ON THE CONJECTURES OF BEILINSON–TATE FOR SIEGEL SIXFOLDS

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Abstract

These are the (extended) notes of a talk given by the author at the conference *Algebraic Number Theory and Related Topics 2023*, held at the Research Institute for Mathematical Sciences, Kyoto University. The author thanks the organizers Masataka Chida, Yoichi Mieda, and Wataru Kai for the wonderful opportunity to speak there. In this survey, we discuss our recent works [BGCLRJ23], [CLJ22] on the Beilinson–Tate conjectures for the Siegel sixfold and the description of Deligne–Beilinson cohomology in terms of tempered currents.

Contents

1. Automorphic L-functions

Let G be a reductive group over $\mathbb{Q}$. The definition of an automorphic L-function requires two data:

- A cuspidal autormophic representation π of $G(\mathbb{A}_{\mathbb{Q}})$, say $\pi = \pi_\infty \otimes \bigotimes'_p \pi_p$, where π_p is unramified at $p \notin S$, for a finite set S of places containing ∞;
- A finite dimensional complex representation r of the Langlands dual group ${}^L G$ of G.

From these data, we can form an Euler product

$$L^S(s,\pi,r) := \prod_{p \in S} L_p(s,\pi,r),$$

where, if s_{π_p} denotes the Frobenius conjugacy class of π_p (a.k.a. its Satake parameters), then

$$L_p(s,\pi,r) := \det\left(I - r(s_{\pi_p})p^{-s}\right)^{-1}.$$

The Euler product $L^S(s,\pi,r)$ converges absolutely for $\mathrm{Re}(s) >> 0$.

Conjecture 1.1 (Langlands)**.** *We can define factors at every place in S so that the resulting completed L-function $\Lambda(s,\pi,r)$ has meromorphic continuation to all $\mathbb{C}$ and it satisfies a functional equation with $s = 1/2$ as center of symmetry.*

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Suppose now that π is tempered everywhere, i.e. π_v is tempered at every place v, then $\Lambda(s,\pi,r)$ converges absolutely (and so it is holomorphic, non-zero) for $\mathrm{Re}(s) > 1$. This fact together with the conjectural functional equation poses some mystery on the values of $\Lambda(s,\pi,r)$ in the strip $\frac{1}{2} \leq s \leq 1$ (equivalently $0 \leq s \leq \frac{1}{2}$). It turns out that the behaviour of the partial L-function $L^S(s,\pi,r)$ (which might coincide with the one of $\Lambda(s,\pi,r)$ if we know how to control the Euler factors at places in S) contains information of analytic and arithmetic nature. For instance, on the analytic side (see [GJS10] for a nice survey on this) the location of poles of $L^S(s,\pi,r)$ is often linked to the non-vanishing of certain periods for π and to the realization of Langlands functoriality. Precisely, one might expect to relate the following three facts:

(1) The L-function $L^S(s,\pi,r)$ has a pole.
(2) A certain period for cusp forms in π is non-zero.
(3) The cuspidal representation π is a (weak) functorial lift with respect to the embedding $H \hookrightarrow {}^LG$, with H being the stabiliser of a generic point of the representation r.

Often, theta correspondence has been used to realise functorial lifts, so one might replace (3) by the existence of such a correspondence.

On the arithmetic side, the behaviour of $L^S(s,\pi,r)$ at $s=1$ contains rich arithmetic information. When $L^S(s,\pi,r)$ is motivic, the conjectures of Beilinson link the trascendental part of the leading coefficients of these L-functions to regulators of motivic cohomology classes, while the conjectures of Tate relate the residues of these L-functions to the existence of algebraic cycles.

1.1. **Some examples of L-functions and their pole at $s = 1$.**

1.1.1. *The exterior square L-function.* Let $\wedge^2 : \mathrm{GL}_4(\mathbf{C}) \to \mathrm{GL}_6(\mathbf{C})$ denote the exterior square of the standard representation V_4 of $\mathrm{GL}_4(\mathbf{C})$; it is an irreducible representation of $\mathrm{GL}_4(\mathbf{C})$ with highest weight $(1,1,0,0)$ (being the second fundamental weight for $\mathrm{GL}_4(\mathbf{C})$). We seek a subgroup $H \hookrightarrow \mathrm{GL}_4(\mathbf{C})$ such that

$$\wedge^2|_H = \mathbf{1} \oplus \rho,$$

with $\mathbf{1}$ the trivial representation of H and ρ a five dimensional irreducible representation of H. Let Sp_4 be the symplectic group of genus 2. Precisely, if I_2 denote the identity 2×2 matrix and let $J := \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$, then

$$\mathrm{Sp}_4(\mathbf{C}) = \{ g \in \mathrm{GL}_4(\mathbf{C}) \, : \, {}^t g J g = J \}.$$

Note that V_4 defines also the standard representation of $\mathrm{Sp}_4(\mathbf{C})$. Moreover, J defines a symplectic pairing $J : V_4 \times V_4 \to \mathbf{C}$ and hence a $\mathrm{Sp}_4(\mathbf{C})$-equivariant map $J : \wedge^2 V_4 \to \mathbf{C}$. Its kernel is the 5 dimensional irreducible $\mathrm{Sp}_4(\mathbf{C})$-representation of highest weight $(1,1)$. Using the exceptional isomorphism $\mathrm{Sp}_4(\mathbf{C}) \simeq \mathrm{Spin}_5(\mathbf{C})$, we can view $\rho_{1,1} := \ker(J)$ as a $\mathrm{Spin}_5(\mathbf{C})$-representation. It coincides with the representation obtained by taking the composition of the standard representation of $\mathrm{SO}_5(\mathbf{C})$ with the projection $\mathrm{Spin}_5(\mathbf{C}) \to \mathrm{SO}_5(\mathbf{C})$. We therefore have the desired decomposition

$$\wedge^2(V_4) = \mathbf{C} \oplus \rho_{1,1}.$$

This can be upgraded to a decomposition for representations of the group of symplectic similitudes GSp_4, but we won't need to get into that at the cost of restricting to automorphic representations with trivial central character. Now suppose that π_p is a spherical admissible representation of $\mathrm{GL}_4(\mathbf{Q}_p)$ with Satake parameters at p which belong to $\mathrm{Sp}_4(\overline{\mathbf{Q}}_p)$, then

$$\begin{aligned} L(s,\pi_p,\wedge^2) &= \det(I-\wedge^2(s_{\pi_p})p^{-s})^{-1} \\ &= (1-p^{-s})\cdot\det(I-\rho_{1,1}(s_{\pi_p})p^{-s})^{-1}. \end{aligned}$$

This implies that, if π is a tempered automorphic representation of $\mathrm{GL}_4(\mathbf{A_Q})$ with trivial central character, unramified outside a finite set S of primes, and with Satake parameters in Sp_4 at every prime not in S, we then have a decomposition of partial L-functions

$$L^S(s,\pi,\wedge^2) = L^S(s,\pi,\rho_{1,1})\zeta^S(s),$$

with $L^S(1,\pi,\rho_{1,1})\neq 0$, and thus $L^S(s,\pi,\wedge^2)$ has a simple pole at $s=1$. One can show that this is equivalent to π being the transfer from a cuspidal generic automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ with trivial central character. Moreover, these facts are also equivalent to asking that Π has non-trivial Shalika periods (*cf.* [JS90]). Shalika periods have shed profound insights into the Langlands program and their arithmetic significance has been explored for instance in [GS23] or [SDW21].

1.1.2. *The Asai L-function and base-change.* Let $F/\mathbf{Q}$ be a quadratic field extension and let us consider $H=\mathrm{PGL}_2$ and $G=\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{PGL}_2)$. The L-group of G is

$${}^LG = (\mathrm{SL}_2(\mathbf{C})\times\mathrm{SL}_2(\mathbf{C}))\rtimes\mathrm{Gal}(F/\mathbf{Q}),$$

where the non-trivial element τ of $\mathrm{Gal}(F/\mathbf{Q})$ acts by permuting the $\mathrm{SL}_2(\mathbf{C})$ factors. Note that the diagonal embedding induces a map $\mathrm{BC}_{F/\mathbf{Q}}: {}^LH\to{}^LG$. The map $\mathrm{BC}_{F/\mathbf{Q}}$ realizes the base change lift from automorphic representations of H to automorphic representations of G, in the sense that the Langlands parameters of a base change lift will factor through $\mathrm{BC}_{F/\mathbf{Q}}$ almost everywhere. Let V_2 denote the standard representation of $\mathrm{SL}_2(\mathbf{C})$ and let $V_4=V_2\otimes V_2$; identify $\mathrm{GL}(V_4)\simeq\mathrm{GL}_4(\mathbf{C})$. We also have a map

$$\mathrm{Asai}_{F/\mathbf{Q}}: {}^LG\to{}^L\mathrm{GL}_4$$

which on pure tensors is defined by

$$\mathrm{Asai}_{F/\mathbf{Q}}(g_1,g_2,\gamma)(v_1\otimes v_2) := \begin{cases} g_1\cdot v_1\otimes g_2\cdot v_2 & \text{if } \gamma=1 \\ g_1\cdot v_2\otimes g_2\cdot v_1 & \text{if } \gamma=\tau. \end{cases}$$

If π is a tempered automorphic representation of $G(\mathbf{A_Q})$, unramified outside a finite set S of primes, the base change lift of a tempered automorphic representation σ of $H(\mathbf{A_Q})$ (we can choose S so that its Satake parameters lie in the image of $\mathrm{BC}_{F/\mathbf{Q}}$ at every prime not in S), the factorization of $\mathrm{Asai}_{F/\mathbf{Q}}\circ\mathrm{BC}_{F/\mathbf{Q}}$ into irreducible ${}^LH$-factors induces

$$L^S(s,\pi,\mathrm{Asai}_{F/\mathbf{Q}}) = L^S(s,\sigma,\mathrm{Sym}^2)L^S(s,\varepsilon_{F/\mathbf{Q}}),$$

with $\varepsilon_{F/\mathbf{Q}}$ the Hecke character associated to the quadratic extension $F/\mathbf{Q}$ by class field theory. The twisted L-function $L^S(s,\pi,\mathrm{Asai}_{F/\mathbf{Q}}\otimes\varepsilon_{F/\mathbf{Q}})$ has thus a pole at $s=1$. The existence of a pole at $s=1$ for $L^S(s,\pi,\mathrm{Asai}_{F/\mathbf{Q}}\otimes\varepsilon_{F/\mathbf{Q}})$ is equivalent to asking that π is in the image of the base change lift. Moreover, these facts are related to the non-vanishing of the Hirzebruch–Zagier cycles for π. Hirzebruch–Zagier cycles and their generating series can be viewed as the prototype for Kudla program and its p-adic counterpart (*cf.* [HZ76], [GG12], [CNR24]). The relevant cases of Beilinson–Tate conjectures pertaining special values of the Asai L-functions have been studied by Kings and Ramakrishnan (*cf.* [Kin98], [Ram87]).

1.2. **An exceptional setting:** GSp_6 **and** G_2**.**

1.2.1. *The group* G_2. Let $\mathbb{H}$ be the algebra of Hamilton quaternions over $\mathbf{Q}$ with the usual basis $\{1, i, j, k\}$. The conjugate $\bar{a}$ of an element $a = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{H}$ is $\bar{a} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$. The split octonion algebra over $\mathbf{Q}$ is $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ with multiplication

$$(a, b) \cdot (c, d) = (ac + d\bar{b}, \bar{a}d + cb).$$

Then $\mathbb{O}$ is a non-commutative, non-associative, alternative $\mathbf{Q}$-algebra. If $x = (a, b)$, let $\overline{x} = (\bar{a}, -b)$. Then $x \mapsto \overline{x}$ is a $\mathbf{Q}$-linear involution on $\mathbb{O}$ satisfying $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$. The norm $\mathrm{N} : \mathbb{O} \to \mathbf{Q}$, $x \mapsto x \cdot \overline{x}$, defines a quadratic form on $\mathbb{O}$. Define

$$G_2 := \{g \in \mathrm{GL}(\mathbb{O}) \mid g(x \cdot y) = (gx) \cdot (gy), \forall x, y \in \mathbb{O}\}.$$

G_2 acts transitively on the set $\mathbb{O}^0$ of trace zero octonions and preserves the quadratic form induced by N. Thus we have an embedding

$$\mathrm{Std} : G_2 \hookrightarrow \mathrm{SO}_7 \to \mathrm{GL}_7,$$

which we denote by the the standard representation of G_2. The dual group ${}^L G_2 = G_2(\mathbf{C})$.

1.2.2. *A key diagram.* Let $J := \begin{pmatrix} & I_3 \\ -I_3 & \end{pmatrix}$ and define $\mathrm{GSp}_6 := \{(g, \nu(g)) \in \mathrm{GL}_6 \times \mathrm{GL}_1 : {}^t g J g = \nu(g) J\}$ and $\mathrm{PGSp}_6 = \mathrm{GSp}_6 / Z_{\mathrm{GSp}_6}$. The dual group of PGSp_6 is $\mathrm{Spin}_7(\mathbf{C})$ and we let $\mathrm{Spin} : \mathrm{Spin}_7(\mathbf{C}) \to \mathrm{SO}_8(\mathbf{C}) \to \mathrm{GL}_8(\mathbf{C})$ be the 8 dimensional Spin representation. Despite G_2 having discrete series at ∞, the locally symmetric space associated to G_2 does not have any complex structure. It is an observation of Gross and Savin [GS98] that the **standard** motive of cuspidal automorphic representations of $G_2(\mathbf{A_Q})$ should be realised inside the cohomology of the Siegel sixfold (i.e. the Shimura variety associated to GSp_6). This is motivated by the following discussion. The étale cohomology of the Siegel sixfold realizes the **Spin** motive of cuspidal automorphic representations of $\mathrm{PGSp}_6(\mathbf{A_Q})$ (*cf.* [KS22]). Now note that the stabilizer of a generic vector in Spin is isomorphic to $G_2(\mathbf{C})$, giving an embedding

$$\xi : G_2(\mathbf{C}) \hookrightarrow \mathrm{Spin}_7(\mathbf{C}).$$

Indeed, one can see that $G_2(\mathbf{C}) = \mathrm{Spin}_7(\mathbf{C}) \cap \mathrm{SO}_7(\mathbf{C})$ and we have the following commutative diagram

$$\begin{tikzcd}
G_2(\mathbf{C}) \arrow[r, hook] \arrow[d, hook, "\xi"'] \arrow[rr, bend left, "\mathrm{Std}"] & \mathrm{SO}_7(\mathbf{C}) \arrow[r, hook] \arrow[d, hook] & \mathrm{GL}_7(\mathbf{C}) \arrow[d, hook] \\
\mathrm{Spin}_7(\mathbf{C}) \arrow[r, hook] \arrow[rr, bend right, "\mathrm{Spin}"'] & \mathrm{SO}_8(\mathbf{C}) \arrow[r, hook] & \mathrm{GL}_8(\mathbf{C}).
\end{tikzcd} \tag{1}$$

Then we have the decomposition

$$\mathrm{Spin}|_{\xi(G_2(\mathbf{C}))} \simeq \mathrm{Std} \oplus \mathbf{C}.$$

If we let π be a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A_Q})$, which is a weak functorial lift (via ξ) of a cuspidal automorphic representation σ of $G_2(\mathbf{A_Q})$, then for a good enough finite set of places S

$$L^S(s, \pi, \mathrm{Spin}) = L^S(s, \sigma, \mathrm{Std}) \zeta^S(s). \tag{2}$$

In general, by Langlands functoriality it is expected that if $L^S(s,\pi,\mathrm{Spin})$ has a simple pole at $s=1$, then π is a functorial lift from G_2 or its compact form. Combining results from [KS22], [Che19], [GS20], [GG09], and [PS18], one can show the following.

Proposition 1.2 ([CLJ22, Proposition 8.1])**.** *Suppose that π is a cohomological cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A_Q})$ which is Steinberg at a finite place. Then π is tempered and the following statements are equivalent:*

(1) *The partial L-function $L^S(s,\pi,\mathrm{Spin})$ has a simple pole at $s=1$,*
(2) *For almost all ℓ, the Satake parameters of π at ℓ lie in $\xi(G_2(\mathbf{C}))$,*
(3) *There exists a cuspidal automorphic representation σ of either G_2 or its form compact at ∞ such that π is a weak functorial lift of σ.*

Moreover, if π supports a Fourier coefficient of rank 2 associated to a quadratic étale $\mathbf{Q}$-algebra F these conditions are equivalent to

(4) *π is* H*-distinguished, with* $\mathrm{H}=\{(g_1,g_2)\in \mathrm{GL}_2\times \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F}\ :\ \det(g_1)=\det(g_2)\}$, *i.e. that there exists a cusp form Ψ in π such that*

$$\mathcal{P}_{\mathrm{H}}(\Psi):=\int_{Z_{\mathrm{GSp}_6}(\mathbf{A})\mathrm{H}(\mathbf{Q})\backslash \mathrm{H}(\mathbf{A})}\Psi(h)dh\neq 0.$$

If one of the first three conditions hold, the residue at $s=1$ of the partial L-function $L^S(s,\pi,\mathrm{Spin})$ is given by

$$\mathrm{Res}_{s=1}L^S(s,\pi,\mathrm{Spin})=L^S(1,\sigma,\mathrm{Std})\prod_{\ell\in S}(1-\ell^{-1}).$$

In (4), H is embedded into GSp_6 as follows. Let $V=V_1\oplus V_2$ be the standard representation of H, with $V_1=\mathbf{Q}e_1\oplus\mathbf{Q}f_1$, resp. $V_2=Fe_2\oplus Ff_2$, the standard representation of GL_2, resp. $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F}$. On V we have a $\mathbf{Q}\times F$-values alternating form $\psi:V\times V\to\mathbf{Q}\times F$ which sends $\psi(e_1,f_1)=(1,0)$ and $\psi(e_2,f_2)=(0,1/2)$. If we regard V as a six dimensional $\mathbf{Q}$-vector space with symplectic form $\psi':=\mathrm{Tr}_{\mathbf{Q}\times F/\mathbf{Q}}\circ\psi:V\times V\to\mathbf{Q}$,

$$\psi:(ae_1+\alpha e_2,bf_1+\beta f_2)\mapsto ab+\frac{1}{2}\mathrm{Tr}_{F/\mathbf{Q}}(\alpha\beta),$$

then we have an embedding $\mathrm{H}\hookrightarrow\mathrm{GSp}(V,\psi')$. The latter can be made isomorphic to GSp_6 after changing the symplectic $\mathbf{Q}$-basis of V. For instance, if $F=\mathbf{Q}(\delta)$ the choice of $\mathbf{Q}$-basis $\{e_1',e_2',e_3',f_1',f_2',f_3'\}:=\{e_1,e_2,\delta e_2,f_1,f_2,\delta^{-1}f_2\}$ of V identifies (V,ψ') with (V,J).

1.2.3. *The integral representation of the Spin L-function by Pollack–Shah.* Pollack–Shah in [PS18], building up on work of Gan–Gurevich in [GG09], give an integral representation of the Spin L-function for cuspidal automorphic representations of PGSp_6 which support Fourier coeffients of rank 2. The results of [BGCLRJ23] and [CLJ22] concerning the Beilinson–Tate conjectures for Siegel sixfolds rely on the arithmetic incarnation of the analytic construction of [PS18]. We now recall their result (which we have implicitly used in Proposition 1.2).

Let $\mathcal{S}(\mathbf{A}^2_\mathbf{Q})$ denote the space of Schwartz-Bruhat functions on $\mathbf{A}^2_\mathbf{Q}$. Let B_2 denote the upper triangular Borel of GL_2 with modulus character $\delta_{\mathrm{B}_2}(\mathrm{diag}(t_1,t_2))=|t_1/t_2|$. Given $\Phi\in\mathcal{S}(\mathbf{A}^2_\mathbf{Q})$, denote by

$$f(g,\Phi,s):=|\det(g)|^s\int_{\mathrm{GL}_1(\mathbf{A_Q})}\Phi((0,t)g)|t|^{2s}d^\times t$$

the normalized Siegel section in $\mathrm{Ind}_{\mathrm{B}_2(\mathbf{A_Q})}^{\mathrm{GL}_2(\mathbf{A_Q})}(\delta_{\mathrm{B}_2}^s)$ and define the associated real analytic Eisenstein series

$$E(g,\Phi,s) := \sum_{\gamma\in \mathrm{B}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{Q})} f(\gamma g,\Phi,s). \tag{3}$$

It has meromorphic continuation to the whole complex plane with at most a simple pole at $s=1$ with constant residue.

Let F be a quadratic étale $\mathbf{Q}$-algebra and let $\mathrm{H} = \{(g_1,g_2) \in \mathrm{GL}_2 \times \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F} : \det(g_1) = \det(g_2)\}$, which embeds into GSp_6 as above. If π is a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A_Q})$, for any factorizable cusp form φ in the space of π and factorizable $\Phi \in \mathcal{S}(\mathbf{A}_\mathbf{Q}^2)$ consider

$$\mathcal{I}(\Phi,\varphi,s) = \int_{Z_{\mathrm{GSp}_6}(\mathbf{A})\mathrm{H}(\mathbf{Q})\backslash \mathrm{H}(\mathbf{A_Q})} E(h_1,\Phi,s)\varphi(h)dh.$$

We take Φ,φ to be factorizable.

Theorem 1.3 ([PS18]). *Let S denote a finite set of places containing ∞ and the ramified places for π. If π supports a Fourier coefficient of rank 2 attached to F, then*

$$\mathcal{I}(\Phi,\varphi,s) = \mathcal{I}_S(\Phi_S,\varphi_S,s)L^S(s,\pi,\mathrm{Spin}),$$

with $\mathcal{I}_S(\Phi_S,\varphi_S,s)$ the integral over the places in S.

Immediately from the theorem, taking the residue at $s=1$ of the equality of Theorem 1.3 yields

$$\frac{\widehat{\Phi}(0)\cdot \mathcal{P}_{\mathrm{H}}(\varphi)}{2} = \mathrm{Res}_{s=1}\left(\mathcal{I}_S(\Phi_S,\varphi_S,s)L^S(s,\pi,\mathrm{Spin})\right),$$

where we have denoted by $\widehat{\Phi}$ the Fourier transform of Φ and we have used that

$$\mathrm{Res}_{s=1}E(g,\Phi,s) = \frac{\widehat{\Phi}(0)}{2}.$$

2. Beilinson–Tate conjectures and main result

2.1. **The analytic class number formula.** Let K be a number field with ring of integers $\mathcal{O}_K$ and degree $[K:\mathbf{Q}] = r_1 + 2r_2$, where r_1 denotes the number of real embeddings of K and $2r_2$ is the number of complex embeddings of K. The Dedekind zeta function of K is the Euler product

$$\zeta_K(s) = \sum_{I\subseteq \mathcal{O}_K} \frac{1}{N_{K/\mathbf{Q}}(I)^s} = \prod_{\mathfrak{p}\subseteq \mathcal{O}_K}(1-N_{K/\mathbf{Q}}(\mathfrak{p})^{-s})^{-1}.$$

It converges for $\mathrm{Re}(s)>1$ and admits meromorphic continuation to $\mathbf{C}$ with a simple pole at $s=1$. Note that

$$(K\otimes_\mathbf{Q}\mathbf{R})^\times \simeq \prod_{\text{real } v|\infty}\mathbf{R}^\times \times \prod_{\text{complex } v|\infty}\mathbf{C}^\times = (\mathbf{R}^\times)^{r_1}\times(\mathbf{C}^\times)^{r_2}$$

and we can define the regulator map $r : (K\otimes_\mathbf{Q}\mathbf{R})^\times \to \mathbf{R}^{r_1+r_2}$, $x \mapsto (\log|x_v|_v)_v$. If $\mathrm{pr} : \mathbf{R}^{r_1+r_2} \to \mathbf{R}^{r_1+r_2-1}$ is any of the natural projections, the regulator of K is the covolume of $(\mathrm{pr}\circ r)(\mathcal{O}_K^\times)$ in $\mathbf{R}^{r_1+r_2-1}$.

Theorem 2.1 (Analytic class number formula)**.**

$$\lim_{s\mapsto 1}\zeta_K(s)(s-1)=\frac{2^{r_1}(2\pi)^{r_2}h_KR_K}{|\mathcal{O}^\times_{K,\mathrm{tors}}|\sqrt{|\Delta_K|}},$$

where h_K is the class number of K, $|\mathcal{O}^\times_{K,\mathrm{tors}}|$ is the number of roots unity in K, Δ_K is the discriminant of K, and R_K is the regulator of K.

In the 70's, the analytic class number formula was extended to other special values of ζ_K; precisely Lichtenbaum noticed that the order of vanishing of $\zeta_K(s)$ at $s=1-m$, for $m>1$, is equal to the dimension d_m of the higher K-group $K_{2m-1}(K)\otimes\mathbf{Q}$. Borel then proved a formula of the leading term of $\zeta_K(s)$ at $s=1-m$ in terms of the covolume of the image of a higher regulator map $r_m: K_{2m-1}(K)\to\mathbf{R}^{d_m}$. In the 80's, Bloch defined a regulator map

$$K_2(E)\to H^1(E(\mathbf{C}),\mathbf{R}),$$

for the second K-group of a complex elliptic curve $E/\mathbf{C}$, and showed a similar formula relating the image of r with the leading term at $s=0$ of the Hasse-Weil L-function of elliptic curves with complex multiplication.

2.2. **The conjecture.** Beilinson conjectures generalize these class number formulas and propose the right framework in which they fit in. We refer to [Nek94], [Sch88] for excellent surveys on these conjectures. In what follows, we restrict to the setting of the conjecture where the underlying pure motive (which we assume to be of automorphic nature) has weight -2 (*cf.* [Nek94, (6.3)]). In this case one has to take into account the regulator from motivic cohomology as well as the contribution of algebraic cycles modulo homological equivalence (and this is why we refer it to Beilinson–Tate conjectures).

Let G be a reductive group over $\mathbf{Q}$, which admits a Shimura datum (G,X_G) and hence has a Shimura variety. For a neat open compact subgroup $K\subseteq G(\mathbf{A}_f)$, we denote by $Y_G(K)$ the attached Shimura variety of level K, which is a smooth, quasi-projective complex variety of dimension d_G. **We assume that d_G is even.** Also assume for simplicity that $Y_G(K)$ has a canonical model, which we still denote $Y_G(K)$, over the rationals $\mathbf{Q}$. Fix an element $h:\mathbb{S}:=\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathrm{GL}_1\to G_{/\mathbf{R}}$ of X_G. Over $\mathbf{C}$, $\mathbb{S}_{/\mathbf{C}}\simeq\mathrm{GL}_1\times\mathrm{GL}_1$ and precomposing h with $\mathrm{GL}_1\to\mathrm{GL}_1\times\mathrm{GL}_1$, $x\mapsto(x,1)$, yields $\mu_h:\mathrm{GL}_{1/\mathbf{C}}\to G_{/\mathbf{C}}$. Up to conjugacy, we can assume that the image of μ_h lands in a maximal torus T of G defined over $\mathbf{Q}$. By our assumption, the conjugacy class of μ_h is defined over $\mathbf{Q}$ and we can further assume that the image of μ_h is contained in a maximal split torus of G. By duality, μ_h defines a character $\widehat{\mu}_h$ of the torus $\widehat{T}$ dual to T and, after choosing a Borel pair for $\widehat{G}$, a dominant weight for $\widehat{G}$. We let $\rho_G:\widehat{G}\to\mathrm{GL}_{n_G}$ be the representation of highest weight $\widehat{\mu}_h$. This representation extends to a representation which we still call ρ_G of the Langlands dual ${}^LG$.

Example 2.2.

(GL_2) Recall that $\widehat{\mathrm{GL}}_2\simeq\mathrm{GL}_2(\mathbf{C})$; $h:\mathbb{S}\to\mathrm{GL}_2$ can be chosen to send $z=a+ib\in\mathbf{C}^\times$ to $\left(\begin{smallmatrix}a&b\\-b&a\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbf{R})$. After base-changing h to $\mathbf{C}$, it can be diagonalized to $(z,z')\mapsto\left(\begin{smallmatrix}z&\\&z'\end{smallmatrix}\right)$. Then $\widehat{\mu}_h:\left(\begin{smallmatrix}z&\\&z'\end{smallmatrix}\right)\mapsto z$ and ρ_{GL_2} is the two dimensional standard representation of $\mathrm{GL}_2(\mathbf{C})$.

($\mathrm{GL}_{2/F}$) Consider the group GL_2 over $\mathrm{Spec}(F)$, for F totally real number field. In this case, the reductive group over $\mathbf{Q}$ is nothing but $G=\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_{2,F})$; as $G(\mathbf{Q})=\mathrm{GL}_2(F)$ and

$$G(\mathbf{R})=\mathrm{GL}_2(F\otimes_{\mathbf{Q}}\mathbf{R})=\mathrm{GL}_2(\mathbf{R})^n,$$

with $n = [F : \mathbf{Q}]$, since $F \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R}^n$, $a \otimes r \mapsto (r\sigma_1(a), \cdots, r\sigma_n(a))$ with σ_i running through $\Sigma_F = \{v : F \hookrightarrow \mathbf{R}\}$. We can therefore define

$$h : \mathbb{S} \to G_{\mathbf{R}} \simeq \mathrm{GL}_{2,\mathbf{R}}^n,$$

by sending $z = a + ib \mapsto \left(\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right), \cdots, \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) \right)$. In this case, ρ_G coincides with the Asai representation $\mathrm{Asai}_{F/\mathbf{Q}}$.

(GSp_6) We can choose $h : \mathbb{S} \to \mathrm{GSp}_6$ so that $z = a + ib \in \mathbf{C}^\times$ to $\left(\begin{smallmatrix} aI_3 & bI_3 \\ -bI_3 & aI_3 \end{smallmatrix} \right) \in \mathrm{GSp}_6(\mathbf{R})$. After diagonalizing it, we can assume $\mu_h : z \mapsto \left(\begin{smallmatrix} I_3 & \\ & zI_3 \end{smallmatrix} \right) \in T_{\mathrm{GSp}_6}(\mathbf{C})$, with T_{GSp_6} being the diagonal maximal torus of GSp_6. The associated character of $\widehat{T}_{\mathrm{GSp}_6}$ is the highest weight of the 8-dimensional representation Spin.

Roughly speaking, the (global) Langlands conjecture predicts that the L-function of the Hecke isotypic components of the cohomology of Y_G should be associated to the representation ρ_G of ${}^LG$ of highest weight $\widehat{\mu}_h$. Let $\pi = \pi_\infty \otimes \pi_f$ be a non-endoscopic tempered cohomological cuspidal automorphic representation of $G(\mathbf{A}_{\mathbf{Q}})$. To π_f one hopes to attach a Chow motive $M(\pi_f)$ pure of weight d_G with coefficients in a big enough number field L_π such that

$$L^S(s, M(\pi_f)(\frac{d_G}{2})^1) = L^S(s, \pi, \rho_G),$$

for S a finite set of places containing ∞ and the ramified primes for π. It should be realized in the interior cohomology of the Shimura variety Y_G, i.e. for $j \in \mathbf{Z}$

$$M(\pi_f)(j) \otimes \pi_f = H_!^{d_G}(Y_G, L_\pi(j))[\pi_f].$$

Denote $j_G := \frac{d_G}{2}$; let $H^1_{\mathcal{M}}(M(\pi_f)(j_G + 1))$ denote the first motivic cohomology group of $M(\pi_f)(j_G+1)$ and let $N(M(\pi_f)(j_G))$ denote the group of algebraic cycles in $M(\pi_f)(j_G)$ up to homological equivalence. Let $H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(j_G + 1))$ be the first Deligne–Beilinson cohomology group of $M(\pi_f)(j_G+1)$; the Betti cycle class cl_B induce a map $r_B : N(M(\pi_f)(j_G)) \to H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(j_G + 1))$ (*cf.* [Nek94, §6.2]); this and Beilinson's regulator $r_{\mathcal{D}}$ induce

$$r = (r_{\mathcal{D}}, r_B) : H^1_{\mathcal{M}}(M(\pi_f)(j_G + 1)) \oplus N(M(\pi_f)(j_G)) \to H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(j_G + 1)).$$

Conjecture 2.3. *(Beilinson–Tate)*

(1) *The map r induces an isomorphism*

$$(H^1_{\mathcal{M}}(M(\pi_f)(j_G + 1)) \oplus N(M(\pi_f)(j_G))) \otimes_{\mathbf{Q}} \mathbf{R} \to H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(j_G + 1)),$$

(2) $\mathrm{ord}_{s=0} L^S(s, \pi, \rho_G) = \dim_{L_\pi} H^1_{\mathcal{M}}(M(\pi_f)(j_G + 1))$,

(3) $-\mathrm{ord}_{s=1} L^S(s, \pi, \rho_G) = \dim_{L_\pi} N(M(\pi_f)(j_G))$,

(4) $\det(\mathrm{Im}\, r) = L^{S,*}(1, M(\pi_f)(j_G))\mathcal{D}(M(\pi_f)(j_G+1))$, *where* $\mathcal{D}(M(\pi_f)(j_G+1))$ *denotes the Deligne L_π-structure of* $\det(H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(j_G + 1))$.

Very few cases of these conjectures (not only in weight -2) are known [Beĭ85], [Beĭ86], [Den89], [Den90], [Ram86], [BC16], [Kin98], [Lem17], and they remain one of the main open problems in arithmetic geometry.

We conclude this section with a speculation. Suppose that we are interested in the arithmetic of L-functions attached to a reductive group G and representation ρ' of ${}^LG$ for which

[1] The twist $M(\pi_f)(\frac{d_G}{2})$ has weight 0 and both L-functions are centred at $s = 1/2$.

either G does not admit a Shimura datum or $\rho' \neq \rho_G$. Is there hope to construct automorphic Chow motives associated to cohomological cuspidal automorphic representations π of G and ρ'? If so, how to approach Beilinson's conjecture in those cases? One possible strategy:

(1) Construct the motive as a piece of the interior cohomology of an auxiliary Shimura variety for a group G' by means of lifting from G to G' (e.g. theta or endoscopic lifting);
(2) Seek for Rankin-Selberg integrals for automorphic representations of G' which naturally detect the lifting from G and calculate the L-function associated to ρ;
(3) Construct the motivic counterpart of the Rankin-Selberg integral of (2).

As briefly commented in §1.2.2, an instance of (1) has been proposed by [GS98] and partially answered by [KS22] for G_2: G_2 does not admit a Shimura variety but its "standard motive" appears as a piece of the cohomology of Y_{GSp_6} by means of an exceptional theta correspondence. In this case, the right Rankin–Selberg integral appears in the literature, but (3) seems out of reach as it is not known at the moment if there exist motivic incarnations of Siegel Eisenstein series for GSp_4. We can similarly analyze the settings described in §1.1 with (G, ρ') equal to either $(\mathrm{GSp}_4, \rho_{1,1} \otimes \varepsilon_{E/\mathbf{Q}})^2$ (in which case $G' = \mathrm{GU}(2,2)$), with $E/\mathbf{Q}$ an auxiliary imaginary quadratic extension and $\varepsilon_{E/\mathbf{Q}}$ the Hecke character associated to it by class field theory, or $(\mathrm{GL}_2, \mathrm{Sym}^2)$ (in which case $G' = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$ with $F/\mathbf{Q}$ real quadratic extension). The "right" Rankin–Selberg integrals appear in the literature (*cf.* [CT24]) and involve again Siegel Eisenstein series for GSp_4.

2.3. **Our case of interest: main results.** Let $\pi = \pi_\infty \otimes \pi_f$ be a cohomological (for trivial coefficients) cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbf{A}_\mathbf{Q})$ which is unramified outside a finite set S of places which we assume to contain ∞. Suppose that at a finite place p the component π_p is the Steinberg representation of $\mathrm{PGSp}_6(\mathbf{Q}_p)$. Then, by results of [KS22], π is tempered with π_∞ in the discrete series L-packet for the trivial representation. Moreover, for a big enough number field L_π we have (*cf.* [CLJ22, §2.8])

$$H^\bullet_B(Y_{\mathrm{GSp}_6}(U), L_\pi(3))[\pi_f] \simeq H^6_{B,!}(Y_{\mathrm{GSp}_6}(U), L_\pi(3))[\pi_f] \simeq M_B(\pi_f)(3) \otimes \pi_f^U,$$

with $M_B(\pi_f)(3)$ a pure Hodge structure of weight 0 such that, if $h^{p,q} = \dim_\mathbf{C}(M_B(\pi_f)(3))^{p,q}$,

$$h^{p,q} = \begin{cases} 1 & \text{if } p \neq 0, \\ 2 & \text{if } p = 0. \end{cases}$$

In particular, we have $\dim_{L_\pi} M_B(\pi_f)(3) = 8$ as expected. Furthermore Kret–Shin [KS22] have shown that

$$L^S(s, M(\pi_f)(3)) = L^S(s, \pi, \mathrm{Spin}).$$

Define $\mathbf{R}_\pi := L_\pi \otimes_\mathbf{Q} \mathbf{R}$. The following lemma calculates the dimension of the π_f-isotypic component of the seventh degree Deligne–Beilinson cohomology group of $Y_{\mathrm{GSp}_6}(K)$.

Lemma 2.4 ([CLJ22, Lemma 2.11]). *We have*

$$\dim_{\mathbf{R}_\pi} H^7_\mathcal{D}(Y_{\mathrm{GSp}_6}(U)/\mathbf{R}, \mathbf{R}_\pi(4))[\pi_f] = \dim_\mathbf{C} \pi_F^U.$$

---

[2] This is a quasi-split version of the setting of §1.1.1, in which if π is a generic cuspidal automorphic representation of $\mathrm{PGU}(2,2)$ in the image of the theta correspondence of a cuspidal automorphic representation σ of PGSp_4 we have a decomposition

$$L^S(s, \pi, \wedge^2) = L^S(s, \pi, \rho_{1,1} \otimes \varepsilon_{E/\mathbf{Q}}) \zeta^S(s).$$

If we write
$$H^7_{\mathcal{D}}(Y_{\mathrm{GSp}_6}(U)/\mathbf{R}, \mathbf{R}_\pi(4))[\pi_f] = H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(4)) \otimes \pi_f^U,$$
then the Lemma implies that $H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(4))$ is one dimensional. Conjecture 2.3 therefore suggests the following:

Expectation 2.5.

(1) *If $L^S(s, \pi, \mathrm{Spin})$ is holomorphic at $s = 1$, there should exist a motivic cohomology class $\mathscr{Z}_{\mathcal{M}} \in H^7_{\mathcal{M}}(Y_{\mathrm{GSp}_6}(U), L_\pi(4))$ such that the projection $r_{\mathcal{D}}(\mathscr{Z}_{\mathcal{M}})(\pi_f)$ to the π_f-isotypic component of its Deligne–Beilinson realization and its Hecke translates generate $H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(4)) \otimes \pi_f^U$.*

(2) *If $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$, there should exist a codimension 3 cycle (modulo homological equivalence) $\mathcal{Z}_{\mathrm{hom}} \in \mathrm{N}^3(Y_{\mathrm{GSp}_6}(U))_{L_\pi}$ such that the projection $r_B(\mathcal{Z}_{\mathrm{hom}})(\pi_f)$ to the π_f-isotypic component of its realization via r_B and its Hecke translates generate $H^1_{\mathcal{D}}(M(\pi_f)_{\mathbf{R}}(4)) \otimes \pi_f^U$.*

In [BGCLRJ23] and [CLJ22], we construct candidates for $\mathscr{Z}_{\mathcal{M}}, \mathcal{Z}_{\mathrm{hom}}$. Before stating our result, we introduce some notation. We fix a sufficiently small open compact $U \subseteq \mathrm{GSp}_6(\mathbf{A}_f)$ such that $\pi_f^U \neq 0$. Let $\pi_\infty^{3,3}$ be the discrete series of $\mathrm{PGSp}_6(\mathbf{R})$ of Hodge type $(3,3)$ and let φ_∞ be the highest weight vector of the minimal K-type of $\pi_\infty^{3,3}$. For a cusp form $\varphi = \varphi_\infty \otimes \varphi_f$, with $\varphi_f \in \pi_f^U$, we have a harmonic differential form ω_φ of Hodge type $(3,3)$. The restriction of $\pi_\infty^{3,3}$ to $\mathrm{Sp}_6(\mathbf{R})$ factors as the direct sum $\pi_{\infty,1}^{3,3} \oplus \overline{\pi}_{\infty,1}^{3,3}$ of the discrete series for $\mathrm{Sp}_6(\mathbf{R})$ of minimal $K \simeq U(3)$-types $\tau_{(2,2,-4)}$ and $\tau_{(4,-2,-2)}$. Each of these representations have a one dimensional weight $(0,0,0)$-eigenspace and we let φ_∞^0 be a vector in the minimal K-type of $\pi_\infty^{3,3}$ whose projection to each of the two factors generates the weight $(0,0,0)$-eigenspace. We will let $\varphi^0 := \varphi_\infty^0 \otimes \varphi_f$. Fix the Schwartz-Bruhat function Φ_∞ on $\mathbf{R}^2$ defined by $(x,y) \mapsto e^{-\pi(x^2+y^2)}$ and, for each L_π-valued Schwartz–Bruhat function $\Phi_f \in \mathcal{S}(\mathbf{A}_f^2, L_\pi)$, we let $\Phi = \Phi_\infty \otimes \Phi_f$. Finally recall that we have introduced the integral representation $\mathcal{I}(\Phi, \varphi, s)$ of the Spin L-function of GSp_6 by Pollack–Shah.

Theorem 2.6 ([BGCLRJ23, Theorem 1.5], [CLJ22, Theorem 1.2])**.**

(1) *If $L^S(s, \pi, \mathrm{Spin})$ is holomorphic at $s = 1$, for suitable $\Phi_f \in \mathcal{S}(\mathbf{A}_f^2, L_\pi)$, there exist $\mathscr{Z}_{\mathcal{M}}^{\Phi_f} \in H^7_{\mathcal{M}}(Y_{\mathrm{GSp}_6}(U), L_\pi(4))$ and a natural map*
$$\langle\ , \omega_\varphi\rangle : H^7_{\mathcal{D}}(Y_{\mathrm{GSp}_6}(U)/\mathbf{R}, \mathbf{R}_\pi(4)) \to \mathbf{C} \otimes_{\mathbf{Q}} L_\pi$$
such that
$$\langle r_{\mathcal{D}}(\mathscr{Z}_{\mathcal{M}}^{\Phi_f})(\pi_f), \omega_\varphi\rangle = C \lim_{s \to 0} \left(\mathcal{I}_S(\Phi_S, \varphi_S^0, s) L^S(s, \pi, \mathrm{Spin})\right),$$
with C an explicit volume factor independent of π.

(2) *Suppose $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$, so that π is a weak functorial lift of a cuspidal automorphic representation σ of $G_2(\mathbf{A}_{\mathbf{Q}})$. Then there exists $\mathcal{Z}_{\mathrm{hom}} \in \mathrm{N}^3(\mathrm{Sh}_{\mathrm{G}}(U))_{L_\pi}$ such that the value of the Poincaré pairing*
$$\begin{aligned}\langle r_B(\mathcal{Z}_{\mathrm{hom}})(\pi_f), \omega_\varphi\rangle &= C' \mathrm{Res}_{s=1} \left(\mathcal{I}_S(\Phi'_S, \varphi_S^0, s) L^S(s, \pi, \mathrm{Spin})\right) \\ &= C' L^S(1, \sigma, \mathrm{Std}) \mathrm{Res}_{s=1} \left(\mathcal{I}_S(\Phi'_S, \varphi_S^0, s) \zeta^S(s)\right),\end{aligned}$$
with C' an explicit non-zero constant independent of π and Φ' a suitable Schwartz–Bruhat function on $\mathbf{A}_{\mathbf{Q}}^2$.

Immediately from the Theorem (plus arguing as in [BGCLRJ23, Corollary 5.23] if in case (1) of Theorem 2.6) yields the following.

Corollary 2.7. *Let φ^0, Φ, Φ' be as in Theorem 2.6. If $\mathcal{I}_S(\Psi_S, \varphi_S^0, 1) \neq 0$ for Ψ equal to Φ' and to the Fourier transform $\widehat{\Phi}$ then Expectation 2.5 holds.*

Understanding the behaviour of the ramified integral $\mathcal{I}_S(\Psi_S, \varphi_S^0, s)$ at $s = 1$ is at the heart of the conjecture and calculating it would strenghten notably our results. At present, this seems however out of reach (at least for us in this generality). The main crux is that the space of Fourier coefficients that the integral $\mathcal{I}(\Psi, \varphi^0, s)$ unfolds to is not finite dimensional in general.

We conclude with a remark on the arithmetic applications of our construction.

(1) In [CRJ20], we showed how the étale realization of $\mathscr{Z}_{\mathcal{M}}^{\Phi_f}$ could be assembled into a norm-compatible tower at p of cohomology classes, giving rise to an element of the Iwasawa cohomology of the local p-adic Spin Galois representation associated with π_f. We expect these classes to form an Euler system for the Spin Galois representation.

(2) If $L^S(s, \pi, \mathrm{Spin})$ has a pole at $s = 1$, Conjecture 2.3 (3) would imply the existence of a Galois invariant vector in the p-adic Spin Galois representation attached to π_f. Inspired by the diagram (1), if σ is a cuspidal automorphic representation of G_2 or its compact form lifting to π, Gross and Savin [GS98] conjectured the existence of the rank 7 "standard" motive $M(\sigma_f)$ attached to σ and the decomposition of the motive $M(\pi_f)(3)$ as the direct sum of $M(\sigma_f)$ and the rank 1 trivial motive generated by the class given in Conjecture 2.3. Moreover, inspired by local calculations, they conjectured that this class should arise from a Hilbert modular threefold. Thanks to the work [KS22], we have a decomposition of Galois representations

$$M_p(\pi_f) \otimes \pi_f^U \simeq [M_p(\sigma_f) \oplus \overline{\mathbf{Q}}_p] \otimes \pi_f^U.$$

Under the hypotheses of Corollary 2.7, in [CLJ22], we show that the étale realization of $\mathcal{Z}_{\mathrm{hom}}$ generates the trivial sub-representation $\overline{\mathbf{Q}}_p$ of $M_p(\sigma_f)$, confirming the conjecture of Gross–Savin at the cost of assuming that the ramified integral is non-zero at $s = 1$.

3. Elements of the proofs and crucial technical innovation

3.1. **The construction of the motivic classes and algebraic cycles.** The idea is to interpret geometrically Pollack–Shah's integral $\mathcal{I}(\Phi, \varphi, s)$ and its residue at $s = 1$. The construction of our motivic class therefore relies on the motivic incarnations of real analytic Eisenstein series on GL_2.

3.1.1. *Modular units.* The input of our construction are the modular units already considered by Beilinson and Kato, which are related to real analytic Eisenstein series by the second Kronecker limit formula which we now recall. Fix the Schwartz-Bruhat function Φ_∞ on $\mathbf{R}^2$ defined by $(x, y) \mapsto e^{-\pi(x^2+y^2)}$ and, for each $\overline{\mathbf{Q}}$-valued function $\Phi_f \in \mathcal{S}(\mathbf{A}_f^2, \overline{\mathbf{Q}})$, the smallest positive integer N_{Φ_f} such that Φ_f is constant modulo $N_{\Phi_f}\widehat{\mathbf{Z}}^2$. Finally, denote $\mathcal{S}_0(\mathbf{A}_f^2, \overline{\mathbf{Q}}) \subset \mathcal{S}(\mathbf{A}_f^2, \overline{\mathbf{Q}})$ the space of elements Φ_f such that $\Phi_f((0,0)) = 0$. The second Kronecker limit formula says the following.

Proposition 3.1. *Let $\Phi_f \in \mathcal{S}_0(\mathbf{A}_f^2, \overline{\mathbf{Q}})$ with $N_{\Phi_f} \geq 3$, then there exists*

$$u(\Phi_f) \in \mathcal{O}(Y_{\mathrm{GL}_2}(U(N_{\Phi_f})))^\times \otimes \overline{\mathbf{Q}} = H^1_{\mathcal{M}}(Y_{\mathrm{GL}_2}(U(N_{\Phi_f})), \overline{\mathbf{Q}}(1))$$

such that for any $g \in \mathrm{GL}_2(\mathbf{A})$ we have

$$\lim_{s\to 0} E(g,\Phi,s) = \log|u(\Phi_f)(g)|,$$

where $\Phi = \Phi_\infty \otimes \Phi_f$ and $E(g,\Phi,s)$ is the Eisenstein series defined in (3).

Here, $U(N_{\Phi_f})$ is the kernel of reduction modulo $N_{\Phi_f}\widehat{\mathbf{Z}}$. Note that when $\Phi_f = \mathrm{char}((0,1) + N\widehat{\mathbf{Z}}^2)$, the unit $u(\Phi_f)$ is a product of Siegel units $g_{0,\star/N}$ as in [Kat04, §1.4].

3.1.2. *Hilbert modular threefolds and the classes.* Let F be a quadratic étale $\mathbf{Q}$-algebra and let

$$\mathrm{H} = \{(g_1,g_2) \in \mathrm{GL}_2 \times \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_{2,F} \,:\, \det(g_1) = \det(g_2)\},$$

which embeds into GSp_6 as in §1.2.2. The associated Shimura variety Y_{H} is a quasi-projective threefold which splits as the product of a modular curve and a PEL-type Hilbert modular surface. The embedding $\mathrm{H} \hookrightarrow \mathrm{GSp}_6$ induces a morphism of Shimura varieties over $\mathbf{Q}$

$$\iota_U : Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f)) \to Y_{\mathrm{GSp}_6}(U).$$

We suppose that U is "nice" (e.g. satisfies the condition of [CRJ20, Lemma 2.1]), so that the morphism ι_U is a closed immersion of codimension 3. Let

$$\iota_{U,*} : H^1_{\mathcal{M}}(Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f)), \mathbf{Q}(1)) \longrightarrow H^7_{\mathcal{M}}(Y_{\mathrm{GSp}_6}(U), \mathbf{Q}(4))$$

be the corresponding pushforward map in cohomology. If we let

$$\mathrm{pr}_1 : Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f)) \twoheadrightarrow Y_{\mathrm{GL}_2}(U_1),$$

with $U \cap \mathrm{H}(\mathbf{A}_f) = U_1 \times_{\det} U_2$, be the map induced by the projection of H onto its GL_2-factor, and let $\Phi_f \in \mathcal{S}_0(\mathbf{A}_f^2, L_\pi)^{U_1}$, we can define

$$\mathscr{Z}_{\mathcal{M}}^{\Phi_f} := \iota_{U,*}(\mathrm{pr}_1^*(u(\Phi_f))) \in H^7_{\mathcal{M}}(Y_{\mathrm{GSp}_6}(U), L_\pi(4)).$$

The class $r_{\mathcal{D}}(\mathscr{Z}_{\mathcal{M}}^{\Phi_f})(\pi_f)$ is the projection of the π_f-isotypic component of the archimedean realization $r_{\mathcal{D}}(\mathscr{Z}_{\mathcal{M}}^{\Phi_f})$.

Similarly, let

$$\mathbf{1}_{Y_{\mathrm{H}}(U\cap \mathrm{H}(\mathbf{A}_f))} \in \mathrm{CH}^0(Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))) = H^0_{\mathcal{M}}(Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f)), \mathbf{Q}).$$

bet the fundamental class of $Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))$, i.e. $\sum_{\mathcal{C}}[\mathcal{C}] \in \mathrm{CH}^0(Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f)))$ with $\mathcal{C}$ running through the set of connected components of $Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))$. Then

$$\iota_{U,*}(\mathbf{1}_{Y_{\mathrm{H}}(U\cap \mathrm{H}(\mathbf{A}_f))}) \in \mathrm{CH}^3(Y_{\mathrm{GSp}_6}(U))$$

is the class associated to the codimension 3 cycle associated to $Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))$ in $Y_{\mathrm{GSp}_6}(U)$ and we define

$$\mathscr{Z}_{\mathrm{hom}} := f_{\mathrm{hom}}(\iota_{U,*}(\mathbf{1}_{Y_{\mathrm{H}}(U\cap \mathrm{H}(\mathbf{A}_f))})) \in \mathrm{N}^3(Y_{\mathrm{GSp}_6}(U))$$

to be its image under the natural map $f_{\mathrm{hom}} : \mathrm{CH}^3(Y_{\mathrm{GSp}_6}(U)) \to \mathrm{N}^3(Y_{\mathrm{GSp}_6}(U))$. The class $r_B(\mathscr{Z}_{\mathrm{hom}})(\pi_f)$ is then defined analogously to $r_{\mathcal{D}}(\mathscr{Z}_{\mathcal{M}}^{\Phi_f})(\pi_f)$.

3.2. **Strategy and a technical difficulty.** Once we have defined the candidates for the two motivic elements which should satisfy Expectation 2.5, we want to

(1) pair their archimedean realization with the the cohomology class attached to the harmonic differential form ω_φ,
(2) relate this to the adelic integral of Pollack–Shah.

In the case of $\mathcal{Z}_{\mathrm{hom}}$, Poincaré duality yields (cf. [CLJ22, Propositions 4.8,4.10])

$$\langle r_B(\mathcal{Z}_{\mathrm{hom}})(\pi_f), \omega_\varphi \rangle = \frac{1}{(2\pi i)^3} \int_{Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))} \iota_U^* \omega_c,$$

where ω_c is a compactly supported differential form such that $\omega_c = \omega_\varphi + d\eta$, with η a degree 5 rapidly decreasing differential form on $Y_{\mathrm{GSp}_6}(U)$. Since, by a result of Borel,

$$\int_{Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))} \iota_U^* d\eta = 0,$$

we get

$$\langle r_B(\mathcal{Z}_{\mathrm{hom}})(\pi_f), \omega_\varphi \rangle = \frac{1}{(2\pi i)^3} \int_{Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))} \iota_U^* \omega_\varphi.$$

It is easy then to see that the latter is equal (up to a non-zero constant) to the residue at $s = 1$ of the integral of Pollack–Shah (*cf.* [CLJ22, Proposition 5.10]). This, Theorem 1.3 and Proposition 1.2 prove Theorem 2.6(2).

What about for $\mathscr{Z}_{\mathcal{M}}^{\Phi_f}$? Here we encounter a technical difficulty. We want to define a map

$$\langle\ , \omega_\varphi \rangle : H^7_{\mathcal{D}}(Y_{\mathrm{GSp}_6}(U)/\mathbf{R}, \mathbf{R}_\pi(4)) \to \mathbf{C} \otimes_{\mathbf{Q}} L_\pi$$

such that

$$\langle r_{\mathcal{D}}(\mathscr{Z}_{\mathcal{M}}^{\Phi_f})(\pi_f), \omega_\varphi \rangle = \int_{Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))} \mathrm{pr}_1^* \log |u(\Phi_f)| \cdot \iota_U^* \omega_\varphi,$$

so that we can conclude the proof of Theorem 2.6(1) by using the second Kronecker limit formula. One way to do it is by expressing Deligne–Beilinson cohomology in terms of currents which can be naturally evaluated against ω_φ. By a result of Jannsen [Jan88], Deligne–Beilinson cohomology can be described by currents on the toroidal compactification $X^{\mathrm{tor}}_{\mathrm{GSp}_6}$ of Y_{GSp_6}. If we knew that ω_φ extends to the toroidal compactification, we would be done. To the best of our knowledge this might not be the case. If ω_φ were holomorphic, then it would be ok but that's not the case as our form is of type $(3,3)$. At this point, it is worth to comment that the other similar cases of Beilinson conjectures appearing in the literature use more or less implicitly the assumption on the extension to the toroidal compactification of the Shimura variety of differential forms attached to cuspidal automorphic forms. When the groups in question are GL_2, $\mathrm{GL}_2 \times \mathrm{GL}_2$ (over $\mathbf{Q}$), then the forms are holomorphic and are known to satisfy the assumption. However, in the cases $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$ of [Kin98], GSp_4 of [Lem17], and $\mathrm{GU}(2,1)$ of [PS17], one is forced to work with cuspidal differential forms which are not holomorphic and it is not clear whether one can extend them or not to the boundary.

Our first attempt to solve this issue (which also appeared in a first draft of [BGCLRJ23]) consisted of approximating the diffential form ω_φ by compactly supported differential forms, which can then be evaluated at the current representing $r_{\mathcal{D}}(\mathscr{Z}_{\mathcal{M}}^{\Phi_f})$. However, this approach is faulty: the final value is equal to

$$\int_{Y_{\mathrm{H}}(U\cap \mathrm{H}(\mathbf{A}_f))} \mathrm{pr}_1^* \log|u(\Phi_f)| \cdot \iota_U^* \omega_\varphi + \mathrm{Err},$$

with Err an error term which we don't know how to control.

The novel idea in [BGCLRJ23] is to give a description of Deligne–Beilinson cohomology in terms of tempered currents on $X^{\mathrm{tor}}_{\mathrm{GSp}_6}$, i.e. sheaves of continuous linear forms on smooth differential forms on $X^{\mathrm{tor}}_{\mathrm{GSp}_6}$ which are rapidly decreasing along $D_{\mathrm{GSp}_6} := X^{\mathrm{tor}}_{\mathrm{GSp}_6} - Y_{\mathrm{GSp}_6}$. Note that this retraces the reason why a Rankin–Selberg integral converges on the first place: the integral of a slowly increasing form (e.g. an Eisenstein series) against a cusp form converges because the cusp form is rapidly decreasing.

In [BGCLRJ23], we carry out the description of Deligne–Beilinson cohomology in terms of tempered currents in a great level of generality so to include and thus fix the gap of the aforementioned cases (i.e. $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_2)$, GSp_4, and $\mathrm{GU}(2,1)$).

3.3. **Tempered currents and Deligne–Beilinson cohomology.** We let Y be a smooth, quasi-projective, complex variety of pure dimension d obtained as the base change of a smooth, quasi-projective scheme over $\mathbf{R}$. Let X be a smooth compactification of Y such that $D = X - Y$ is a simple normal crossing divisor and denote by $j : Y \hookrightarrow X$ the open embedding. Let Ω^*_Y be the sheaf of holomorphic differential forms on Y and let $\Omega^*_X(\log D)$ be the sheaf of holomorphic differential forms on Y with logarithmic poles along D. Recall we have quasi-isomorphisms of complexes

$$Rj_*\mathbf{C} \to Rj_*\Omega^*_Y \leftarrow \Omega^*_X(\log D).$$

For any $p \in \mathbf{Z}$, the Deligne–Beilinson cohomology groups $H^*_{\mathcal{D}}(Y, \mathbf{R}(p))$ are defined as the hypercohomology groups of the complex

$$\mathbf{R}(p)_{\mathcal{D}} := \mathrm{cone}(Rj_*\mathbf{R}(p) \oplus F^p\Omega^*_X(\log D) \to Rj_*\Omega^*_Y)[-1]. \tag{4}$$

The cohomology groups $H^*_{\mathcal{D}}(Y/\mathbf{R}, \mathbf{R}(p))$ are then defined as $H^*_{\mathcal{D}}(Y, \mathbf{R}(p))^{\overline{F^*_\infty}=1}$, with $\overline{F^*_\infty} = F^*_\infty \otimes c$ being the de Rham involution given by the action of the complex conjugation on Y and on the coefficients.

To give the desired presentation of Deligne–Beilinson cohomology, we replace the complexes appearing in Equation (4) by quasi-isomorphic complexes of tempered currents. Locally around any point, we can find a coordinate system $(z_1, \ldots, z_k, z_{k+1}, \ldots z_d)$ such that X is isomorphic to a polydisc of dimension d and some radius $r > 0$ and that the normal crossing divisor D is given by the equations $z_1 \ldots z_k = 0$. Slowly increasing (resp. rapidly decreasing) functions on X are then defined locally by asking that

$$|f(z)| \leq C \left(\prod_{i=1}^{k} |\log|z_i|| \right)^N$$

for some $N \geq 0$ (resp. for all $N \leq 0$) and some constant C, and if similar conditions holds for the derivatives of f as well. Let $\mathscr{A}^*_{si}$, $\mathscr{A}^*_{rd}$ be the complexes of sheaves on X of slowly increasing and rapidly decreasing differential forms respectively. These are complexes of fine sheaves equipped with a natural Hodge structure, given by the type of a differential form, and with a real structure, given by real valued smooth differential forms. We define the complex $\mathscr{D}^*_{si}$ of sheaves on X of tempered currents as follows. Define $\mathscr{D}^{p,q}_{si}$ as the sheaves $U \mapsto \Gamma_c(U, \mathscr{A}^{d-p,d-q}_{rd})^*$ of continuous linear forms on compactly supported sections on U of rapidly decreasing differential forms, where $U \subseteq X$ is an open set. Here we have used a natural Fréchet topology on rapidly decreasing differential forms. From these, we define $\mathscr{D}^*_{si}$, which is so equipped with a Hodge filtration as well as with a natural real structure. Note that there there is a filtered morphism of complexes

$$\iota : \mathscr{A}^*_{si} \to \mathscr{D}^*_{si}$$

coming from the fact that, for any open subset $U \subseteq X$, any form $\omega \in \mathscr{A}^{p,q}_{si}(U)$ defines a current $T_\omega \in \mathscr{D}^{p,q}_{si}(U)$ by

$$T_\omega(\eta) = \frac{1}{(2\pi i)^d} \int_U \omega \wedge \eta, \quad (\eta \in \Gamma_c(U, \mathscr{A}^{d-p,d-q}_{rd})).$$

Let $\mathscr{D}^*_{si,\mathbf{R}(p-1)}$ be given by $\mathbf{R}(p-1)$-valued tempered currents and let $\pi_{p-1} : \mathbf{C} \to \mathbf{R}(p-1)$ be the projection defined by $\pi_{p-1}(z) = \frac{z+(-1)^{p-1}\overline{z}}{2}$.

Theorem 3.2 ([BGCLRJ23, Theorem 2.25])**.** *We have*

$$H^n_{\mathcal{D}}(Y, \mathbf{R}(p)) \simeq \frac{\{(S,T) : dS = 0, dT = \pi_{p-1}(S)\}}{d(\widetilde{S}, \widetilde{T})},$$

where $(S,T) \in F^p\mathscr{D}^n_{si}(X) \oplus \mathscr{D}^{n-1}_{si,\mathbf{R}(p-1)}(X)$ *and* $d(\widetilde{S}, \widetilde{T}) = (dS, dT - \pi_{p-1}(S))$.

Let us define Gysin morphisms. Let $\iota : Y \hookrightarrow Y'$ be a closed immersion of codimension c and assume that the smooth compactifications X, X' are chosen so that ι extends to $\iota : X \hookrightarrow X'$ with $\iota^{-1}(D') = D$. We can then define a Gysin morphism

$$\iota_* : H^n_{\mathcal{D}}(Y, \mathbf{R}(p)) \to H^{n+2c}_{\mathcal{D}}(Y', \mathbf{R}(p+c))$$

by setting $\iota_*[S,T] := [\iota_* S, \iota_* T]$ (where given a current X, $\iota_* X : \omega \mapsto X(\iota^*\omega)$).

We can finally get to the construction of the linear form associated to certain rapidly decreasing differential forms at the cost of restricting their possible Hodge types.

Proposition 3.3 ([BGCLRJ23, Proposition 2.27])**.** *Let* $n \in \mathbf{N}, p \in \mathbf{Z}$ *and let* $\omega \in \mathscr{A}^{2d-n}_{rd}(X)$ *be a smooth closed rapidly decreasing differential form of Hodge type components inside* $\{(a,b) : a, b > d-p\}$. *Then the assignment* $(S,T) \mapsto T(\omega)$ *induces a map*

$$\langle\, , \omega\rangle : H^{n+1}_{\mathcal{D}}(Y, \mathbf{R}(p)) \to \mathbf{C}.$$

3.4. **End of proof of Theorem 2.6(1) and remarks.** Proposition 3.3 is enough for our applications: let $n = d = 6$, $p = 4$, $Y = Y_{\mathrm{GSp}_6}$. Then $\{(a,b) : a, b > 2\} = \{(3,3)\}$ and so we have a linear pairing

$$\langle\ , \omega_\varphi\rangle : H^7_{\mathcal{D}}(Y_{\mathrm{GSp}_6}(U), \mathbf{R}(4)) \to \mathbf{C}.$$

Moreover, with some diagram chasing, one sees that $r_{\mathcal{D}}(\mathscr{Z}^{\Phi_f}_{\mathcal{M}})$ is represented by the pair of tempered currents

$$(\iota_{U,*} T_{\mathrm{pr}_1^* d\log u(\Phi_f)}, \iota_{U,*} T_{\mathrm{pr}_1^* \log|u(\Phi_f)|}).$$

By the explicit description of the Gysin morphism and the pairing, one gets

$$\begin{aligned}
\langle r_{\mathcal{D}}(\mathscr{Z}^{\Phi_f}_{\mathcal{M}}), \omega_\varphi\rangle &= \iota_{U,*} T_{\mathrm{pr}_1^* \log|u(\Phi_f)|}(\omega_\varphi) \\
&= T_{\mathrm{pr}_1^* \log|u(\Phi_f)|}(\iota_U^* \omega_\varphi) \\
&= \int_{Y_{\mathrm{H}}(U \cap \mathrm{H}(\mathbf{A}_f))} \mathrm{pr}_1^* \log|u(\Phi_f)| \cdot \iota_U^* \omega_\varphi,
\end{aligned}$$

as desired. Proposition 3.1 and Theorem 1.3 finish the proof.

Let us conclude with a remark on how the similar gap in the constructions of [Kin98] and [PS17] can be fixed; the case of [Lem17], which involves cup products of Siegel units, has been discussed in [CLJ23, Theorem 3.15].

[Kin98]: Consider the motivic class constructed as pushforward of a Siegel unit with respect to the closed immersion (for nice enough level U) of

$$\iota : Y_{\mathrm{GL}_2}(U \cap \mathrm{GL}_2(\mathbf{A}_f)) \hookrightarrow Y_{\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2}(U),$$

with $F/\mathbf{Q}$ a real quadratic field, associated to the natural embedding $\mathrm{GL}_2 \hookrightarrow \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$. This defines a class in $H^3_{\mathcal{M}}(Y_{\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2}(U), \overline{\mathbf{Q}}(2))$. If we let ω be the differential form of Hodge type $(1,1)$ associated to a Hilbert modular cusp form $\mathcal{F}$ of level U, by Proposition 3.3 (with $d = n = p = 2$) we can pair it with the archimedean realization of the motivic class yielding the value

$$\int_{Y_{\mathrm{GL}_2}(U \cap \mathrm{GL}_2(\mathbf{A}_f))} \log|u(\Phi_f)| \cdot \iota^*\omega.$$

By Proposition 3.1, this is related to the Rankin–Selberg integral that calculates the Asai L-function of $\mathcal{F}$.

[PS17]: let $E/\mathbf{Q}$ be an imaginary quadratic extension defining the unitary group $\mathrm{GU}(2,1)$. The associated Shimura variety is the so called Picard modular surface. Denote by H its subgroup

$$H = \{(g,t) \in \mathrm{GL}_2 \times \mathrm{Res}_{E/\mathbf{Q}}\mathrm{GL}_1 \, : \det(g) = \mathrm{Nm}(t)\}.$$

Then we have a diagram

$$Y_{\mathrm{GL}_2} \twoheadleftarrow Y_H \hookrightarrow Y_{\mathrm{GU}(2,1)}$$

and one defines a motivic element in $H^3_{\mathcal{M}}(Y_{\mathrm{GU}(2,1)}, \overline{\mathbf{Q}}(2))$ as the pushforward of the pullback of a Siegel unit. By Proposition 3.3 (same numerology of case above) its archimedean realization can be paired with a differential form of Hodge type (1,1) associated to cusp forms of $\mathrm{GU}(2,1)$. The value of the resulting integral is meticulously studied in [PS17], where they relate it to a non-critical value of the degree six standard L-function of $\mathrm{GU}(2,1)$.

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