenstein series which are* not *of the form*

$$G^f\binom{1,\dots,1,k_1,\dots,k_r}{d_1,\dots,d_s,0,\dots,0},$$

for some $r,s\geq 0$ *and* $k_1,\dots,k_r\geq 1, d_1,\dots,d_s\geq 0$.

In fact, we will define the algebra of formal multiple zeta values as the quotient by the above ideal (Definition 2.10) and then show that this algebra is isomorphic to the classical definition of formal multiple zeta values. Theorem 1.1 offers a new perspective on the extended double shuffle relations. Namely, it states that these relations are equivalent to the combination of swap invariance, the stuffle product, and the relations derived from dividing out the mentioned ideal.

The main result of [BIM] focuses on the derivations of the algebra $\mathcal{G}^f$. This is motivated by the derivations for classical (quasi)modular forms, which we recall now. For this, we first introduce another normalization of the Eisenstein series and define

$$G_k := (-2\pi i)^{-k}\mathbb{G}_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{m,n\geq 1} m^{k-1}\,q^{mn}, \qquad (B_k = k\text{th Bernoulli number}).$$

As is well-known, for even k the relations

$$\begin{aligned}\frac{k+1}{2}G_k &= \frac{1}{k-2}\,q\frac{d}{dq}G_{k-2} + \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 2 \text{ even}}} G_{k_1}G_{k_2}\,, \qquad (k\geq 4)\\ \frac{(k+1)(k-1)(k-6)}{12}\,G_k &= \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 4 \text{ even}}} (k_1-1)(k_2-1)\,G_{k_1}G_{k_2}\,, \qquad (k\geq 6)\end{aligned} \tag{1.5}$$

suffice to write every Eisenstein series G_k with $k\geq 4$ as a polynomial in G_4 and G_6. Moreover, the relations (1.5) imply that the space $\widetilde{\mathcal{M}} = \mathbb{Q}[G_2,G_4,G_6]$ of quasimodular forms (with rational coefficients) is closed under the operator $q\frac{d}{dq}$. Even more, the algebra $\widetilde{\mathcal{M}}$ is an $\mathfrak{sl}_2$-algebra (see [Z]). In general, an $\mathfrak{sl}_2$-*algebra* is an algebra together with three derivations W, D, δ acting on this algebra and satisfying the following commutator relations

$$[W, D] = 2D, \quad [W, \delta] = -2\delta, \quad [\delta, D] = W.$$

In the case of quasimodular forms, these derivations are given by $D = q\frac{d}{dq}$ and the other two derivations are determined by $\delta G_2 = -\frac{1}{2}$, $\delta G_4 = \delta G_6 = 0$ and $WG_k = kG_k$. As one of the main results of [BIM] (Theorem 3.1), the algebra $\mathcal{G}^f$ is an $\mathfrak{sl}_2$-algebra which can be seen as a natural generalization of the $\mathfrak{sl}_2$-algebra of quasimodular forms:

Theorem 1.2 ([BIM, Theorem 1.2])**.** *There exist derivations W, D, δ on $\mathcal{G}^f$ such that*
(i) *$\mathcal{G}^f$ is an $\mathfrak{sl}_2$-algebra;*
(ii) *the subalgebra $\widetilde{\mathcal{M}}^f = \mathbb{Q}[G^f(2), G^f(4), G^f(6)] \subset \mathcal{G}^f$ is isomorphic to $\widetilde{\mathcal{M}}$ as an $\mathfrak{sl}_2$-algebra.*

We refer to $\widetilde{\mathcal{M}}^f$ as the algebra of *formal quasimodular forms*. The derivations W, D, δ are given explicitly. The $\mathfrak{sl}_2$-algebra structure of $\mathcal{G}^f$ gives a natural definition of the space of *formal modular forms* $\mathcal{M}^f := \ker \delta_{|\widetilde{\mathcal{M}}^f}$ and a notion of Rankin–Cohen brackets. The projection π in Theorem 1.1 then also naturally leads to the space of *formal cusp forms* $\mathcal{S}^f := \ker \pi_{|\mathcal{M}^f}$. These spaces are isomorphic to their classical counterparts $\mathcal{M}$ and $\mathcal{S}$.

Acknowledgements. The author thanks the organizers of the conference "Algebraic Number Theory and Related Topics 2023" for giving him the opportunity to present the result of [BIM]. This project was partially supported by JSPS KAKENHI Grant 23K03030.

2. Formal multiple Eisenstein series

In this Section, we recall the basic algebraic setup of [BIM]. Define the set $\mathcal{A}$, whose elements we call *letters*, by

$$\mathcal{A} = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \mid k \geq 1, \, d \geq 0 \right\}.$$

We are interested in $\mathbb{Q}$-linear combinations of words in the letters of $\mathcal{A}$, i.e., in elements of $\mathbb{Q}\langle \mathcal{A} \rangle$. Here and in the following, we call the monic monomials in $\mathbb{Q}\langle \mathcal{A} \rangle$ *words*. For $k_1, \dots, k_r \geq 1$ and $d_1, \dots, d_r \geq 0$, we will use the following notation to write words in $\mathbb{Q}\langle \mathcal{A} \rangle$:

$$\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} := \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \cdots \begin{bmatrix} k_r \\ d_r \end{bmatrix},$$

where the product on the right is the usual non-commutative product in $\mathbb{Q}\langle \mathcal{A} \rangle$. The space $\mathfrak{H}^1$ (defined in [IKZ] or [BIM, Introduction]) can be viewed naturally as a subspace of $\mathbb{Q}\langle \mathcal{A} \rangle$ via the inclusion

$$\begin{aligned} \mathfrak{H}^1 &\longrightarrow \mathbb{Q}\langle \mathcal{A} \rangle \\ z_{k_1} \cdots z_{k_r} &\longmapsto \begin{bmatrix} k_1, \dots, k_r \\ 0, \dots, 0 \end{bmatrix}. \end{aligned} \tag{2.1}$$

We will extend the stuffle product defined on $\mathfrak{H}^1$ to $\mathbb{Q}\langle \mathcal{A} \rangle$ in the following way.

Definition 2.1. Define the *stuffle product* $*$ on $\mathbb{Q}\langle\mathcal{A}\rangle$ as the $\mathbb{Q}$-bilinear product, which satisfies $1 * w = w * 1 = w$ for any word $w \in \mathbb{Q}\langle\mathcal{A}\rangle$ and

$$\begin{bmatrix}k_1\\d_1\end{bmatrix} w * \begin{bmatrix}k_2\\d_2\end{bmatrix} v = \begin{bmatrix}k_1\\d_1\end{bmatrix}\left(w * \begin{bmatrix}k_2\\d_2\end{bmatrix} v\right) + \begin{bmatrix}k_2\\d_2\end{bmatrix}\left(\begin{bmatrix}k_1\\d_1\end{bmatrix} w * v\right) + \begin{bmatrix}k_1+k_2\\d_1+d_2\end{bmatrix}(w * v)$$

for any letters $\left[\begin{smallmatrix}k_1\\d_1\end{smallmatrix}\right], \left[\begin{smallmatrix}k_2\\d_2\end{smallmatrix}\right] \in \mathcal{A}$ and words $w, v \in \mathbb{Q}\langle\mathcal{A}\rangle$.

This turns $\mathbb{Q}\langle\mathcal{A}\rangle$ into a commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle\mathcal{A}\rangle, *)$, as shown in [H]. The algebra $(\mathfrak{H}^1, *)$ can be viewed as a subalgebra of $(\mathbb{Q}\langle\mathcal{A}\rangle, *)$ via (2.1). Most of the time, we will work with the generating series of our objects. Let $\mathcal{B}_0(\mathbb{Q}\langle\mathcal{A}\rangle) = \mathbb{Q}\langle\mathcal{A}\rangle$ and for $r \geq 1$ set

$$\mathcal{B}_r = \mathcal{B}_r(\mathbb{Q}\langle\mathcal{A}\rangle) := \mathbb{Q}\langle\mathcal{A}\rangle[\![X_1, Y_1, \ldots, X_r, Y_r]\!] \quad \text{and} \quad \mathcal{B} = \mathcal{B}(\mathbb{Q}\langle\mathcal{A}\rangle) := \bigoplus_{r \geq 0} \mathcal{B}_r(\mathbb{Q}\langle\mathcal{A}\rangle).$$

For $r \geq 0$ consider the family of formal power series $\mathfrak{A} = (\mathfrak{A}_0, \mathfrak{A}_1, \ldots) \in \mathcal{B}$, where $\mathfrak{A}_0 = 1$ and for $r \geq 1$ one has

$$\mathfrak{A}_r\begin{pmatrix}X_1, \ldots, X_r\\Y_1, \ldots, Y_r\end{pmatrix} := \sum_{\substack{k_1,\ldots,k_r \geq 1\\d_1,\ldots,d_r \geq 0}} \begin{bmatrix}k_1, \ldots, k_r\\d_1, \ldots, d_r\end{bmatrix} X_1^{k_1-1} \cdots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \cdots \frac{Y_r^{d_r}}{d_r!}.$$

Definition 2.2. We define the *swap* as the linear map $\sigma : \mathcal{B} \to \mathcal{B}$ given by $\sigma(f_r) = (\sigma f_r)$ with

$$\sigma\left(f_r\begin{pmatrix}X_1, \ldots, X_r\\Y_1, \ldots, Y_r\end{pmatrix}\right) := f_r\begin{pmatrix}Y_1 + \cdots + Y_r, \ldots, Y_1 + Y_2, Y_1\\X_r, X_{r-1} - X_r, \ldots, X_1 - X_2\end{pmatrix}. \tag{2.2}$$

Note that the swap σ is an involution.

Definition 2.3. By applying a map $\rho : \mathcal{B} \to \mathcal{B}$ to the power series $\mathfrak{A}$ one obtains a map $f : \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}\langle\mathcal{A}\rangle$ by comparing coefficients, i.e., $\rho\left[\begin{smallmatrix}k_1,\ldots,k_r\\d_1,\ldots,d_r\end{smallmatrix}\right]$ is defined as the coefficient of $X_1^{k_1-1} \cdots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \cdots \frac{Y_r^{d_r}}{d_r!}$ of $\rho\mathfrak{A}_r$.

In particular, we can think of the swap $\sigma : \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}\langle\mathcal{A}\rangle$ as the linear map defined in the above sense. Formal multiple Eisenstein series are objects whose generating series are invariant under the change of variables in (2.2).

Definition 2.4. We define the space of *formal multiple Eisenstein series* as

$$\mathcal{G}^f := {}^{(\mathbb{Q}\langle\mathcal{A}\rangle, *)}\big/_{\mathcal{I}},$$

where $\mathcal{I}$ is the ideal in $(\mathbb{Q}\langle\mathcal{A}\rangle, *)$ generated by $\sigma(w) - w$ for all $w \in \mathbb{Q}\langle\mathcal{A}\rangle$. For $k_1, \ldots, k_r \geq 1$ and $d_1, \ldots, d_r \geq 0$ we denote the class of the word $w = \left[\begin{smallmatrix}k_1,\ldots,k_r\\d_1,\ldots,d_r\end{smallmatrix}\right]$ in $\mathcal{G}^f$ by $G^f(w) = G^f\left(\begin{smallmatrix}k_1,\ldots,k_r\\d_1,\ldots,d_r\end{smallmatrix}\right)$. We define a weight grading and two filtrations on $\mathcal{G}^f$, as follows:

(i) Write

$$\begin{aligned}\mathrm{wt}(w) &:= k_1 + \cdots + k_r + d_1 + \cdots + d_r,\\ \mathrm{lwt}(w) &:= d_1 + \cdots + d_r,\\ \mathrm{dep}(w) &:= r\end{aligned}$$

for the *weight*, *lower weight* and *depth* of the word w respectively.

(ii) Write $\mathcal{G}^f_k$ for the subspace of $\mathcal{G}^f$ generated by $G^f(w)$ with $\mathrm{wt}(w) = k$.

(iii) Write $\mathrm{Fil}_d^{\mathrm{lwt}}\,\mathcal{G}^f$, $\mathrm{Fil}_r^{\mathrm{dep}}\,\mathcal{G}^f$ for the lower weight and depth filtration on $\mathcal{G}^f$ respectively. We shorten the notation when considering two filtrations at the same time, that is

$$\begin{aligned}\mathrm{Fil}_d^{\mathrm{lwt}}\,\mathcal{G}^f &= \langle\, G^f(w) \mid w \in \mathcal{A}^*, \mathrm{lwt}(w) \leq d \,\rangle_{\mathbb{Q}}\,,\\ \mathrm{Fil}_{d,r}^{\mathrm{lwt,dep}}\,\mathcal{G}^f &= \langle\, G^f(w) \mid w \in \mathcal{A}^*, \mathrm{lwt}(w) \leq d \text{ and } \mathrm{dep}(w) \leq r \,\rangle_{\mathbb{Q}}\,, \text{ etc.}\end{aligned}$$

In the special case $\mathrm{lwt}(w) = 0$, we write

$$G^f(k_1, \dots, k_r) \coloneqq G^f\binom{k_1, \dots, k_r}{0, \dots, 0}.$$

The space of formal multiple Eisenstein series is a commutative $\mathbb{Q}$-algebra $(\mathcal{G}^f, *)$ where each element is swap invariant. Notice that the $\mathbb{Q}$-linear map

$$\begin{aligned} G^f : (\mathfrak{H}^1, *) &\longrightarrow \mathcal{G}^f \\ z_{k_1} \cdots z_{k_r} &\longmapsto G^f(k_1, \dots, k_r) \end{aligned} \tag{2.3}$$

is an algebra homomorphism. Even though the double indices are crucial for the definition, a non-trivial conjecture is that the space $\mathcal{G}^f$ is already spanned by the singles indexed $G^f(k_1, \dots, k_r)$, i.e., that the mapping (2.3) is surjective. This conjecture is the formal version of the conjectures in [Ba2, Conjecture 4.3], [BK, Conjecture 5 (B2)] and [BI, Conjecture 3.15], of which only special cases are known (cf. [Ba2, Bu1, V]).

Conjecture 2.5. *The map* (2.3) *is surjective, i.e.*

$$\mathcal{G}^f \simeq \mathrm{Fil}_0^{\mathrm{lwt}}\,\mathcal{G}^f.$$

A more refined version of the conjecture is that

$$\mathrm{Fil}_{d,r}^{\mathrm{lwt,dep}}\,\mathcal{G}^f \subset \mathrm{Fil}_{0,d+r}^{\mathrm{lwt,dep}}\,\mathcal{G}^f \qquad \text{for all } d, r \geq 0.$$

Denote the generating series of the formal multiple Eisenstein series is denoted by

$$\mathfrak{G}_r\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{\substack{k_1,\dots,k_r \geq 1\\ d_1,\dots,d_r \geq 0}} G^f\binom{k_1, \dots, k_r}{d_1, \dots, d_r} X_1^{k_1-1} \cdots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \cdots \frac{Y_r^{d_r}}{d_r!}.$$

Since the formal multiple Eisenstein series are swap invariant we have

$$\mathfrak{G}_r\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \mathfrak{G}_r\binom{Y_1 + \dots + Y_r, \dots, Y_1 + Y_2, Y_1}{X_r, X_{r-1} - X_r, \dots, X_1 - X_2}.$$

On $\mathbb{Q}\langle\mathcal{A}\rangle$ we can define another product $\circledast$ by $w \circledast v = \sigma(\sigma(w) * \sigma(v))$ for $w, v \in \mathbb{Q}\langle\mathcal{A}\rangle$. One can easily check that since σ is an involution, this product is commutative and associative. Due to the swap invariance of $\mathcal{G}^f$, this product is the same as the product $*$. This implies a large family of relations among elements in $\mathcal{G}^f$, i.e., $f * g - f \circledast g = 0$ for all $f, g \in \mathcal{G}^f$. These relations can be seen as an analogue of the double shuffle relations for multiple zeta values. If f and g are of depth 1, these relations are given as follows.

Proposition 2.6 ([BIM, Proposition 2.9]). *For $k_1, k_2 \geq 1, d_1, d_2 \geq 0$ we have*

$$G^f\binom{k_1}{d_1} G^f\binom{k_2}{d_2} = G^f\binom{k_1, k_2}{d_1, d_2} + G^f\binom{k_2, k_1}{d_2, d_1} + G^f\binom{k_1 + k_2}{d_1 + d_2}$$

$$= \sum_{\substack{l_1+l_2=k_1+k_2 \\ e_1+e_2=d_1+d_2}} \left(\binom{l_1-1}{k_1-1}\binom{d_1}{e_1}(-1)^{d_1-e_1} + \binom{l_1-1}{k_2-1}\binom{d_2}{e_1}(-1)^{d_2-e_1}\right) G^f\binom{l_1, l_2}{e_1, e_2}$$

$$+ \frac{d_1! d_2!}{(d_1+d_2+1)!}\binom{k_1+k_2-2}{k_1-1} G^f\binom{k_1+k_2-1}{d_1+d_2+1},$$

where we sum over all $l_1, l_2 \geq 1$ and $e_1, e_2 \geq 0$, subject to $l_1 + l_2 = k_1 + k_2$ and $e_1 + e_2 = d_1 + d_2$, in the second expression.

Proposition 2.6 shows that the formal multiple Eisenstein series in depth two give a realization of the formal double Eisenstein space introduced in [BKM, Definition 2.1], since the latter are formal symbols satisfying the above relations. It was then shown in [BKM, Theorem 4.4], that these relations can be used to obtain the following relations.

Theorem 2.7. *For all $k_1, k_2 \geq 1$ with $k = k_1 + k_2 \geq 4$ even we have*

$$\frac{1}{2}\left(\binom{k_1+k_2}{k_2} - (-1)^{k_1}\right) G^f(k) = \sum_{\substack{j=2 \\ j\,even}}^{k-2} \left(\binom{k-j-1}{k_1-1} + \binom{k-j-1}{k_2-1} - \mathbf{1}_{j,k_1}\right) G^f(j)\, G^f(k-j)$$

$$+ \frac{1}{2}\left(\binom{k-3}{k_1-1} + \binom{k-3}{k_2-1} + \mathbf{1}_{k_1,1} + \mathbf{1}_{k_2,1}\right) G^f\binom{k-1}{1}.$$

The following relations are special cases of Theorem 2.7, which will be used later when dealing with formal (quasi)modular forms. They can be seen as the formal version of the classical recursive formulas for Eisenstein series given in (1.5).

Corollary 2.8. (i) *For even $k \geq 4$ we have*

$$\frac{k+1}{2} G^f(k) = G^f\binom{k-1}{1} + \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 2\ even}} G^f(k_1)\, G^f(k_2).$$

(ii) *For all even $k \geq 6$ we have*

$$\frac{(k+1)(k-1)(k-6)}{12} G^f(k) = \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 4\ even}} (k_1-1)(k_2-1)\, G^f(k_1)\, G^f(k_2).$$

Due to Euler we know that for $m \geq 1$ we have

$$\zeta(2m) = -\frac{B_{2m}}{2(2m)!}(-2\pi i)^{2m} = -\frac{B_{2m}}{2(2m)!}(-24\zeta(2))^m.$$

As an analogue, in our formal setup we can show the following.

Corollary 2.9 ([BIM, Corollary 2.12]). *For $m \geq 1$ we have*

$$G^f(2m) = -\frac{B_{2m}}{2(2m)!}(-24 G^f(2))^m + Q_{2m},$$

for $Q_{2m} \in D\widetilde{\mathcal{M}}^f$*, where* $\widetilde{\mathcal{M}}^f = \mathbb{Q}[G^f(2), G^f(4), G^f(6)]$ *is the space of formal quasimodular forms.*

We now mention the connection of formal multiple Eisenstein series to of formal multiple zeta values. Conjecturally, these satisfy exactly the same relations as multiple zeta values. The definition in [BIM] is equivalent to the usual definition of formal multiple zeta values as formal symbols modulo the extended double shuffle relations. The difference is that in [BIM] the authors define the space of formal multiple zeta values as a quotient of $\mathcal{G}^f$. This approach has the benefit of allowing connections to q-analogues of multiple zeta values and modular form on a formal level, which is not directly possible with the usual approach. Define the following two subsets of the alphabet $\mathcal{A}$.

$$\mathcal{A}_0 := \left\{ \begin{bmatrix} k \\ 0 \end{bmatrix} \mid k \geq 1 \right\}, \qquad \mathcal{A}^1 := \left\{ \begin{bmatrix} 1 \\ d \end{bmatrix} \mid d \geq 0 \right\}.$$

With this we define the following ideal in $(\mathbb{Q}\langle\mathcal{A}\rangle, *)$ generated by the set $\mathcal{A}^* \backslash ((\mathcal{A}^1)^*(\mathcal{A}_0)^*)$

$$\mathfrak{N} := \left(\mathcal{A}^* \backslash ((\mathcal{A}^1)^*(\mathcal{A}_0)^*)\right)_{\mathbb{Q}\langle\mathcal{A}\rangle},$$

where for an alphabet $\mathcal{L}$ by $\mathcal{L}^*$ we denote the set of words in the letters $\mathcal{L}$, i.e., the free monoid generated by the elements in $\mathcal{L}$. The generators of $\mathfrak{N}$ are exactly those elements which are <u>not</u> of the form

$$G^f\begin{pmatrix} 1, \dots, 1, k_1, \dots, k_r \\ d_1, \dots, d_s, 0, \dots, 0 \end{pmatrix},$$

for some $k_1, \dots, k_r \geq 1, d_1, \dots, d_s \geq 0$.

Definition 2.10. The algebra of *formal multiple zeta values* is defined by

$$\mathcal{Z}^f := \mathcal{G}^f \big/ \mathfrak{N}.$$

The justification for the name formal multiple zeta values comes from the fact that our notion is equivalent, up to the non-vanishing of $\zeta^f(1)$ in our case, to the one by Racinet [Rac] (see Theorem 2.13 below), which consists of formal symbols satisfying the extended double shuffle relations. In particular, we expect $\mathcal{Z}^f \cong \mathcal{Z}[T]$. Note that this definition does not coincide with the definition of formal multiple zeta values in the Introduction: it is the content of Theorem 2.13 that both definitions are equivalent.

We denote the canonical projection of the space of formal multiple Eisenstein series into the space of formal multiple zeta values by

$$\pi : \mathcal{G}^f \longrightarrow \mathcal{Z}^f. \tag{2.4}$$

This projection can be seen as a formal version of the 'projection onto the constant term'.

Proposition 2.11. *The map* $\pi_{|\operatorname{Fil}_0^{\mathrm{lwt}} \mathcal{G}^f} : \operatorname{Fil}_0^{\mathrm{lwt}} \mathcal{G}^f \to \mathcal{Z}^f$ *is surjective.*

Proof. All non-zero elements in $\mathcal{Z}^f$ are linear combinations of elements of the form

$$f = G^f\begin{pmatrix} 1, \dots, 1, & 1, \dots, 1 & , k_1, \dots, k_r \\ d_1, \dots, d_s, & \underbrace{0, \dots, 0}_{j} & , 0, \dots, 0 \end{pmatrix} \tag{2.5}$$

with $s,r\geq 0$, $j\geq 0$, $d_s\geq 1$ and $k_1\geq 2$. By induction on j one can apply the usual calculation used for the stuffle regularization ([BI, Proposition 4.18]) to show that f can be written as

$$f\equiv \sum_{m=0}^{j} f_m\, G^{\mathrm{f}}(1)^m \qquad \mathrm{mod}\ \mathfrak{N}\,,$$

where the f_m are have the same shape as (2.5) with $j=0$. Such elements can be expressed as products $G^{\mathrm{f}}\binom{1,\dots,1}{d_1,\dots,d_s}G^{\mathrm{f}}\binom{k_1,\dots,k_r}{0,\dots,0}$ modulo $\mathfrak{N}$. By the definition of σ it is easy to see that $\sigma(\mathbb{Q}\langle\mathcal{A}^1\rangle)=\mathbb{Q}\langle\mathcal{A}_0\rangle$, i.e., $\sigma G^{\mathrm{f}}\binom{1,\dots,1}{d_1,\dots,d_s}\in \mathrm{Fil}_0^{\mathrm{lwt}}\,\mathcal{G}^{\mathrm{f}}$. Therefore, we can find a representative for the class of f in $\mathcal{Z}^f$ which is an element in $\mathrm{Fil}_0^{\mathrm{lwt}}\,\mathcal{G}^{\mathrm{f}}$. □

Definition 2.12. For $k_1,\dots,k_r\geq 1$ we define the *formal multiple zeta value* $\zeta^{\mathrm{f}}(k_1,\dots,k_r)$ by

$$\zeta^{\mathrm{f}}(k_1,\dots,k_r):=\pi(G^{\mathrm{f}}(k_1,\dots,k_r))\,.$$

The $\mathbb{Q}$-linear map defined on the generators by

$$\begin{aligned}\zeta^f:\mathfrak{H}^1&\longrightarrow \mathcal{Z}^f\\ z_{k_1}\cdots z_{k_r}&\longmapsto \zeta^{\mathrm{f}}(k_1,\dots,k_r)\end{aligned}\tag{2.6}$$

is an algebra homomorphism with respect to the stuffle product $*$. This follows from the fact that (2.3) is an algebra homomorphism and from the definition of $\zeta^{\mathrm{f}}(k_1,\dots,k_r)$. The justification for calling ζ^f formal multiple zeta values comes from the following theorem, stating that they conjecturally satisfy the same relations as ($*$-regularized) multiple zeta values, namely the extended double shuffle relations (cf. [IKZ, Rac]). Notice that, in contrast to [Rac], we have $\zeta^{\mathrm{f}}(1)\neq 0$.

Theorem 2.13 ([BIM, Theorem 2.17])**.** *The formal multiple zeta values ζ^f satisfy exactly the extended double shuffle relations, i.e. the kernel of the map* (2.6) *is the ideal generated by $w*v-w\sqcup\!\sqcup\, v$ for $w\in\mathfrak{H}^1$ and $v\in\mathfrak{H}^0$.*

3. Derivations for formal multiple Eisenstein series

Now we introduce the $\mathfrak{sl}_2$-structure on $\mathcal{G}^{\mathrm{f}}$. For this, the authors in [BIM, Section 3] introduced various families of derivations on quasi-shuffle algebras, which are σ-equivariant. We will leave out the details and just give the definition of the derivations explicitly.

An algebra A is called an $\mathfrak{sl}_2$*-algebra* if there exists a Lie algebra homomorphism $\mathfrak{sl}_2\to \mathrm{Der}(A)$. More explicitly, there exist three derivations $D,W,\delta\in\mathrm{Der}(A)$ such that (D,W,δ) forms an $\mathfrak{sl}_2$*-triple*. This means that the satisfy the commutator relations

$$[W,D]=2D,\quad [W,\delta]=-2\delta,\quad [\delta,D]=W\,.$$

The algebra of quasimodular forms $\mathbb{Q}[G_2,G_4,G_4]$ is our main example of an $\mathfrak{sl}_2$-algebra. The differential operator

$$D:=\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\tau}=q\frac{\mathrm{d}}{\mathrm{d}q}$$

preserves the space of quasimodular forms. Besides D, an important differential operator on quasimodular forms is the operator δ defined by $\delta G_2=-\frac{1}{2}$ and $\delta G_4=\delta G_6=0$. Lastly, ones defines the operator W by $W(G_k)=kG_k$ for $k=2,4,6$. The algebra of quasimodular forms is an $\mathfrak{sl}_2$-algebra with respect to these three operators.

We will now introduce three derivations D, W and δ on $\mathcal{G}^f$ giving it the structure of an $\mathfrak{sl}_2$-algebra. These can be seen as natural extensions of the derivations on quasimodular forms described above. This is supported by the fact that there is a $\mathfrak{sl}_2$-subalgebra of $\mathcal{G}^f$ isomorphic to the $\mathfrak{sl}_2$-algebra of quasimodular forms. This algebra of formal quasimodular forms will be defined in the next section. For motivation of the definitions of the following operators, we refer to [BIM, Section 4.4]).

The derivations W and D on $\mathcal{G}^f$ are given as the $\mathbb{Q}$-linear maps defined on the generators by

$$
\begin{aligned}
DG^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r} &:= \sum_{j=1}^{r} k_j\, G^f\binom{k_1,\dots,k_j+1,\dots,k_r}{d_1,\dots,d_j+1,\dots,d_r},\\
WG^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r} &:= (k_1+\dots+k_r+d_1+\dots+d_r)\binom{k_1,\dots,k_r}{d_1,\dots,d_r}.
\end{aligned}
$$

The definition of δ is in comparison much more complicated. Its discovery is based on extensive numerical experiments done by the authors of [BIM]. We write it as a sum of five derivations as follows

$$
\delta := \delta^1 - \frac{1}{2}\left(\delta^2+\delta^3+\delta^4+\delta^5\right),
$$

where $\delta^j : \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}\langle\mathcal{A}\rangle$ $(1 \le j \le 5)$ are the linear maps defined on the generators by

$$
\begin{aligned}
\delta^1\begin{bmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{bmatrix} &:= \sum_{j=1}^{r} \mathbf{1}_{k_j>1}\, d_j \begin{bmatrix}k_1,\dots,k_j-1,\dots,k_r\\ d_1,\dots,d_j-1,\dots,d_r\end{bmatrix}\\
&\quad - \frac{1}{2}\sum_{j=1}^{r-1} \mathbf{1}_{k_{j+1}=1}\, d_{j+1} \begin{bmatrix}k_1,\dots,k_j,k_{j+2},\dots,k_r\\ d_1,\dots,d_j+d_{j+1}-1,d_{j+2},\dots,d_r\end{bmatrix}\\
&\quad - \frac{1}{2}\sum_{j=1}^{r-1} \mathbf{1}_{k_j=1}\, d_j \begin{bmatrix}k_1,\dots,k_{j-1},k_{j+1},\dots,k_r\\ d_1,\dots,d_{j-1},d_j+d_{j+1}-1,\dots,d_r\end{bmatrix},\\
\delta^2\begin{bmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{bmatrix} &:= \mathbf{1}_{\substack{k_r=2\\ d_r=0}} \begin{bmatrix}k_1,\dots,k_{r-1}\\ d_1,\dots,d_{r-1}\end{bmatrix} - \frac{1}{2}\mathbf{1}_{\substack{k_{r-1}=k_r=1\\ d_{r-1}=d_r=0}} \begin{bmatrix}k_1,\dots,k_{r-2}\\ d_1,\dots,d_{r-2}\end{bmatrix},\\
\delta^3\begin{bmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{bmatrix} &:= \mathbf{1}_{\substack{k_r=1\\ d_r=1}} \begin{bmatrix}k_1,\dots,k_{r-1}\\ d_1,\dots,d_{r-1}\end{bmatrix},\\
\delta^4\begin{bmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{bmatrix} &:= \sum_{j=2}^{r} \mathbf{1}_{\substack{k_j=1\\ k_{j-1}>1}} \begin{bmatrix}k_1,\dots,k_{j-1}-1,k_{j+1},\dots,k_r\\ d_1,\dots,d_{j-1}+d_j,d_{j+1},\dots,d_r\end{bmatrix}\\
&\quad - \sum_{j=1}^{r-1} \mathbf{1}_{\substack{k_j=1\\ k_{j+1}>1}} \begin{bmatrix}k_1,\dots,k_{j-1},k_{j+1}-1,\dots,k_r\\ d_1,\dots,d_{j-1},d_j+d_{j+1},\dots,d_r\end{bmatrix},
\end{aligned}
$$

$$\delta^5\begin{bmatrix}k_1,\dots,k_r\\d_1,\dots,d_r\end{bmatrix} := \sum_{j=1}^{r-1}\mathbf{1}_{k_j=2}\begin{bmatrix}k_1,\dots,k_{j-1},k_{j+1},\dots,k_r\\d_1,\dots,d_{j-1},d_j+d_{j+1},\dots,d_r\end{bmatrix}$$
$$-\sum_{j=1}^{r-1}\mathbf{1}_{k_{j+1}=2}\begin{bmatrix}k_1,\dots,k_j,k_{j+2},\dots,k_r\\d_1,\dots,d_j+d_{j+1},d_{j+2},\dots,d_r\end{bmatrix}$$
$$+\frac{1}{2}\sum_{j=1}^{r-2}\mathbf{1}_{k_{j+1}=k_{j+2}=1}\begin{bmatrix}k_1,\dots,k_j,k_{j+3},\dots,k_r\\d_1,\dots,d_j+d_{j+1}+d_{j+2},d_{j+3},\dots,d_r\end{bmatrix}$$
$$-\frac{1}{2}\sum_{j=1}^{r-2}\mathbf{1}_{k_j=k_{j+1}=1}\begin{bmatrix}k_1,\dots,k_{j-1},k_{j+2},\dots,k_r\\d_1,\dots,d_{j-1},d_j+d_{j+1}+d_{j+2},\dots,d_r\end{bmatrix}.$$

Theorem 3.1 ([BIM, Theorem 4.12])**.** *With the maps D, W, δ defined above, the space $\mathcal{G}^f$ is an $\mathfrak{sl}_2$-algebra.*

On the single indexed G^f the derivation δ has the much easier form

$$\delta\, G^f(k_1,\dots,k_r) = -\frac{1}{2}\mathbf{1}_{k_1=2}\, G^f(k_2,\dots,k_r) + \frac{1}{4}\mathbf{1}_{k_1=k_2=1}G^f(k_3,\dots,k_r)$$
$$+\frac{1}{2}\sum_{j=1}^{r-1}\mathbf{1}_{\substack{k_j=1\\k_{j+1}>1}}\, G^f(k_1,\dots,k_{j-1},k_{j+1}-1,\dots,k_r)$$
$$-\frac{1}{2}\sum_{j=2}^{r}\mathbf{1}_{\substack{k_j=1\\k_{j-1}>1}}\, G^f(k_1,\dots,k_{j-1}-1,k_{j+1},\dots,k_r).$$

Moreover, following the same proof as in [BB, Proposition 6.30 and Corollary 6.31] one has the following nice expression for the operator D

$$DG^f(k_1,\dots,k_r) = G^f\big(z_2 * z_{k_1}\cdots z_{k_r} - z_2 \sqcup\!\sqcup\, z_{k_1}\cdots z_{k_r}\big).$$

This leads to the following:

Proposition 3.2 ([BIM, Proposition 4.15])**.** *The subspace $\mathrm{Fil}_0^{\mathrm{lwt}}\,\mathcal{G}^f$ is an $\mathfrak{sl}_2$-subalgebra of $\mathcal{G}^f$.*

4. Formal quasimodular forms

In this section, present the definition of formal analogues of classical quasimodular forms as a subalgebra of the space $\mathcal{G}^f$.

Definition 4.1. We define the algebra of *formal quasimodular forms* $\widetilde{\mathcal{M}}^f$ as the smallest $\mathfrak{sl}_2$-subalgebra of $\mathcal{G}^f$ which contains $G^f(2)$.

Combining all the results proven for formal multiple Eisenstein series, the authors of [BIM] obtain the following.

Corollary 4.2 ([BIM, Corollary 5.5]). *The Ramanujan differential equations are satisfied in $\mathcal{G}^f$, i.e.,*

$$DG^f(2) = 5G^f(4) - 2G^f(2)^2\,,$$
$$DG^f(4) = 14G^f(6) - 8G^f(2)\,G^f(4)\,,$$
$$DG^f(6) = \frac{120}{7}G^f(4)^2 - 12G^f(2)\,G^f(6)\,.$$

Theorem 4.3 ([BIM, Theorem 5.4]). *We have $\widetilde{\mathcal{M}}^f \cong \widetilde{\mathcal{M}}$ as $\mathfrak{sl}_2$-algebras.*

Corollary 4.4. *The algebra of formal quasimodular forms $\widetilde{\mathcal{M}}^f$ satisfies:*

(i) $G^f(k) \in \widetilde{\mathcal{M}}^f$ *for all even* $k \geq 2$.

(ii) $\widetilde{\mathcal{M}}^f = \mathbb{Q}[G^f(2), G^f(4), G^f(6)] = \mathbb{Q}[G^f(2), DG^f(2), D^2G^f(2)]$.

(iii) *The Chazy equation is satisfied, i.e.,*

$$D^3G^f(2) + 24\,G^f(2)\,D^2G^f(2) - 36(DG^f(2))^2 = 0\,.$$

Definition 4.5. We define the algebra of *formal modular forms* by $\mathcal{M}^f = \ker \delta_{|\widetilde{\mathcal{M}}^f}$. Write $\mathcal{M}^f_k$ to denote the space of all formal modular forms of weight $k \geq 0$.

Proposition 4.6. *We have $\mathcal{M}^f = \mathbb{Q}[G^f(4), G^f(6)] \cong \mathcal{M}$.*

Recall the projection $\pi : \mathcal{G}^f \to \mathcal{Z}^f$ given by (2.4).

Definition 4.7. We define the algebra of *formal cusp forms* by $\mathcal{S}^f = \ker \pi_{|\mathcal{M}^f}$. Write $\mathcal{S}^f_k$ to denote the space of all formal cusp forms of weight $k \geq 0$.

The first example of a non-zero formal cusp form appears in weight $12 = \mathrm{lcm}(4, 6)$.

$$\Delta^{\mathfrak{f}} = \frac{e(4)^3 - e(6)^2}{1728} = 2400 \cdot 6! \cdot G^f(4)^3 - 420 \cdot 7! \cdot G^f(6)^2\,,$$

where we write $e(k) = -\frac{2k!}{B_k}G^f(k)$ for even $k \geq 2$. These elements correspond to the Eisenstein series E_k, which all have constant term 1.

Proposition 4.8. *We have $\Delta^{\mathfrak{f}} \in \mathcal{S}^f_{12}$, $D\Delta^{\mathfrak{f}} = e(2)\Delta^{\mathfrak{f}}$ and $\mathcal{M}^f_k = \mathbb{Q}G^f(k) \oplus \mathcal{S}^f_k$.*

References

[Ba1] H. Bachmann: *Multiple Zeta-Werte und die Verbindung zu Modulformen durch Multiple Eisensteinreihen*, Master thesis, Universität Hamburg (2012) (available at `http://www.henrikbachmann.com`).

[Ba2] H. Bachmann: *The algebra of bi-brackets and regularized multiple Eisenstein series*, J. Number Theory **200** (2019), 260–294.

[Ba5] H. Bachmann: *Stuffle regularized multiple Eisenstein series revisited*, RIMS Kôyûroku **2238** (2023), 73–86.

[BB] H. Bachmann, A. Burmester: *Combinatorial multiple Eisenstein series*, Res. Math. Sci. **10**:3, (2023), no. 5, 32 pp.

[BKM] H. Bachmann, U. Kühn, N. Matthes: *Realizations of the formal double Eisenstein space*, preprint, arXiv:2109.04267 (2021), 16 pp.

[BI] H. Bachmann, J.-W. van Ittersum: *Partitions, multiple zeta values and the q-bracket*, Selecta Math. **30**:3 (2024), 46 pp.

[BIM] H. Bachmann, J.-W. van Ittersum, N. Matthes: *Formal multiple Eisenstein series and their derivations*, preprint, arXiv:2312.04124 (2023)

[BK] H. Bachmann, U. Kühn: *A dimension conjecture for q-analogues of multiple zeta values*, Theory and Arithmetic, Springer Proceedings in Mathematics & Statistics **314** (2020), 237–258.

[Bu1] A. Burmester: *An algebraic approach to multiple q-zeta values*, PhD Thesis, Universität Hamburg (2023), 232 pp.

[GKZ] H. Gangl, M. Kaneko, D. Zagier: *Double zeta values and modular forms*, in "Automorphic forms and zeta functions" World Sci. Publ., Hackensack, NJ (2006), 71–106.

[H] M. E. Hoffman: *Quasi-shuffle products*, J. Algebraic Combin. **11** (2000), 49-68.

[IKZ] K. Ihara, M. Kaneko and D. Zagier: *Derivation and double shuffle relations for multiple zeta values*, Compositio Math. **142** (2006), 307–338.

[Rac] G. Racinet: *Series generatrices non-commutatives de polyzetas et associateurs de Drinfeld*, Thesis, Laboratoire Amienois de Mathematique Fondamentale et Appliquee (2000), xi+171 pp.

[V] A. Vleeshouwers: *Multiple zeta values and their q-analogues*, Master thesis, Radboud University (2020), arXiv:2007.08865, 56 pp.

[Z] D. Zagier: *Elliptic modular forms and their applications*, In *The 1-2-3 of modular forms*, K. Ranestad, ed., Universitext, pp. 1–103. Springer, Berlin, 2008.

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