SINGULARITIES IN ARITHMETIC GEOMETRY

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Abstract. This is a report of the recent developments on the study of singularities in positive and mixed characteristics based on the talk delivered by the author in the workshop "Algebraic Number Theory and Related Topics 2023" in Kyoto University. Since Y. André proved the direct summand conjecture via perfectoid spaces, many interesting and exciting results have been obtained, which led to a deep understanding of singularities in mixed characteristic and certain non-Noetherian rings.

1. Review of the homological conjecture

Let us recall the statement of the Direct Summand Conjecture by Hochster in 1969 as formulated in [14].

Conjecture 1 (Hochster). *Let R be a Noetherian regular local ring and let $R \to S$ be a module-finite extension of domains. Then there exists an R-module N such that $S \cong R \oplus N$ as R-modules.*

This is a conjecture on regular rings. If the reader is interested in regular rings, it is not too much to say that the next theorem of Serre is of historical importance:

Theorem 2 (Serre). *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then the following statements are equivalent to one another.*

(1) R is regular.
(2) Every finitely generated R-module has finite free resolution.
(3) $\mathrm{proj.dim}_R(k) < \infty$.

Using this result, one can prove that if R is a regular local ring and $\mathfrak{p}$ is a prime ideal, then the localization $R_{\mathfrak{p}}$ is also regular local. Serre's proof is totally characteristic-free. However, the Direct Summand Conjecture has gone through a roundabout path toward the settlement (see [2] and [3]).

Theorem 3 (André, Heitmann, Hochster). *The Direct Summand Conjecture is true.*

Although the Direct Summand Conjeture is a characteristic-free statement, the proof depends on the characteristic of the regular local ring R. There are three cases that we need to take care of separately. Let us assume that $(R, \mathfrak{m}, k)$ is a Noetherian local domain. Namely, R has a unique maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m} = k$ is its residue field.

- R has equal characteristic 0, or equivalently, $\mathbf{Q} \subseteq R$.
- R has equal characteristic $p > 0$ for a prime p, or equivalently, $\mathbf{F}_p \subseteq R$.
- R has mixed characteristic $p > 0$ for a prime p, or equivalently, $\mathbf{Z}_{(p)} \subseteq R$ and $pR \neq R$.

We make a few remarks on the proof of the Direct Summand Conjecture. If the regular local ring R has equal characteristic 0, then we have a trace map $\mathrm{Tr} : S \to R$ in the sense of field theory. Set $d := [Q(S) : Q(R)]$. Since $\mathbf{Q} \subseteq R$, it makes sense to define the R-module map

$$\frac{1}{d}\mathrm{Tr} : S \to R$$

which gives an R-splitting of the inclusion $R \to S$. This proof reveals that it suffices to assume that R is an *integrally closed domain*.

Next let us assume $\mathbf{F}_p \subseteq R$. This case is a bit tricky. The Direct Summand Conjecture is equivalent to the Monomial Conjecture (see [14, Theorem 1]), which is now a theorem.

Theorem 4 (Monomial Conjecture)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with a system of parameters $x_1, \ldots, x_d$. Then for all positive integers t, we have*

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1}).$$

If we assume that R is *Cohen-Macaulay*, then the Monomial Conjecture is quite an easy exercise. Bhatt pushed further the Direct Summand Conjecture in the framework of the derived category. Namely, he proved the following (see [5] for the proof).

Theorem 5 (Bhatt)**.** *Let R be a regular ring and let $f : X \to \mathrm{Spec}\,(R)$ be a proper surjective morphism of schemes. Then in the derived category $\mathrm{D}(R)$, the natural map $f^* : R \to \mathbf{R}\Gamma(X, \mathcal{O}_X)$ splits.*

In particular, if $f : X \to \mathrm{Spec}\,(R)$ is finite, then $X = \mathrm{Spec}\,(S)$ for some module-finite extension $R \to S$ and $\mathbf{R}\Gamma(X, \mathcal{O}_X) \simeq_{q.i.} S$. So this case comes down to the usual statement of the Direct Summand Conjecture.

2. Cohen-Macaulay algebras

Let us recall the definition of Cohen-Macaulay rings/modules.

Definition 6. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let M be an R-module. If M is a nonzero finitely generated module such that some (equivalently, every) system of parameters of R is a regular sequence on M, we say that M is a *maximal Cohen-Macaulay module*. In particular, if $R = M$, then we say that R is a *Cohen-Macaulay ring*. If $M \neq \mathfrak{m}M$ and some system of parameters of R is a regular sequence on M, we say that M is a *big Cohen-Macaulay R-module*. Moreover, if every system of parameters of R is a regular sequence on M, then we say that M is a *balanced big Cohen-Macaulay R-module*.

Cohen-Macaulay rings are a cornerstone of all singularities in commutative algebra, algebraic geometry, and number theory. In fact, certain normal Cohen-Macaulay singularities show up in the local models of Shimura varieties (see [12] for example). Big Cohen-Macaulay modules are not necessarily assumed to be finitely generated. In other words, even if $M \neq 0$, it can happen that $M = \mathfrak{m}M$. For example, let M be the field of fractions of a local domain R. Then M is not a finitely generated R-module. In fact, if M is finitely generated, $M \neq 0$ holds if and only if $M \neq \mathfrak{m}M$. The following question is, however, largely open except for the case that the dimension is at most 2.

Question 7 (Hochster)**.** *Does every complete Noetherian local ring have a maximal Cohen-Macaulay module?*

Let R be a complete Noetherian local domain of dimension at most 2. Then the integral closure $\overline{R}$ of R in the total ring of fractions is normal. By Serre's regularity theorem, we see that $R \to \overline{R}$ is module-finite and $\overline{R}$ is maximal Cohen-Macaulay over R. On the other hand, big Cohen-Macaulay algebras exist in general.

Theorem 8 (André, Hochster-Huneke)**.** *Every Noetherian local ring has a big Cohen-Macaulay algebra.*

Using this theorem, the Monomial Conjecture can be shown easily. Indeed, let M be a big Cohen-Macaulay R-module. Suppose that $x_1^t \cdots x_d^t \in (x_1^{t+1}, \ldots, x_d^{t+1})$ for some t. Then we can map this relation to M, so that $x_1^t \cdots x_d^t \in (x_1^{t+1}, \ldots, x_d^{t+1})M$. Since $x_1, \ldots, x_d$ is a regular sequence on M, we can get a contradiction. There are other homological conjectures which are equivalent to the Direct Summand Conjecture. Most of the homological conjectures follow from Theorem 8. We are curious to know an answer of the following question.

Question 9. *Is there a characteristic-free approach to the Direct Summand Conjecture?*

Unlike Serre's regularity characterization, the currently known proof of the Direct Summand Conjecture depends on the characteristic of the ring. All known constructions of big Cohen-Macaulay algebras also depend on the characteristic, and we will discuss the construction of big Cohen-Macaulay algebras.

3. Perfectoid rings and absolute integral closure

We start giving the definition of perfectoid rings which are building blocks of perfectoid spaces. The foundations on the geometry of perfectoid spaces are built in [24] and [33]. The reference [36] is written in the style of commutative algebra. The method of perfectoid geometry has brought many fruitful applications to commutative algebra, which is the main theme of this expository paper.

Definition 10. Let A be a ring with a regular element $\varpi \in A$. We say that A is an *(integral) perfectoid ring* if $p \mid \varpi^p$, A is ϖ-adically complete and separated, and the ring map $A/\varpi A \to A/\varpi^p A$ defined by $x \mapsto x^p$ is bijective.

The above definition is a special case of more general perfectoid rings studied in [8]. For our purpose, this is sufficient.

Definition 11. Let A be an integral domain with field of fractions K. Let $\overline{K}$ be the algebraic closure of K. Then the *absolute integral closure* of A is defined to be the integral closure of A in $\overline{K}$.

For example, the absolute integral closure $\mathbf{Z}^+$ of the ring of integers is a non-Noetherian principal ideal domain; every finitely generated ideal of $\mathbf{Z}^+$ is principal. Let A be a domain such that a prime $p \in A$ is a nonzero non-unit element. Then the p-adic completion $\widehat{A^+}$ of A^+ is perfectoid. Using absolute integral closure, we have the following deep result (see [16], [17] and [31]).

Theorem 12 (Hochster-Huneke, Hyuneke-Lyubeznik, Quy)**.** *Assume $(R, \mathfrak{m}, k)$ is a Noetherian local domain of equal characteristic $p > 0$. If R is a homomorphic image of a Cohen-Macaulay local ring, then R^+ is a balanced big Cohen-Macaulay algebra.*

We strongly recommend to the reader [17] for a readable proof. In the mixed characteristic case, we have a similar result [6] whose proof requies advanced techniques, including prismatic cohomology and p-adic Riemann-Hilbert correspondence.

Theorem 13 (Bhatt)**.** *Let $(R, \mathfrak{m}, k)$ be an excellent local domain of mixed characteristic $p > 0$. Then the p-adic completion of the absolute integral closure R^+ is a perfectoid balanced big Cohen-Macaulay algebra.*

It is important to have sufficiently many big Cohen-Macaulay algebras (see [4] for the proof of the following result and references therein).

Theorem 14 (André, Hochster-Huneke)**.** *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local map of Noetherian complete local domains. Then there exists a commutative diagram of rings:*

$$\begin{array}{ccc} \mathcal{B}(R) & \longrightarrow & \mathcal{B}(C) \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that $\mathcal{B}(R)$ (resp. $\mathcal{B}(S)$) is a big Cohen-Macaulay algebra over R (resp. S).

4. Almost Cohen-Macaulay algebras

In mixed characteristic, it is instructive to try to construct almost Cohen-Macaulay algebras, which we define below. Let us say that a pair (A, I) is a *basic setup* if A is a ring with an ideal $I \subseteq A$ such that $I^2 = I$.

Definition 15. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with a system of parameters $x_1, \ldots, x_d$. Assume that (A, I) is a basic setup such that A is an R-algebra. Then an A-module M is an *I-almost Cohen-Macaulay R-module* (with respect to $x_1, \ldots, x_d$) if the following conditions hold:

1. $M/\mathfrak{m}M$ is not an I-almost zero A-module.
2. For any $i = 0, \ldots, d-1$, we have

$$I \cdot \frac{((x_1, \ldots, x_i)M :_M x_{i+1})}{(x_1, \ldots, x_i)M} = 0.$$

Recall that for an $\mathbf{F}_p$-algebra A, we define the *perfect closure* of A to be $A_{\text{perf}} := \bigcup_{e>0} (A_{\text{red}})^{1/p^e}$. In positive characteristic, we have the following result, which is relatively easy to prove, using the Frobenius map and Cohen's structure theorem on complete local rings.

Theorem 16 (Singh-Srinivas-Roberts)**.** *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local $\mathbf{F}_p$-domain. Then there exist a nonzero element $g \in A$ and a system of parameters $x_1, \ldots, x_d$ of R such that R_{perf} is an I-almost Cohen-Macaulay algebra with respect to $x_1, \ldots, x_d$, where $I = \bigcup_{e>0} g^{1/p^e} A_{\text{perf}}$.*

It seems to be quite difficult to find an almost Cohen-Macaulay algebra that is not Cohen-Macaulay in the equal characteristic zero case. In the mixed characteristic case, the following question was raised.

Question 17 (Roberts)**.** *Assume $(R, \mathfrak{m})$ is a complete Noetherian local domain in mixed characteristic and let R^+ be the integral closure of R in a fixed algebraic closure of the fraction field of R. Then does there exist an R-algebra B such that $R \subset B \subset R^+$ and B is almost Cohen-Macaulay?*

Of course, Bhatt gave an answer to this question by taking B to R^+. However, there is another answer which is stated as follows (see [30]).

Theorem 18 (Nakazato-Shimomoto)**.** *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain of mixed characteristic $p > 0$ with perfect residue field k. Let $p, x_2, \ldots, x_d$ be a system of parameters and let R^+ be the absolute integral closure of R. Then there exists an R-algebra T together with a nonzero element $g \in R$ such that the following hold:*

1. *T admits compatible systems of p-power roots $p^{\frac{1}{p^n}}, g^{\frac{1}{p^n}} \in T$ for all $n > 0$.*
2. *The Frobenius endomorphism $Frob : T/(p) \to T/(p)$ is surjective.*

(3) *T is a $(pg)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay normal domain with respect to the system of parameters $p, x_2, \ldots, x_d$ and $R \subset T \subset R^+$.*
(4) *The p-adic completion $\widehat{T}$ is integral perfectoid.*
(5) *$R[\frac{1}{pg}] \to T[\frac{1}{pg}]$ is an ind-étale extension. In other words, $T[\frac{1}{pg}]$ is a filtered colimit of finite étale $R[\frac{1}{pg}]$-algebras contained in $T[\frac{1}{pg}]$.*

The proof of this theorem relies on the decompleted version of André's Perfectoid Abhyankar's Lemma and Riemann's extension theorem (Hebbarkeitssatz). I am interested in the connection of the above theorem with Theorem 13. Using Hochster's *partical algebra modifications*, one can find a map $T \to B$ such that B is an honest big Cohen-Macaulay R-algebra. This method was developed after Heitmann proved the Direct Summand Conjecture in dimension 3 (see [15] and [35] for details).

5. Problems on regular rings

The most fundamental result in the singularities of commutative rings in characteristic $p > 0$ is Kunz' theorem (see [25] and [26]).

Theorem 19 (Kunz)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local $\mathbf{F}_p$-algebra. Then the following assertions are equivalent.*

(1) *R is regular.*
(2) *The Frobenius map $F : R \to R$ is flat.*
(3) *The natural ring map $R \to R_{\mathrm{perf}}$ is flat.*

Although there is NO Frobenius map in mixed or equal characteristic zero, we have the following results, which stem from Theorem 19 (see [7] and [29]).

Theorem 20 (Bhatt-Iyengar-Ma)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of mixed characteristic. Then the following assertions are equivalent.*

(1) *R is regular.*
(2) *There exist a perfectoid ring A and a faithfully flat ring map $R \to A$.*

Theorem 21 (Ma-Schwede)**.** *Let $(R, \mathfrak{m}, k)$ be a local domain which is essentiall of finite type over a field K. Then the following assertions are equivalent.*

(a) *R is regular.*
(b) *Let $f : X \to \mathrm{Spec}\,(R)$ be a regular alteration, that is, X is regular and f is a proper surjective, generically finite map. Then $\mathrm{proj.dim}_R \mathbf{R} f_* \mathcal{O}_X < \infty$.*

Moreover, if R contains $\mathbf{Q}$, then the above conditions are also equivalent to

(c) *Let $f : X \to \mathrm{Spec}\,(R)$ be a regular alteration. Then $\mathrm{proj.dim}_R \mathbf{R} f_* \mathcal{O}_X = 0$.*

More recently, Ishizuka and Nakazato proved a mixed characteristic Kunz' theorem via prismatic cohomology (see [22] for the precise statement). Many interesting problems on regular rings have been solved via perfectoid geometry. However, there are still open questions. Let me list some of them.

Conjecture 22 (Claborn-Fossum)**.** *Let* $(R, \mathfrak{m}, k)$ *be a regular local ring and let* $\mathrm{CH}^p(R)$ *denote the Chow group of codimension-p cycles on* $\mathrm{Spec}\,(R)$*. Then* $\mathrm{CH}^p(R) = 0$ *for all* $p > 0$*.*

This conjecture has been settled in the equal characteristic case, but remains open in mixed characteristic. One wonders whether some sort of the tilting methods in perfectoid geometry could be useful. See [37] for recent advances, including the connection with Gersten's conjecture. Some problems are solved for *unramified* regular local rings. For instance, the following conjecture is open for ramified regular local rings, thus in mixed characteristic (see [38] for the unramified case).

Conjecture 23 (Hartshorne)**.** *Let* $(R, \mathfrak{m}, k)$ *be a d-dimensional complete regular local ring with separably closed residue field. Let* $I \subseteq \mathfrak{m}$ *be an ideal. Then the following statements are equivalent.*

(1) $H^j_I(M) = 0$ *for all* $j > d - 2$ *and R-modules M.*
(2) $\dim(R/I) \geq 2$ *and the punctured spectrum of* R/I *is connected in Zariski topology.*

6. Arithmetic deformation of singularities

For a surjective scheme map $f : X \to \mathrm{Spec}\,(\mathbf{Z})$ of finite type, we can consider various fibers: the generic fiber $f^{-1}(\mathrm{Spec}\,(\mathbf{Q}))$ and closed fibers $f^{-1}(\mathrm{Spec}\,(\mathbf{F}_p))$. There is a quite interesting connection on the singularities appearing in these fibers and the configuration of points of X valued in a finite ring.

Theorem 24 (Aizenbud)**.** *Let the notation be as above. Assume that* $f^{-1}(\mathrm{Spec}\,(\mathbf{Q}))$ *is reduced, absolutely irreducible, and local complete intersection. Then the following assertions are equivalent.*

(1) Fix any integer $m > 0$*. Then*

$$\lim_{p \to \infty} \frac{|X(\mathbf{Z}/p^m\mathbf{Z})|}{p^{m \cdot \dim f^{-1}(\mathrm{Spec}\,(\mathbf{Q}))}} = 1.$$

(2) $f^{-1}(\mathrm{Spec}\,(\mathbf{Q}))$ *has only rational singularities.*

This is the main result of [1] which contains more detailed information. This says that singularities in the generic fiber can affect the behavior of the set of rational points. Another interesting result is due to Ma and Schwede (see [28]).

Theorem 25 (Ma-Schwede)**.** *Under the notation as above, assume that there is a prime p such that* $f^{-1}(\mathrm{Spec}\,(\mathbf{F}_p))$ *has only F-rational singularities. Then the generic fiber* $f^{-1}(\mathrm{Spec}\,(\mathbf{Q}))$ *has rational singularities.*

F-rational singularities are defined on Noetherian rings in characteristic $p > 0$ via the tight closure theory. There is an open question on the arithmetic deformation of singularities. For example, the following conjecture is stated in [12, Conjecture 3.6].

Conjecture 26. *Let $(R, \mathfrak{m}, k)$ be an excellent normal local domain of mixed characteristic admitting a dualizing complex. Assume that R/xR is an $\mathbf{F}_p$-algebra that is F-finite and F-injective, and $R[\frac{1}{x}]$ is pseudo-rational for some nonzero element $x \in \mathfrak{m}$. Then R has only pseudo-rational singularities.*

Some interesting results are obtained in the work of Sato and Takagi in [32].

7. Related topics

We understand the methods of perfectoid geometry are quite powerful in solving problems in commutative ring and algebraic geometry. However, as perfectoid rings are usually not Noetherian, we need to find a refined theory which approximates perfectoid rings in terms of Noetherian rings. Let $R_0 = W(k)[[x_2, \ldots, x_d]]$. Then we have a tower consisting of regular local rings:

$$R_0 \hookrightarrow R_1 := R_0[p^{1/p}, x_2^{1/p}, \ldots, x_d^{1/p}] \hookrightarrow \cdots \hookrightarrow R_n := R_0[p^{1/p^n}, x_2^{1/p^n}, \ldots, x_d^{1/p^n}] \hookrightarrow \cdots.$$

This is the simplest example of a *perfectoid tower*. The point is that each R_n is Noetherian, $R_n \to R_{n+1}$ is a module-finite flat extension, and the p-adic completion of $R_\infty := \bigcup_{n>0} R_n$ is perfectoid. A complete treatment is found in [20]. Perfectoid towers are good for *log-regular* rings.

Theorem 27 (Gabber-Ramero, Ishiro-Nakazato-Shimomoto)**.** *Let $(R, \mathfrak{m}, k)$ be a complete log-regular local ring of mixed characteristic with perfect residue field k. Then there is a perfectoid tower $R_0 := R \hookrightarrow R_1 \hookrightarrow \cdots$ with certain good properties with respect to the tilting operation and calculating étale cohomology. As an application, assume that $\widehat{R^{\mathrm{sh}}}[\frac{1}{p}]$ is locally factorial, where $\widehat{R^{\mathrm{sh}}}$ is the completion of the strict Henselization of R. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{p}]$ is a finite group, where $\mathrm{Cl}(R)_{\mathrm{tor}}$ is the torsion part of the divisor class group of R.*

The finiteness result on the divisor class group is proved as an application of tilting étale cohomology of the tower. The finiteness of the divisor class group is fully settled by Ishiro using the technique of calculating the class group of a monoid (see [19] for the proof).

Theorem 28 (Ishiro)**.** *Let $(R, \mathfrak{m}, k)$ be a log-regular local ring (of any characteristic). Then the divisor class group $\mathrm{Cl}(R)$ is a finitely generated Abelian group.*

Here is an interesting question (see [10] and [11] for the purity of Brauer groups of regular schemes). Perhaps, the purity in the context of logarithmic algebraic geometry is a right path (see [18]).

Question 29. *Is it possible to use Theorem 27 to prove purity-type result of the Brauer group of regular log schemes in mixed characteristic?*

The notion of perfectoid towers is quite useful for the study of singularities of arithmetic schemes.

Question 30. *Other than log-regular rings, can one find Noetherian rings of mixed characteristic admitting perfectoid towers?*

Another impetus in the recent developments of p-adic geometry is the theory of prismatic cohomology (see [9] for this exciting mathematics). As an analog of the perfection, the *perfectoidization* was introduced in [9], which has proved to be quite powerful in arithmetic geometry, algebraic geometry and commutative algebra. Unlike the perfection, the perfectoidization of a ring is defined as a certain cosimplicial commutative ring and is therefore hard to grasp. It is shown by Bhatt and Scholze that the perfectoidization of a ring that is integral over a perfectoid ring is discrete, thus a usual ring.[1] Recently, the work of Ishizuka has successfully done the job of presenting the perfectoidization in terms of some classical ring operations, such as integral closure.

Theorem 31 (Ishizuka)**.** *Let p be a prime and let R be a p-torsion free ring and assume that the following properties hold.*

1. *The p-adic completion $\widehat{R}$ has a ring map from some p-adic integral perfectoid ring.*
2. *The perfectoidization $\widehat{R}_{\mathrm{perfd}}$ of the p-adic completion $\widehat{R}$ is an integral perfectoid ring.*
3. *Let $C(R)$ be the p-root closure of R in $R[\frac{1}{p}]$. Then the p-adic completion $\widehat{C(R)}$ is an integral perfectoid ring.*

Then there is a ring isomorphism $\widehat{R}_{\mathrm{perfd}} \cong \widehat{C(R)}$.

See [21] for the proof of this theorem and related results and [23] for an application to the construction of almost Cohen-Macaulay algebras.

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