RATIONAL ORBITS AND DENSITY THEOREMS RELATED TO PREHOMOGENEOUS VECTOR SPACES

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1. GIT and prehomogeneous vector spaces

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In early 1980's, the auther noticed a relation between GIT (geometric invariant theory) and the theory of prehomogeneous vector spaces. We explain the starting point in this section. We use classical language in the following.

The convergence of the zeta function was proved in Shintani's paper [19] as follows. Let $G = \mathrm{GL}_2(\mathbb{R})$, $V = \mathrm{Sym}^3\mathbb{R}^2$ be the space of binary cubic forms and $\Phi \in \mathcal{S}(V_\mathbb{R})$ the space of rapidly decreasing functions. Let $\Delta(x)$ be the discriminant of $x \in V$. Put $V'_\mathbb{Z} = \{x \in V_\mathbb{Z} \mid \Delta(x) \neq 0\}$. The zeta function for (G, V) is defined as follows

$$Z(\Phi, s) = \int_{G_\mathbb{R}/G_\mathbb{Z}} |\det g|^s \sum_{x \in V'_\mathbb{Z}} \Phi(gx) dg.$$

This function of $s \in \mathbb{C}$ converges absolutely and locally uniformly if $\mathrm{Re}(s)$ is sufficiently large. The proof of the convergence is reduced to the convergence of the following kind of integrals by considering a Siegel set for $G_\mathbb{R}/G_\mathbb{Z}$

$$\int_{\substack{\lambda_1 \in \mathbb{R}, \\ \lambda_2 \geqq \epsilon}} \lambda_1^{2\mathrm{Re}(s)} \lambda_2^{-2} \sum_{x \in V'_\mathbb{Z}} \Phi(\lambda_1^3 \lambda_2^3 x_0, \ldots, \lambda_1^3 \lambda_2^{-3} x_3) d^\times \lambda_1 d^\times \lambda_2$$

where $\epsilon > 0$ is a constant and $d^\times \lambda_i = \lambda_i^{-1} d\lambda_i$ for $i = 1, 2$.

The crucial observation is that if $x \in V'_\mathbb{Z}$, then $(x_0, x_1) \neq (0, 0)$ and $(x_2, x_3) \neq (0, 0)$. Since Φ is rapidly decreasing, there exists a polynomial $P(\lambda_1^{\pm 1}, \lambda_2^{\pm 1})$ such that for any $N_1, N_2 \geqq 1$,

$$\int_{\substack{\lambda_1 \in \mathbb{R}, \\ \lambda_2 \geqq \epsilon}} \lambda_1^{2s} \lambda_2^{-2} P(\lambda_1^{\pm 1}, \lambda_2^{\pm 1}) (\lambda_1^3 \lambda_2)^{-N_1} (\lambda_1^3 \lambda_2^{-1})^{-N_2} d^\times \lambda.$$

If $\lambda_1 \geqq 1$, we can choose $N_1 \gg 0$ depending on s and the integral converges absolutely. If $\lambda_1 \leqq 1$, we can choose $N_1 \gg 0$ so that the integral with respect to λ_2 converges. Then if $\mathrm{Re}(s)$ is large enough, the integral converges absolutely. It is possible to refine the argument so that the integral converges absolutely for $\mathrm{Re}(s) > 6$.

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The point of the above argument is that if $x \in V'$ implies that $(x_0, x_1) \neq (0,0)$ and $(x_2, x_3) \neq (0,0)$. The weights of coordinates with respect to the action of

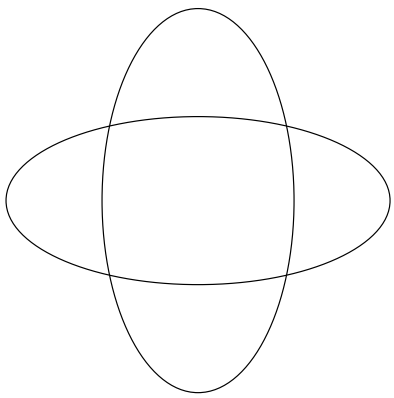
$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

are $3, 1, -1, -3$. So the above condition says that if $x \in V'$, there are non-zero coordinates for both positive and negative weights. This is obviously the stability in GIT. This was the starting point of my investigation.

2. Rational orbits of prehomogeneous vector spaces

At first I did not think that prehomogeneous vector spaces are interesting since the set of generic points is a single orbit over algebraically closed fields. It is possible to define the zeta function and compute some poles for the representation $V = \mathrm{Sym}^n \mathrm{Aff}^2$ of $G = \mathrm{GL}_2$ ([32]). However, it is difficult to interpret the poles at present. At some point I was looking at the case $G = \mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{Sym}^2 \mathrm{Aff}^3 \otimes \mathrm{Aff}^2$. Points of V are expressed as $x = (Q_1, Q_2) \in V$ where Q_1, Q_2 are ternary quadratic forms. A single ternary quadratic form $Q \in \mathrm{Sym}^2 k^3$ defines a conic in $\mathbb{P}^2$ which is isomorphic to $\mathbb{P}^1$ over algebraically closed field. I could not think it is such an interesting object geometrically.

However, for $x = (Q_1, Q_2) \in V$, we can consider the intersection of two conics as follows.



Coordinates of intersection points are solutions of quartic equations. 11th century mathematician Omar Khayaam used this fact to describe solutions of cubic equations with extra points at infinity. This case turns out to parametrize quartic extensions as follows ([29]). Let k be a field.

Theorem 2.1 (D. Wright–A.Y.). *$G_k \backslash V'_k \cong \mathrm{H}^1(k, \mathfrak{S}_4)$ where $\mathrm{H}^1(k, \mathfrak{S}_4)$ is the Galois cohomology set with the trivial action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\mathfrak{S}_4$.*

For $x = (Q_1, Q_2) \in V'_k$, we denote

$$\mathrm{Zero}(x) = \{p \in \mathbb{P}^2(\overline{k}) \mid Q_1(p) = Q_2(p) = 0\}.$$

It turns out that points in $\mathrm{Zero}(x)$ are defined over k^{sep}. Moreover, if we consider the subset of $x \in V'_k$ for which the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\mathrm{Zero}(x)$ is transitive, then the correspondence between rational orbits and isomorphism classes of quartic extensions is bijective.

At this point, the importance of prehomogeneous vector spaces became obvious. There is no geometric moduli, but the set of rational orbits is something like an arithmetic moduli. We now state the definition of prehomogeneous vector spaces and describe interpretations of rational orbits for some cases.

Definition 2.2. Let G be a connected reductive group and V a finite dimensional representation of G both over a field k. Then (G, V) is called a prehomogeneous vector space if the following conditions are satisfied.

1. There exists $w \in V$ such that $G \cdot w \subset V$ is Zariski open.
2. There exists a non-constant polynomial $\Delta(x) \in k[V] \setminus k$ and a character $\chi : G \to \mathrm{GL}_1$ such that $\Delta(gx) = \chi(g)\Delta(x)$.

The set $\{x \in V_k \mid \Delta(x) \neq 0\}$ is denoted by V_k^{ss}.

The notion of regularity was introduced in [18, p.60, DEFINITION 7] for prehomogeneous vector spaces over $\mathbb{C}$. To ensure expected properties for prehomogeneous vector spaces over arbitrary fields, the definition of the regularity in [27, p.217, Definition 4.5] is more convenient. Irreducible regular prehomogeneous vector spaces over $\mathbb{C}$ was classified in [18]. There are 29 cases. These prehomogeneous vector spaces are defined over aribitary fields except that there are restrictions for $\mathrm{ch}\, k$ in some cases. We list interpretations of rational orbits as far as the author understands.

(1) GL_n, $V = \mathrm{M}(n)$, 1 point (classical)
(2) $\mathrm{GL}_1 \times \mathrm{GL}_n$, $\mathrm{Sym}^2 k^n$, isomorphism classes of $\mathrm{PSO}(Q)$, (classical)
(3) GL_{2n}, $\wedge^2 k^{2n}$, 1 point (Witt)
(4) $\mathrm{GL}_1 \times \mathrm{GL}_2$, $\mathrm{Sym}^3 k^2$, $\mathrm{H}^1(k, \mathfrak{S}_3)$ (classical)
(5) $\mathrm{GL}_1 \times \mathrm{GL}_6$, $\wedge^3 k^6$, $\mathrm{H}^1(k, \mathfrak{S}_2)$ ([28, p.1692, Proposition (1.12)])
(6) $\mathrm{GL}_1 \times \mathrm{GL}_7$, $\wedge^3 k^7$, k-forms of the Octonion ([28, p.1696, Proposition (2.24)])
(7) $\mathrm{GL}_1 \times \mathrm{GL}_8$, $\wedge^3 k^8$, k-forms of SL_3 ([30, p.127, Theorem 1.5])
(8) $\mathrm{GL}_3 \times \mathrm{GL}_2$, $\mathrm{Sym}^2 k^3 \otimes k^2$, $\mathrm{H}^1(k, \mathfrak{S}_4)$ ([29, p.311, Proposition 5.4])
(9) $\mathrm{GL}_6 \times \mathrm{GL}_2$, $\wedge^2 k^6 \otimes k^2$, $\mathrm{H}^1(k, \mathfrak{S}_3)$ ([29, p.311, Proposition 5.4])
(10) $\mathrm{GL}_5 \times \mathrm{GL}_3$, $\wedge^2 k^5 \otimes k^3$, quaternion algebras [17, p.68, Theorem 3.4]
(11) $\mathrm{GL}_5 \times \mathrm{GL}_4$, $\wedge^2 k^5 \otimes k^4$, $\mathrm{H}^1(k, \mathfrak{S}_5)$ ([29, p.311, Proposition 5.4])
(12) $\mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathrm{GL}_2$, $\mathrm{M}(3) \otimes k^2$, $\mathrm{H}^1(k, \mathfrak{S}_3)$ ([29, p.311, Proposition 5.4])
(13) $\mathrm{Sp}(2n) \times \mathrm{GL}_{2m}$, $\mathrm{M}(2n, 2m)$?
(14) $\mathrm{GL}_1 \times \mathrm{GSp}(6)$, $[0, 1, 0] = k^{14}$, SU(3,Q) ([34, p.47, Theorem 4.11])
(15) $\mathrm{GO}(n) \times \mathrm{GL}_m$, $\mathrm{M}(n, m)$? partially known.
(16) $\mathrm{GL}_1 \times \mathrm{Spin}(7)$, ([10, p.1015, PROPOSITION 4])
(17) $\mathrm{Spin}(7) \times \mathrm{GL}_2$, spin$\otimes k^2$?
(18) $\mathrm{Spin}(7) \times \mathrm{GL}_3$, spin$\otimes k^3$?
(19) $\mathrm{GL}_1 \times \mathrm{Spin}(9)$, spin k^{16}, ([10, p.1016, PROPOSITION 5])
(20) $\mathrm{Spin}(10) \times \mathrm{GL}_2$, half spin$\otimes k^2$?
(21) $\mathrm{Spin}(10) \times \mathrm{GL}_3$, half spin$\otimes k^3$?
(22) $\mathrm{GL}_1 \times \mathrm{Spin}(11)$, spin k^{32}, ([10, p.1018, PROPOSITION 6])

(23) $\mathrm{GL}_1 \times \mathrm{Spin}(12)$, spin k^{32}, ([10, p.1012, PROPOSITION 3])
(24) $\mathrm{GL}_1 \times \mathrm{Spin}(14)$, half spin k^{64} ?
(25) $\mathrm{GL}_1 \times G_2$, k^7, $\mathrm{SU}(2,1)$ ([31, p.116, Theorem (0.3)])
(26) $G_2 \times \mathrm{GL}_2$, $k^7 \otimes k^2$, $\mathrm{SU}(\begin{pmatrix} -s & \\ & 1 \end{pmatrix})$ ([31, p.116, Theorem (0.3)])
(27) $\mathrm{GL}_1 \times E_6$, $J = k^{27}$, 1 point ([11, p.272, Theorem 4.1])
(28) $E_6 \times \mathrm{GL}_2$, $J \otimes k^2 = k^{54}$ (E_6 split), $\mathrm{H}^1(k, \mathfrak{S}_3)$ ([12, p.309, Theorem 1.19])
(29) $\mathrm{GL}_1 \times E_7$, k^{56}, k-forms of E_6 (the correspondence may not be bijective) ([22, p.280, PROPOSITION 5.5])

3. Prehomogeneous vector spaces and density theorems

What makes the theory of prehomogeneous vector spaces special is that one can expect density theorems. The origin of density theorems related to prehomogeneous vector spaces probably goes back to Gauss.

Let $G = \mathrm{GL}_1 \times \mathrm{GL}_2$, $V = \mathrm{Sym}^2\mathbb{Q}^2$. Let h_D be the narrow class number of orders of discriminat D of quadratic fields. The following statement

$$\sum_{-X<D<0} h_D \sim \frac{4\pi}{21\zeta(3)} X^{\frac{3}{2}}$$

was called the Gauss conjecture. This conjecture was proved in 1865 by Lipschutz. It is possible to interpret h_D by the number of equivalence classes of integral binary quadratic forms.

Siegel [21] generalized the above result to quadratic froms in arbitrary number n of variables in 1944 including the case $D > 0$ for binary quadratic forms. As far as the space of binary quadratic forms is concerned, the error estimate has been improved by many people including Shintani [20], Chamizo-Iwaniec [4]. However, these results are results over $\mathbb{Z}$. The set of equivalence classes over $\mathbb{Q}$ is more sparse and generally speaking, it is more difficult to count sparse objects.

Counting integral equivalence classes of binary quadratic forms with some weights is essentially the same as counting $h_D, h_D R_D$ of orders of quadratic fields (R_D is the analogue of the regulator). Goldfeld-Hoffstein [7] proved (not by the method of prehomogeneous vector spaces) in 1985 the density theorem which corresponds to quivalence classes of binary quadratic forms over $\mathbb{Q}$ as follows.

$$\lim_{X\to\infty} X^{-3/2} \sum_{\substack{[F:\mathbb{Q}]=2 \\ 0<\Delta_F\le X}} h_F R_F = \frac{\pi^2}{36} \prod_p (1 - p^{-2} - p^{-3} + p^{-4}),$$

$$\lim_{X\to\infty} X^{-3/2} \sum_{\substack{[F:\mathbb{Q}]=2 \\ 0<-\Delta_F\le X}} h_F = \frac{\pi}{18} \prod_p (1 - p^{-2} - p^{-3} + p^{-4})$$

where Δ_F is the absolute discriminant of F. Their result include an error term estimate. The numbers $1/18$ or $1/36$ and $4/21$ are different since the integral structure considered by Gauss corresponds to forms $au^2 + 2buv + cv^2$ where $a, b, c \in \mathbb{Z}$. The proof of the above statement by the theory of prehomogeneous vector spaces (without the error term estimate) was given by Datskovsky [5] in 1993.

It seems that considering equivalence classes over $\mathbb{Q}$ provvide more interesting density theorems. If we consider the prehomogeneous vector space $G = \mathrm{GL}_1 \times \mathrm{GL}_2$, $V = \mathrm{Sym}^3\mathrm{Aff}^2$, the classical theorem of Davenport and Heilbronn [6] is obtained as follows.

$$\lim_{X\to\infty} \frac{1}{X} \sum_{\substack{[F:\mathbb{Q}]=3 \\ |\Delta_F|\le X}} 1 = \frac{1}{\zeta(3)}$$

where Δ_F is the absolute discriminant of F.

The reason why one can prove this kind of density theorems is that the set of rational orbits $G_k\backslash V_k^{\mathrm{ss}}$ parametrizes interesting objects. As we stated in (4) of the previous section, $G_k\backslash V_k^{\mathrm{ss}}$ is in bijective correspondence with $\mathrm{H}^1(k, \mathfrak{S}_3)$ in the above case. If (G, V) is a prehomogeneous vector space, we expect a density theorem of counting elements of $x \in G_k\backslash V_k^{\mathrm{ss}}$ with weight $\mathrm{vol}(G^\circ_{x\,\mathbb{A}}/G^\circ_{x\,k})$.

The following density theorems has been proved.

(1) $\mathrm{Sym}^2\mathrm{Aff}^3 \otimes \mathrm{Aff}^2 \to$ quartic fields (Bhargava [1])
(2) $\wedge^2\mathrm{Aff}^5 \otimes \mathrm{Aff}^4 \to$ quintic fields (Bhargava [2])
(3) $\mathrm{Sym}^2\mathrm{Aff}^n \to \mathrm{vol}(\mathrm{PSO}(Q)_\mathbb{A}/\mathrm{PSO}(Q)_\mathbb{Q})$ (Hayasaka-A.Y. [8], [9], [36])
(4) Non-split form of $k^2 \otimes k^2 \otimes k^2 \to [k_1 : \mathbb{Q}] = 3$ fixed, $[F : \mathbb{Q}] = 2$, density of $h_L R_L/h_F R_F$ $(L = k_1 \cdot F)$ (A.Y. [33], [35])

This result (3) is a $\mathbb{Q}$ version of Siegel's result in 1944. If n is odd (resp. even), the density is in the form

$$\text{constant} \times \frac{X^{\frac{n+1}{2}}}{\sqrt{\log X}} \quad (\text{resp. constant} \times X^{\frac{n+1}{2}}).$$

4. Zeta functions

We cannot explain all the steps of the zeta function theory on the process to go from $\mathbb{Z}$-equivalence classes to $\mathbb{Q}$-equivalence classes. Roughly speaking, one has to know the principal part of the zeta function or something which is equivalence. Then the local theory and the computation of the "local density" proves the density theorem of $\mathbb{Q}$-equivalence classes. We explain what has to be done for the zeta function theory.

We compute the poles by a wrong argument in the case of $G = \mathrm{GL}_2$, $V = \mathrm{Sym}^3\mathbb{Q}^2$. The zeta function in this case is defined for a Schwartz–Bruhat function Φ on $V_\mathbb{A}$ and $s \in \mathbb{C}$ as follows

$$Z(\Phi, s) = \int_{G_\mathbb{A}/G_\mathbb{Q}} |\det g|^s \sum_{x\in V_\mathbb{Q}^{\mathrm{ss}}} \Phi(gx)\, dg$$

where dg is an invariant measure on $G_\mathbb{A}$.

The standard argument is to use the Poisson summation formula for the sum $\sum_{x\in V_\mathbb{Q}^{\mathrm{ss}}} \Phi(gx)$ for $g \in G_\mathbb{A}$ such that $|\det g| = 1$. The difficulty is that we have to consider all points of $V_\mathbb{Q}$. However, the action of the group may not be the best on the set $V_\mathbb{Q} \setminus V_\mathbb{Q}^{\mathrm{ss}}$. It turns out that this set has a stratification called the GIT stratification, which we explain in the next section.

One of the strata in this case is $S_{\mathbb{Q}} = \{x \in V_{\mathbb{Q}} \mid x \text{ has triple roots}\}$. Let

$$
\begin{aligned}
&Z' = \{(0,0,0,x) \mid x \neq 0\},\\
&a(t_1,t_2) = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix},\ n(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},\\
&B_{\mathbb{Q}} = \{a(t_1,t_2)n(u)\}\,.
\end{aligned}
$$

Then $S_{\mathbb{Q}} \cong G_{\mathbb{Q}} \times_{B_{\mathbb{Q}}} Z'_{\mathbb{Q}}$. (GIT stratification)

We illustrate the computation of the contribution from $S_{\mathbb{Q}}$ by a WRONG argument. We assume that Φ is invariant by the action of the standard maximal compact subgroup of $G_{\mathbb{A}}$. Then

$$
\begin{aligned}
\int_{\substack{G_{\mathbb{A}}/G_{\mathbb{Q}} \\ |\det g|=1}} \sum_{x \in S_{\mathbb{Q}}} \Phi(gx)dg &= \int_{\substack{G_{\mathbb{A}}/B_{\mathbb{Q}} \\ |\det g|=1}} \sum_{x \in Z'_{\mathbb{Q}}} \Phi(gx)dg \\
&\doteqdot \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \sum_{x \in Z'_{\mathbb{Q}}} \Phi(a(t_1,t_1^{-1})x)|t_1|^{-2}d^\times t_1 \\
&\doteqdot \frac{1}{3}\zeta(\Phi(0,0,0,*),\tfrac{2}{3}) \quad \text{this is a lie.}
\end{aligned}
$$

Repalce Φ by $\Phi_\lambda(*) = \Phi(\lambda *)$ $(\lambda > 0)$. Then

$$
\begin{aligned}
\frac{1}{9}\int_0^1 \lambda^{\frac{2s}{3}}\zeta(\Phi_\lambda,\tfrac{2}{3})d^\times\lambda &= \frac{1}{9}\int_0^1 \lambda^{\frac{2(s-1)}{3}}\zeta(\Phi,\tfrac{2}{3})d^\times\lambda \\
&= \frac{\zeta(\Phi,\frac{2}{3})}{9}\frac{3}{2(s-1)} \\
&= \frac{\zeta(\Phi,\frac{2}{3})}{6}\frac{1}{s-1}.
\end{aligned}
$$

This above argument is of course wrong since the integral diverges. However, it can be justifed by a similar argument as in the trace formula using the smoothed version of Eisenstein series.

The inductive struture $S_{\mathbb{Q}} \cong G_{\mathbb{Q}} \times_{B_{\mathbb{Q}}} Z'_{\mathbb{Q}}$ was essential in the above computation. It is possible to verify this statement explicitly in this case. However, it becomes difficult to show such a stratification explicitly for bigger prehomogeneous vector spaces. It is possible to apply what we call the GIT stratification, which we will explain in the next section.

5. GIT stratification

The notion of the GIT stratification is based on a certain "convexity". It was established in the context of GIT by Ness [14], Kempf [13] and Kirwan [15] (see [16] also). For the rationality questions, see [24].

Let k be a perfect field and $\overline{k}$ its algebraic closure. Even though, the rationality question is answered for non-split groups in [24], we assume that algebraic groups are split in this section for simplicity. We use the notation $\mathrm{diag}(g_1,\ldots,g_m)$ for the block diagonal matrix whose diagonal blocks are $g_1, \ldots, g_m$. We are mainly interested in prehomogeneous vector spaces, but we first consider a general situation.

Let G be a connected reductive group, V a finite dimensional representation of G both defined over k. As we mentioned above, G is assumed to be split. We assume that there is a connected split reductive subgroup G_1 of G, a split torus $T_0 \subset \mathrm{Z}(G)$ (the center of G), such that $T_0 \cap G_1$ is finite and $G = T_0 G_1$ as algebraic groups. We assume that there is a rational character χ of T_0 such that the action of $t \in T_0$ is given by the scalar multiplication by $\chi(t)$.

Let $(T_0 \cap G_1) \subset T \subset G_1$ be a maximal split torus, $X_*(T), X^*(T)$ the groups of one parameter subgroups (abbreviated as 1PS from now on) and the group of rational characters respectively. We put

$$\mathfrak{t} = X_*(T) \otimes \mathbb{R},\ \mathfrak{t}_{\mathbb{Q}} = X_*(T) \otimes \mathbb{Q},\ \mathfrak{t}^* = X^*(T) \otimes \mathbb{R},\ \mathfrak{t}^*_{\mathbb{Q}} = X^*(T) \otimes \mathbb{Q}.$$

Let $\mathbb{W} = N_G(T)/T$ be the Weyl group of G. $\mathbb{W}$ acts on $\mathfrak{t}^*$ also.

There is a natural pairing $\langle\ ,\ \rangle_T : X^*(T) \times X_*(T) \to \mathbb{Z}$ defined by $t^{\langle \chi, \lambda \rangle_T} = \chi(\lambda(t))$ for $\chi \in X^*(T), \lambda \in X_*(T)$. This is a perfect paring ([3, pp.113–115]).

There exists an inner product $(\ ,\ )$ on $\mathfrak{t}$ which is invariant under the actions of $\mathbb{W}$ and the Galois group $\mathrm{Gal}(\overline{k}/k)$. We may assume that this inner product is rational, i.e., $(\lambda, \nu) \in \mathbb{Q}$ for all $\lambda, \nu \in \mathfrak{t}_{\mathbb{Q}}$. Let $\|\ \|$ be the norm on $\mathfrak{t}$ defined by $(\ ,\ )$. We choose a Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$ for the action of $\mathbb{W}$.

For $\lambda \in \mathfrak{t}$, let $\beta = \beta(\lambda)$ be the element of $\mathfrak{t}^*$ such that $\langle \beta, \nu \rangle = (\lambda, \nu)$ for all $\nu \in \mathfrak{t}$. The map $\lambda \mapsto \beta(\lambda)$ is a bijection and we denote the inverse map by $\lambda = \lambda(\beta)$. There is a unique positive rational number a such that $a\lambda(\beta) \in X_*(T)$ and is indivisible. We use the notation λ_β for $a\lambda(\beta)$.

Identifying $\mathfrak{t}$ with $\mathfrak{t}^*$ we have a $\mathbb{W}$-invariant inner product $(\ ,\ )_*$ on $\mathfrak{t}^*$, the norm $\|\ \|_*$ determined by $(\ ,\ )_*$ and a Weyl chamber $\mathfrak{t}^*_+$.

Let $N = \dim V$. We choose a coordinate system $v = (v_1, \ldots, v_N)$ on V by which T acts diagonally. Let $\gamma_i \in \mathfrak{t}^*$ and $\mathbb{a}_i$ be the weight and the coordinate vector which corresponds to i-th coordinate. Let $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$. For a subset $\mathfrak{I} \subset \Gamma$, we denote the convex hull of $\mathfrak{I}$ by $\mathrm{Conv}\,\mathfrak{I}$. Let $\mathbb{P}(V)$ be the projective space associated with V and $\pi_V : V \setminus \{0\} \to \mathbb{P}(V)$ the natural map. For $\mathfrak{I} \subset \Gamma$ such that $0 \notin \mathrm{Conv}\,\mathfrak{I}$, let β be the closest point of $\mathrm{Conv}\,\mathfrak{I}$ to the origin. Then β lies in $\mathfrak{t}^*_{\mathbb{Q}}$. Let $\mathfrak{B}$ be the set of all such β which lies in $\mathfrak{t}^*_+$.

We define

$$Y_\beta = \mathrm{Span}\{\mathbb{a}_i \,|\, (\gamma_i, \beta)_* \geq (\beta, \beta)_*\}, \quad Z_\beta = \mathrm{Span}\{\mathbb{a}_i \,|\, (\gamma_i, \beta)_* = (\beta, \beta)_*\},$$
$$W_\beta = \mathrm{Span}\{\mathbb{a}_i \,|\, (\gamma_i, \beta)_* > (\beta, \beta)_*\}$$

where Span is the spanned subspace. Clearly $Y_\beta = Z_\beta \oplus W_\beta$.

If λ is a 1PS of G, we define

$$P(\lambda) = \left\{ p \in G \,\middle|\, \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists} \right\},\ M(\lambda) = Z_G(\lambda) \text{ (the centralizer)},$$
$$U(\lambda) = \left\{ p \in G \,\middle|\, \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} = 1 \right\}.$$

The group $P(\lambda)$ is a parabolic subgroup of G ([23, p.148]) with Levi part $M(\lambda)$ and unipotent radical $U(\lambda)$. We put $P_\beta = P(\lambda_\beta)$, $M_\beta = Z_G(\lambda_\beta)$ and $U_\beta = U(\lambda_\beta)$.

Let χ_β be the indivisible rational character of M_β such that the restriction of χ_β^a to T coincides with $b\beta$ for some positive integers a, b. We define $G_\beta = \{g \in M_\beta \mid \chi_\beta(g) = 1\}^\circ$ (the identity component). Then G_β acts on Z_β. Note that M_β and G_β are defined over k, and since $\langle \chi_\beta, \lambda_\beta \rangle$ is a positive multiple of $\|\beta\|_*$, $M_\beta = G_\beta \mathrm{Im}(\lambda_\beta)$. Moreover, if ν is any rational 1PS in G_β, $(\nu, \lambda_\beta) = 0$.

Let $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ be the set of semi-stable points of $\mathbb{P}(Z_\beta)$ with respect to the action of $G_\beta^1 \stackrel{\mathrm{def}}{=} G_\beta \cap G_1$. Since there is a difference between Z_β and $\mathbb{P}(Z_\beta)$, we remove appropriate scalar directions from G_β to consider stability. For the notion of semi-stable points, see [16]. We regard $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ as a subset of $\mathbb{P}(V)$. Put

$$Z_\beta^{\mathrm{ss}} = \pi_V^{-1}(\mathbb{P}(Z_\beta)^{\mathrm{ss}}),\ Y_\beta^{\mathrm{ss}} = \{(z, w) \mid z \in Z_\beta^{\mathrm{ss}}, w \in W_\beta\}.$$

We define $S_\beta = GY_\beta^{\mathrm{ss}}$. Note that S_β can be the empty set. We denote the set of k-rational points of S_β, etc., by $S_{\beta\, k}$, etc.

The following theorem is COROLLARY 1.4 [24, p.264].

Theorem 5.1. *Suppose that k is a perfect field. Then we have*

$$V_k \setminus \{0\} = V_k^{\mathrm{ss}} \coprod \coprod_{\beta \in \mathfrak{B}} S_{\beta\, k}.$$

Moreover, $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$.

We call this stratification *the GIT stratification*. The importance of the above theorem is the rationality of the inductive structure of S_β. Obviously, we can use computer to determine $\mathfrak{B}$. The computer computation of $\mathfrak{B}$ was carried out for the following cases in [25].

(1) $G = \mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{Aff}^3 \otimes \mathrm{Aff}^3 \otimes \mathrm{Aff}^2$.
(2) $G = \mathrm{GL}_6 \times \mathrm{GL}_2$, $V = \wedge^2 \mathrm{Aff}^6 \otimes \mathrm{Aff}^2$.
(3) $G = \mathrm{GL}_5 \times \mathrm{GL}_4$, $V = \wedge^2 \mathrm{Aff}^5 \otimes \mathrm{Aff}^4$.
(4) $G = \mathrm{GL}_8$, $V = \wedge^3 \mathrm{Aff}^8$.

There may be S_β which is the empty set. It takes some work to determine β such that $S_\beta \neq \emptyset$. This has been carried out for the cases (1)–(3) in [26], [27].

References

[1] M. Bhargava. Higher composition laws. III. The parametrization of quartic rings. *Ann. of Math. (2)*, 159(3):1329–1360, 2004.
[2] M. Bhargava. The density of discriminants of quartic rings and fields. *Ann. of Math. (2)*, 162(2):1031–1063, 2005.
[3] A. Borel. *Linear algebraic groups*. Springer-Verlag, Berlin, Heidelberg, New York, 2nd edition, 1991.
[4] F. Chamizo and H. Iwaniec. On the gauss mean-value formula for class number. *Nagoya Math. J.*, 151:199–208, 1998.
[5] B. Datskovsky. A mean value theorem for class numbers of quadratic extensions. *Contemporary Mathematics*, 143:179–242, 1993.

[6] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. II. *Proc. Royal Soc.*, A322,:405–420, 1971.
[7] D. Goldfeld and J. Hoffstein. Eisenstein series of 1/2-integral weight and the mean value of real Dirichlet series. *Invent. Math.*, 80:185–208, 1985.
[8] N. Hayasaka and A. Yukie. On the density of unnormalized tamagawa numbers of orthogonal groups I. *Publ. Res. Inst. Math. Sci.*, 44(2):545–607, 2008.
[9] N. Hayasaka and A. Yukie. On the density of unnormalized Tamagawa numbers of orthogonal groups. II. *Amer. J. Math.*, 131(3):683–730, 2009.
[10] J. Igusa. A classification of spinors of dimension up to twelve. *Amer. J. Math.*, 92:997–1028, 1970.
[11] J. Igusa. On a certain class of prehomogeneous vector spaces. *J. of Pure and Applied Algebra*, 47:265–282, 1987.
[12] R. Kato and A. Yukie. Rational orbits of the space of pairs of exceptional Jordan algebras. *J. Number Theory*, 189:304–352, 2018.
[13] G. Kempf. Instability in invariant theory. *Ann. of Math.*, 108:299–316, 1978.
[14] G. Kempf and L. Ness. The length of vectors in representation spaces. In *Algebraic Geometry, Proceedings, Copenhagen*, volume 732 of *Lecture Notes in Mathematics*, pages 233–242. Springer-Verlag, Berlin, Heidelberg, New York, 1978.
[15] F.C. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*. Mathematical Notes. Princeton University Press, 1984.
[16] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*. Springer-Verlag, Berlin, Heidelberg, New York, 3rd edition, 1994.
[17] H. Ochiai. Quotients of some prehomogeneous vector spaces. *J. Algebra*, 192(1):61–73, 1997.
[18] M. Sato and T. Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya Math. J.*, 65:1–155, 1977.
[19] T. Shintani. On Dirichlet series whose coefficients are class-numbers of integral binary cubic forms. *J. Math. Soc. Japan*, 24:132–188, 1972.
[20] T. Shintani. On zeta-functions associated with vector spaces of quadratic forms. *J. Fac. Sci. Univ. Tokyo, Sect IA*, 22:25–66, 1975.
[21] C.L. Siegel. The average measure of quadratic forms with given discriminant and signature. *Ann. of Math.*, 45:667–685, 1944.
[22] T. A. Springer. Some groups of type E_7. *Nagoya Math. J.*, 182:259–284, 2006.
[23] T.A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1998.
[24] K. Tajima and A. Yukie. Stratification of the null cone in the non-split case. *Comment. Math. Univ. St. Pauli*, 63(1-2):261–276, 2014.
[25] K. Tajima and A. Yukie. On the GIT stratification of prehomogeneous vector spaces I. *Comment. Math. Univ. St. Pauli*, 68(1):1–48, 2020.
[26] K. Tajima and A. Yukie. On the GIT stratification of prehomogeneous vector spaces II. *Tsukuba J. Math.*, 44(1):1–62, 2020.
[27] K. Tajima and A. Yukie. On the GIT stratification of prehomogeneous vector spaces III. *Kyushu J. Math.*, 76(2):209–368, 2022.
[28] D Witte, A. Yukie, and R. Zierau. Prehomogeneous vector spaces and ergodic theory II. *Trans. Amer. Math. Soc.*, 352(4):1687–1708, 2000.
[29] D.J. Wright and A. Yukie. Prehomogeneous vector spaces and field extensions. *Invent. Math.*, 110(2):283–314, 1992.
[30] A. Yukie. Prehomogeneous vector spaces and ergodic theory I. *Duke Math. J.*, 90(1):123–148, 1997.
[31] A. Yukie. Prehomogeneous vector spaces and field extensions III. *J. Number Theory*, 67(1):115–137, 1997.

[32] A. Yukie. On Shintani zeta functions for GL(2). *Trans. Amer. Math. Soc.*, 350(12):5067–5094, 1998.

[33] A. Yukie. On the density theorem related to the space of non-split tri-Hermitian forms I. *J. Number Theory*, 194:117–169, 2019.

[34] A. Yukie. Rational orbits of primitive trivectors in dimension six. *Tohoku Math. J. (2)*, 71(1):35–52, 2019.

[35] A. Yukie. On the density theorem related to the space of non-split tri-Hermitian forms II. *Manuscripta Math.*, 162(1):221–239, 2020.

[36] Akihiko Yukie. On the density of unnormalized Tamagawa numbers of orthogonal groups III. *J. Number Theory*, 238:557–610, 2022.

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