距離関数の空間はベールである
-SPACES OF METRICS ARE BAIRE-

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Contents

1. This manuscript is a survey on comeager subsets in spaces of metrics

This manuscript is devoted to a survey on comeager subsets in spaces of metrics. In this survey, we first explain the history of spaces of metrics. Next, we exhibit results on comeager subsets in spaces of metrics. Combining these results and the fact that spaces of metrics are Baire, then we can obtain the existence and abundance of spaces satisfying specific properties.

2. Preliminaries

We first introduce the basic notations of spaces of metrics. Most of parts are the same to several sections of the author's preprint [25].

For a set X, a map $d\colon X \times X \to [0, \infty)$ is called a *pseudometric* if the following conditions are true:

(1) for every $x \in X$, we have $d(x, x) = 0$;

(2) for every pair $x, y \in X$, we have $d(x, y) = d(y, x)$;

(3) for every triple $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X, d) is called a *pseudometric space*. If d satisfies the additional condition:

(4) for every pair $x, y \in X$, the equality $d(x, y) = 0$ implies $x = y$,

then d is called a *metric*.

For a topological space X, let $\mathrm{CPM}(X)$ denote the set of all continuous maps $d\colon X \times X \to [0, \infty)$ such that d is a pseudometric on X. We also denote by $\mathrm{Met}(X)$ the set of all metrics d on X generating the same topology of X. This space is our main subject. Notice that $\mathrm{Met}(X) \subseteq \mathrm{CPM}(X)$, and X is metrizable if and only if $\mathrm{Met}(X) \neq \emptyset$. Next we introduce the topology to $\mathrm{Met}(X)$. We define $\mathcal{D}_X\colon \mathrm{CPM}(X)^2 \to [0, \infty]$ by $\mathcal{D}_X(d, e) = \sup_{x,y \in X} |d(x, y) - e(x, y)|$. Note that $\mathcal{D}_X$ can take the value ∞, but, using ϵ-open balls, we can define the topology induced by $\mathcal{D}_X$ as in the cases of ordinary metrics. This topology coincides with the topology of uniform convergence. In this paper, we represent the restricted metric $\mathcal{D}_X |_{\mathrm{Met}(X)^2}$ as the original symbol $\mathcal{D}_X$. In what follows, we consider that $\mathrm{CPM}(X)$ and $\mathrm{Met}(X)$ are equipped with the topologies induced by $\mathcal{D}_X$. Namely, we consider $\mathrm{CPM}(X)$ and $\mathrm{Met}(X)$ have the topologies of uniform convergence.

For a pseudometric space (X, d), for a point $x \in X$, and for $r \in (0, \infty)$, put $U(x, r; d) = \{ p \in X \mid d(x, p) < r \}$, which is the open ball centered at x with radius r.

Next, let us review ultrametrics (non-Archimedean metrics).

A pseudometric $d\colon X \times X \to [0, \infty)$ is said to be a *pseudo-ultrametric* or *non-Archimedean pseudometric* if d satisfies the so-called the strong triangle inequality: $d(x, y) \leq d(x, z) \vee d(z, y)$ for all $x, y, z \in X$, where the symbol "$\vee$" means the maximum operator on $\mathbb{R}$, i.e., $x \vee y = \max\{x, y\}$. A pair (X, d) is called a *pseudo-ultrametric space*. A pseudo-ultrametric d on X is called an *ultrametric* or *non-Archimedean metric* if the equality $d(x, y) = 0$ implies $x = y$. Of course, every ultrametric is a metric.

A set R is said to be a *range set* if $R \subseteq [0, \infty)$ and $0 \in R$. We say that a range set R is *characteristic* if for every $z \in (0, \infty)$, there exists $r \in R \setminus \{0\}$ such that $r < z$. This condition is equivalent to $\inf(R \setminus \{0\}) = 0$. A metric d on X is said to be *R-valued* if $d(x, y) \in R$ for all $x, y \in X$.

For a topological space X, and for a range set R, we denote by $\mathrm{UCPM}(X, R)$ the all R-valued continuous maps $d\colon X \times X \to [0, \infty)$ for which d is a pseudo-ultrametric on X. We also denote by $\mathrm{UMet}(X; R)$ the all R-valued ultrametrics d on X. Notice that $\mathrm{UMet}(X; R) \subseteq \mathrm{UCPM}(X, R)$. When considering non-Archimedean analogues, it is often more effective to give a limitation on the range of metrics (see, for example, [10]). Namely, we will think not only ($[0, \infty)$-valued) ultrametrics but also R-valued ultrametrics for an arbitrary range set R.

For a range set R, a topological space X is said to be *R-valued ultrametrizable* if $\mathrm{UMet}(X; R) \neq \emptyset$. When $R = [0, \infty)$, the space X is simply said to be *ultrametrizable*.

Remark 2.1. In [18, Proposition 2.14], it was shown that X is ultrametrizable if and only if for every characteristic range set R, the space X is R-valued ultrametrizable ($\mathrm{UMet}(X; R) \neq \emptyset$).

We define $\mathcal{UD}_X^R\colon \mathrm{UCPM}(X,R)^2 \to [0,\infty]$ by declaring that $\mathcal{UD}_X^R(d,e)$ is the infimum of all $\epsilon \in R$ such that $d(x,y) \leq e(x,y) \vee \epsilon$ and $e(x,y) \leq d(x,y) \vee \epsilon$ for all $x, y \in X$. Then $\mathcal{UD}_X^R$ is an ultrametric on $\mathrm{UCPM}(X,R)$ taking values in $[0,\infty]$. Since $\mathcal{D}_X(d,e)$ is the same to the infimum ϵ such that $d(x,y) \leq e(x,y) + \epsilon$ and $e(x,y) \leq d(x,y) + \epsilon$, the ultrametric $\mathcal{UD}_X^R$ is a non-Archimedean analogue of the supremum metric $\mathcal{D}_X(d,e)$ in the sense of replacing "+" with "$\vee$". Similarly to $\mathcal{D}_X$, we can define the topology induced by $\mathcal{UD}_X^R$ using open balls. In this paper, we represent the restricted metric $\mathcal{UD}_X^R|_{\mathrm{UMet}(X;R)^2}$ as the original symbol $\mathcal{UD}_X^R$. In what follows, we consider that $\mathrm{UCPM}(X,R)$ and $\mathrm{UMet}(X;R)$ are equipped with the topologies induced by $\mathcal{UD}_X^R$. This topology is strictly stronger than the topology of uniformly convergence. It could be called the *topology of non-Archimedean uniformly convergence.*

Remark 2.2. Let R be a range set, and X be an R-valued ultrametrizable space. Then we have the inclusions $\mathrm{UMet}(X;R) \subseteq \mathrm{Met}(X)$ and $\mathrm{UCPM}(X,R) \subseteq \mathrm{CPM}(X)$. For every pair $d, e \in \mathrm{UCPM}(X,R)$, we also obtain $\mathcal{D}_X(d,e) \leq \mathcal{UD}_X^R(d,e)$. Except for the case where X is empty or one-point, the topology generated by $\mathcal{UD}_X^R(d,e)$ is always strictly stronger than that generated by $\mathcal{D}_X$.

For a topological space X, for a range set R, and for an open covering $\mathcal{C}$ of X, we define $\mathrm{UL}(\mathcal{C};R) = \mathrm{UCPM}(X,R) \cap \mathrm{L}(\mathcal{C})$.

3. History of space of metrics

3.1. **History.** Next let us review the history of research on spaces of metrics.

As long as the author knows, the concept of spaces of metrics first appeared in 1944, as a Shanks' work [36] proving that for every pair X and Y of compact metrizable spaces, $\mathrm{Met}(X)$ is isometric to $\mathrm{Met}(Y)$ if and only if X is homeomorphic to Y([36, Theorem 3.2]). This result is an analogue of Banach–Stone–Eilenberg theorem, which states that for every pair of compact Hausdorff spaces X and Y, the spaces $C(X)$ and $C(Y)$ of real-valued continuous functions with supremum metrics are isometric to each other if and only of X is homeomorphic to Y.

About half of a century later, Šalát, Tóth, and Zsilinszky [43] began to investigate spaces of all possible metrics on given sets. During the 1990s, some mathematicians follows this subject (see [43], [44], [42], [6], and [40]). Remark that this space of metrics depends only on the cardinality of an underlying set. Under our notation, they considered that the space $(\mathrm{CPM}(X), \mathcal{D}_X)$ for a discrete topological space X. Let us explain some of their results. For example, Šalát, Tóth, and Zsilinszky [43] proved that, for a discrete space (a set) X, the set of uniformly discrete metrics (metrics which positive values are uniformly bounded below) is open dense in $\mathrm{CPM}(X)$.

Starting in 2020, in contrast, the author (Y. Ishiki) considered the set of topological metrics; namely, for a metrizable space X, the space $\mathrm{Met}(X)$ of metrics generating the same topology of X equipped with the supremum distance $\mathcal{D}_X$. Although it was not known whether $\mathrm{Met}(X)$ is Baire or not, the author clarified the denseness and Borel hierarchy of a subset $\{\, d \in \mathrm{Met}(X) \mid (X,d) \text{ satisfies } \mathcal{P} \,\}$ for a certain property $\mathcal{P}$ on metric spaces, and proved that some subsets are comeager in $\mathrm{Met}(X)$ ([17], [18], [19], [21], [22], and [26]). We give explanation focusing on the author's papers.

The paper [17] was a first one investigating $\mathrm{Met}(X)$. In that paper, the author showed that the set of all metrics in $\mathrm{Met}(X)$ having Assouad dimension ∞ (equivalently, non-doubling) is dense and G_δ. It was also show that if X is locally compact and second-countable, then $\mathrm{Met}(X)$ is completely metrizable, in particular, it is Baire. Note that, except when X is the empty set or the one-point space, the supremum metric $\mathcal{D}_X$ is not complete on $\mathrm{Met}(X)$. We will describe known results on comeager subsets in Section 5. The paper [18] handled non-Archimedean analogues of Hausdorff's metric extension and theorem appearing in the previous paper [17]. The author [19] clarified the denseness and Borel hierarchy of the sets of doubling metrics, uniformly disconnected metrics, and uniformly perfect metrics. The paper [22] proved that, the set of metrics taking values in a disconnected subset of reals is comeager. At the same time, the author [21] showed the extension theorem for proper functions and proper metrics. This paper indicated the possibility of the theory of spaces of proper metrics. In [20] and [23], the author proved that the space $(\mathrm{UCPM}(X, R), \mathcal{UD}_X^R)$ of continuous pseudo-ultrametrics is isometric to the Urysohn universal ultrametric sapce. This work does not directly relate to the theory of spaces of metrics. However, the idea of using continuous pseudometrics was applied in the later paper [25], where the author showed that spaces of metrics are Baire, and the set of complete metrics is comeager in the space of metrics. This paper [25] established the author's theory of comeager subsets of metrics.

As applications of infinite-dimensional topology, recently, Koshino researched topological types of spaces of metrics equipped with not only the uniform topologies but also the compact-open topologies ([30], [31], and [32]). Let us assert one of Koshino's results.

Theorem 3.1 ([30, Corollary 1])**.** *If X is a separable metrizable space of cardinality κ, then*

(1) $\mathrm{Met}(X)$ *is homeomorphic to* $[0,1)^{\kappa(\kappa-1)/2}$ *if* $\kappa < \infty$*;*

(2) $\mathrm{Met}(X)$ *is homeomorphic to* ℓ^2 *if X is compact;*

(3) $\mathrm{BMet}(X)$ *is homeomorphic to* ℓ^∞ *if X is not compact, where* $\mathrm{BMet}(X)$ *is the set of all bounded metrics in the space* $\mathrm{Met}(X)$*.*

In the context of Lipschitz-free Banach spaces (it is also called Arens–Eells spaces, or 1-Wasserstein spaces), there are several works on spaces of metrics (see [37], [38], and [11, Problem 6.6]).

3.2. **Why** $\mathrm{Met}(X)$**?** In this subsection, we review the author's observation in 2020, made while preparing the preprint [17], in which we first investigated comeager subsets in the space $\mathrm{Met}(X)$ of metrics. The author's motivation of research on $\mathrm{Met}(X)$ stems from the following mathematical subjects.

(A) The theory of moduli spaces. Specifically, the Gromov–Hausdorff space (see [5]), and space of Riemannian metrics (see [8]).

(B) The theory of Baire spaces (see [2]). In particular, Banach and Mazurkiewicz's proofs of the existence (denseness) of nowhere differentiable functions (see [3] and [33]).

(C) Differential Topology. Specifically, transversality theorems, and Sard's theorem (see, for example, [14]). For the difference between measure and category, see [35].

(D) Vershik's result [41] on universal metric in the space of metrics on $\mathbb{Z}_{\geq 0}$ equipped with a measure.

(E) Hausdorff's metric extension theorem [13], and its improvements.

In what follows, we explain each item of (A)–(E).

3.2.1. *Item* (A). Since there already have existed a theory of moduli spaces related to metric spaces, such as the Gromov–Hausdorff space, and spaces of Riemannian metrics, the author thought that we can make the theory of moduli spaces of metrics. Considering the author's paper [16] on quasi-symmetric maps, which is a generalization of (quasi-)conformal maps appearing in Teichmüller spaces, we could say our research also comes from the theory of Teichmüller spaces.

3.2.2. *Item* (B). Let us recall the Banach and Mazurkiewicz theorem asserting that the set of nowhere differential functions is comeager in the function space on $[0, 1]$. As a corollary of this theorem, we can obtain the existence of nowhere differentiable functions. Of course, we can make those functions concretely, using, for example, Weierstrass' method. The author would like to emphasizes that Banach and Mazurkiewicz's theorem indicates that the theory of Baire spaces is a framework that gives us a systematic method to show the existence and the abundance of special objects in topological spaces. Based on this observation, the author planned to prove the abundance of "strange" metrics in spaces of metrics using Baire spaces.

3.2.3. *Item* (C). There is another branch of methods to show the existence and the abundance of special objects in terms of measure theory. In particular, we focus on applications of measure theory to differentiable topology. Sard's theorem states that for a sufficiently smooth map $f\colon M \to N$, the set of critical values of f is small in the sense of measure. Transversality theorem is a development of Sard's theorem, which, roughly speaking, says that under certain conditions, there are so many "good" maps between differentiable manifolds. For example, the abundance of Morse function can be obtained as a consequence of transversality theorems. The author did not major in differential topology; however, these theorems in differential topology inspired the author to construct the theory of spaces of metrics, and to show the abundance of "good" metrics in spaces of metrics, contrasting with (B). Remark that transversality theorems and its corollaries state the abundance of "good" objects whereas Baire category theorem and its corollaries implies that the abundance of "bad" objects. Here, we notice that there is a binary opposition, an analogy, or a duality, between the theory of Baire spaces and measure theory as aspects of method to show the abundance of special objects. The Oxtoby's book [35] deals with analogues of the theory of Baire spaces and measure theory.

3.2.4. *Item* (D). Vershik [41, Theorem 4] proved that, almost all (in the sense of measure) elements d of $\mathrm{CPM}(\mathbb{Z}_{\geq 0})$ satisfy that the completion of (X, d) is isometric to the Urysohn universal ultrametric space. Based on Vershik's result, and considering a duality between Baire spaces and measure theory explained above, there should be a theory of spaces of metrics from a point of view of Baire spaces. The author's theory on spaces of metrics can be regarded as one of counterparts of Vershiks' result.

3.2.5. *Item* (E). The author believes that Item (E) is most important among those items. Let us recall Hausdorff's metric extension theorem.

Theorem 3.2 ([13]). *Let X be a metrizable space, and A be a closed subset of X. If d is a metric on A that generates the same topology of A, then there exists a metic D on X that generates the same topology of X and satisfies that $D|_{A^2} = d$.*

This theorem is an analogue of the Tietze–Urysohn extension theorem. The Tietze–Urysohn theorem, or the existence of a partition of unity, is used to investigate the function spaces on a normal spaces. Thus, when the author knew this theorem, the author thought there should be spaces of metrics because we already have got an extension theorem, and Hausdorff's theorem would be useful for a study of spaces of metrics. However, it was an optimistic consideration. Indeed, to research the space $\mathrm{Met}(X)$ of metrics in the author's first paper [17] on spaces of metrics, we need more strong form of extension theorem of metrics:

Theorem 3.3 ([17, Theorem 1.1]). *Let X be a metrizable space, and let $\{A_i\}_{i\in I}$ be a discrete family of closed subsets of X. Then for every metric $d \in \mathrm{Met}(X)$, and for every family $\{e_i\}_{i\in I}$ of metrics with $e_i \in \mathrm{Met}(A_i)$, there exists a metric $m \in \mathrm{Met}(X)$ satisfying the following:*

(1) *for every $i \in I$ we have $m|_{A_i^2} = e_i$;*

(2) $\mathcal{D}_X(m, d) = \sup_{i\in I} \mathcal{D}_{A_i}(e_{A_i}, d|_{A_i^2})$.

Moreover, if X is completely metrizable, and if each $e_i \in \mathrm{Met}(A_i)$ is a complete metric, then we can choose $m \in \mathrm{Met}(X)$ as a complete metric on X.

To prove Theorem 3.3, the author used analogues between extensions of metrics and extensions of functions. Hausdorff's theorem (Theorem 3.2) is an analogue of Tietze–Urysohn's theorem, and there exists a proof of Theorem 3.2 using Dugunsji's extension theorem, which is a generalization of Tietze–Urysohn's theorem. Theorem 3.3 can be considered as an analogue of Katětov–Tong's insertion theorem of real-valued functions (see [28] and [39]). Thus, the author thought that we could make use of a generalization of Tietze–Urysohn's theorem. In fact, Michael's continuous selection theorem enables us to show Theorem 3.3. Subsequently, we took the first step in researching comeager subsets of spaces of metrics.

For more information on relationships between extensions of metrics and functions, we refer the readers to [15]. We exhibits the relationships between extensions of metrics and extensions of functions (see Figure 1) under the following abbreviations:

- Theorems on extension of functions:

  (TU): Tietze–Urysohn theorem (see [46, Theorem 15.8]).

  (D): Dugundji's extension theorem [7].

  (KT): Katětov and Tong's Insertion theorem ([28] and [39]).

  (M): Michael's continuous selection theorems for paracompactness [34].

  (F) and (Y): Franz's theorem [9] and Yamazaki's theorem [47] on extensions of functions involving zero sets.

  (P): Tietze–Urysohn theorem for proper functions [21].

- Theorems on extension of metrics:

  [H]: Hausdorff's metric extension theorem [13].

  [S]: Simultaneous extension of metrics [29].

  [I]: Insertion theorem of metrics (an interpolation theorem of metrics) [17].

  [PM]: an extension theorem on proper metrics [21].

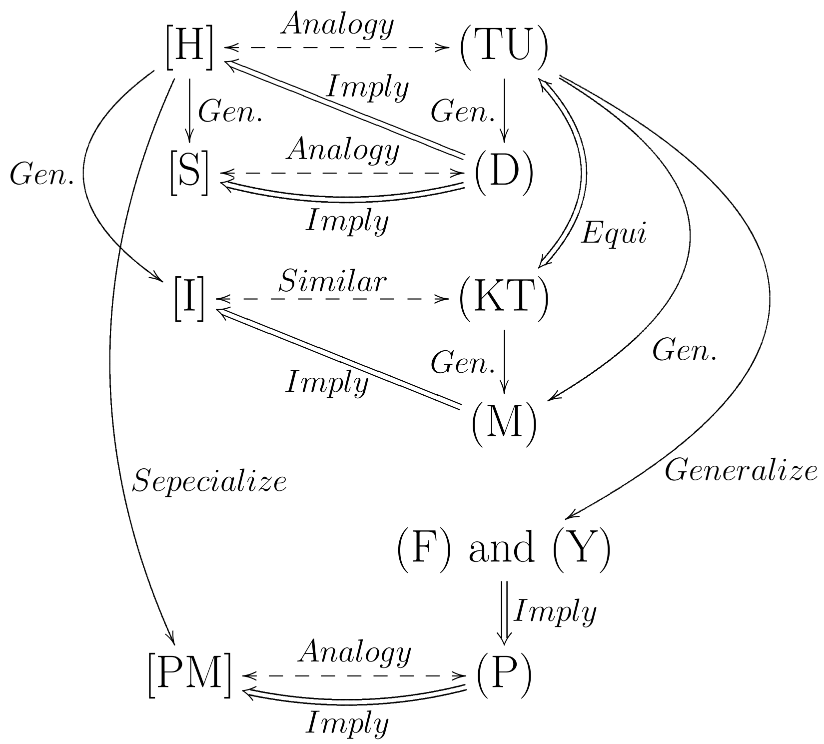

Figure 1. Relationships between extension theorems of functions and metrics

4. Recent progress in spaces of metrics

After RIMS symposium, in July 2024, for every metrizable space X, and every closed subset A of X, the author [24] recently constructed an extensor of metrics

$$E\colon \mathrm{Met}(A) \to \mathrm{Met}(X)$$

such that $\mathcal{D}_X(E(d), E(e)) = \mathcal{D}_A(d, e)$ for all $d, e \in \mathrm{Met}(A)$. The proof is based on three constructions, ℓ^1-products, Wasserstein spaces, and L^1-like spaces, and also based on Whitney–Dugundji decomposition of metric spaces.

In September 2024, as an application of the author's extension theorem of metrics, the author and Katsuhisa Koshino established a joint work [27], which including, for example, as one of main results, a theorem asserting that every compact metric space can be isometrically embedded into $(\mathrm{Met}(X), \mathcal{D}_X)$, where X is an arbitrary uncountable Polish space.

5. Comeager subsets in spaces of metrics

5.1. **Spaces of metrics are Baire.** In this subsection, we discuss Baire-ness of spaces of metrics.

We first explain the partial results on Baire-ness of spaces of metrics.

In 2020, the author proved that the following theorems on the complete metrizability of $\mathrm{Met}(X)$.

Theorem 5.1 ([17, Lemma 5.1]). *Let X be a second-countable locally compact Hausdorff space. Then $\mathrm{Met}(X)$ is completely metrizable. Specifically, it is Baire.*

Soon afterwards, Koshino established the next stronger result.

Theorem 5.2 ([30, Proposition 3]). *Let X be a σ-compact metrizable space. Then $\mathrm{Met}(X)$ is completely metrizable. Specifically, it is Baire.*

We next explain the result on Baire-ness of spaces metrics obtained in [25]. For a pseudometric space (X, d), and a covering $\mathcal{C} = \{C_i\}_{i\in I}$ of X, we say that a positive real number $r \in (0, \infty)$ is a *Lebesgue number of* $\mathcal{C}$ if for every $x \in X$ there exists $i \in I$ such that $U(x, r; d) \subseteq C_i$.

For a topological space X, and for a covering $\mathcal{C}$ of X, we denote by $\mathrm{L}(\mathcal{C})$ the set of all $d \in \mathrm{CPM}(X)$ such that $\mathcal{C}$ has a (positive) Lebesgue number with respect d.

Theorem 5.3 ([25, Theorem 1.1]). *Let X be a metrizable space, and $\mathcal{C}$ an open cover. Then the set $\mathrm{L}(\mathcal{C})$ is open dense in $\mathrm{CPM}(X)$.*

Let X be a metrizable space, and $w \in \mathrm{CPM}(X)$. We define $\mathrm{I}(w)$ the set of all $d \in \mathrm{CPM}(X)$ such that $1_X\colon (X, d) \to (X, w)$ is uniformly continuous, where 1_X stands for the identity map. Namely, $d \in \mathrm{I}(w)$ if and only if for every $\epsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that for every pair $x, y \in X$, the inequality $d(x, y) < \delta$ implies $w(x, y) < \epsilon$. As a consequence of Theorem 5.3, we prove that $\mathrm{I}(w)$ is comeager in $\mathrm{CPM}(X)$ (compare with the proof of [30, Proposition 3]).

Theorem 5.4 ([25, Theorem 1.2]). *Let X be a metrizable space, $w \in \mathrm{CPM}(X)$. Then the set $\mathrm{I}(w)$ is comeager in $\mathrm{CPM}(X)$.*

Theorem 5.4 implies that the following two theorems:

Theorem 5.5 ([25, Theorem 1.3])**.** *Let X be a metrizable space. Then* $\mathrm{Met}(X)$ *is comeager in* $\mathrm{CPM}(X)$*. In particular, the space* $\mathrm{Met}(X)$ *is Baire itself.*

Theorem 5.6 ([25, Theorem 1.4])**.** *Let X be a completely metrizable space. Then the set* $\mathrm{Comp}(X)$ *is comeager in* $\mathrm{CPM}(X)$*. Hence, it is also comeager in* $\mathrm{Met}(X)$*.*

We also obtain a non-Archimedean analogues of those theorems. For a topological space X, for a range set R, and for an open covering $\mathcal{C}$ of X, we define $\mathrm{UL}(\mathcal{C}; R) = \mathrm{UCPM}(X, R) \cap \mathrm{L}(\mathcal{C})$.

The next theorem is a non-Archimedean analogue of Theorem 5.3. *Ultraparacompactness* is a non-Archimedean analogue of paracompactness. A space if ultraparacompact if and only if it is paracompact and has covering dimension 0.

Theorem 5.7. *Let R be a range set, X an ultraparacompact Hausdorff space, and $\mathcal{C}$ an open covering of X. Then the set* $\mathrm{UL}(\mathcal{C}; R)$ *is open and dense in* $\mathrm{UCPM}(X, R)$*.*

Let R be a range set, and X an R-valued metrizable space, and take $w \in \mathrm{CPM}(X)$. Notice that w is not necessarily non-Archimedean. We define $\mathrm{UI}(w, R)$ the set of all $d \in \mathrm{UCPM}(X, R)$ such that $1_X \colon (X, d) \to (X, w)$ is uniformly continuous.

We also obtain an analogue of Theorem 5.4 for ultrametrics.

Theorem 5.8. *Let R be a range set, and X an R-valued ultrametrizable space, and take $w \in \mathrm{CPM}(X)$ (w is not necessarily non-Archimedean). Then the set* $\mathrm{UI}(w, R)$ *is comeager in* $\mathrm{UCPM}(X, R)$*.*

The following theorem is corresponding to Theorem 5.5.

Theorem 5.9. *Let R be a range set, X an R-valued ultrametrizable space. Then* $\mathrm{UMet}(X; R)$ *is comeager in* $(\mathrm{UCPM}(X, R), \mathcal{UD}_X^R)$*. In particular, the moduli space* $\big(\mathrm{UMet}(X; R), \mathcal{UD}_X^R\big)$ *is Baire.*

For a topological space X, and for a range set R, put

$$\mathrm{UComp}(X; R) = \mathrm{UMet}(X; R) \cap \mathrm{Comp}(X).$$

The next result is an analogue of Theorem 5.6.

Theorem 5.10. *Let R be a range set, X a completely metrizable and R-valued ultrametrizable space. Then* $\mathrm{UComp}(X; R)$ *is comeager in* $(\mathrm{UCPM}(X, R), \mathcal{UD}_X^R)$*. Moreover, the set* $\mathrm{UComp}(X; R)$ *is also comeager in* $(\mathrm{UMet}(X; R), \mathcal{UD}_X^R)$*.*

Recently, Koshino proved a duality of absolute Borel complexity of X and $\mathrm{Met}(X)$, and, as a corollary, he obtain:

Theorem 5.11 ([32, Corollary])**.** *Let X be a separable metrizable space. Then X is σ-compact if and only if* $\mathrm{Met}(X)$ *is completely metrizable.*

5.2. **Comeager subsets in spaces of metrics.** In this section, we exhibit known comeager subsets in spaces of metrics.

For a property $\mathcal{P}$ on metric spaces, we consider that

$$\{\, d \in \mathrm{Met}(X) \mid (X, d) \text{ has the property } \mathcal{P} \,\}.$$

We exhibit the table of properties $\mathcal{P}$ such that the comeager-ness of the set $\{\, d \in \mathrm{Met}(X) \mid (X, d) \text{ has } \mathcal{P} \,\}$ is already known.

Table 1: Table of comeager sets in $\mathrm{Met}(X)$

Reference	Assumptions on X	Property $\mathcal{P}$
[17, Theorem 4.12.]	X is not discrete.	d is non-doubling.
[17, Theorem 4.12.]	X is not discrete.	d is non-uniformly disconnected
[17, Cor 4.17, and Prop 4.18]	X is not discrete.	d is not satisfying the strong triangle inequality.
[17, Cor 4.17 and Prop 4.19]	X is not discrete.	d is not satisfying the Ptolemy inequality.
[17, Cor 4.17 and Prop 4.20.]	X is not discrete.	d is not satisfying the Gromov $\mathrm{Cycl_m}(0)$ condition.
[17, Theorem 4.15]	X is not discrete.	d is having rich pseudo-cones property. Namely, the set of all pseudo-cone of (X, d) is the same to the whole of Gromov–Hausdorff space.
[17, Thm 1.3 and Exam 1.1]	X is locally non-discrete. Namely, every open subsets is non-discrete.	every open subset is (1) non-doubling, (2) non-uniformly disconnected, (3) not satisfying the strong triangle inequality, (4) not satisfying Ptolemy inequality, (5) not satisfying the Gromov $\mathrm{Cycl_m}(0)$ condition, and (6) having rich pseudo-cones property.
[19, Thm 1.4]	X is the Cantor set.	d is non-uniformly perfect.

Table 1: Table of comeager sets in $\mathrm{Met}(X)$

[22, Thm 1.1]	X is metrizable and having the large inductive dimension 0.	The set $\{ d(x,y) \mid x, y \in X \}$ is closed and totally disconnected in $\mathbb{R}$.
[22, Thm 1.2]	X is metrizable and having the large inductive dimension 0.	d is *gap-like*, i.e., for every $p \in X$, the set $\{ d(p,x) \mid x \in X \}$ is not dense in any neighborhood of 0 in $[0, \infty)$.
[26, Thm 1.2]	X is metrizable and having the large inductive dimension 0 with $\mathrm{Card}(X) \leq 2^{\aleph_0}$.	d is *strongly rigid*, i.e., for all distinct $\{x,y\}, \{u,v\} \in [X]^2$, we have $d(x,y) \neq d(u,v)$.
[26, Thm 1.3]	X is σ-compact, metrizable and having the large inductive dimension 0 with $3 \leq \mathrm{Card}(X) \leq 2^{\aleph_0}$.	d is *rigid*, i.e., every isometric bijection $f\colon (X,d) \to (X,d)$ must be the identity map. Equivalently, the self-isometry group of (X,d) is trivial.
[25, Thm 1.4]	X is completely metrizable	d is complete.

Combining results placed in the table and Theorem 5.5, we obtain the following result on the abundance "strange" metrics in spaces of metrics.

Corollary 5.12. *Let X be a σ-compact, non-discrete, metrizable space have large inductive dimension* 0 *and satisfying* $3 \leq \mathrm{Card}(X) \leq 2^{\aleph_0}$*. Then the set of all $d \in \mathrm{Met}(X)$ satisfying the following conditions is comeager in X, especially, it is non-empty.*

(1) *non-doubling;*
(2) *non-uniformly disconnected;*
(3) *not being ultrametrics;*
(4) $\{d(x,y) \mid x, y \in X\}$ *is closed and totally disconnected in* $[0, \infty)$*;*
(5) *strongly rigid;*
(6) *complete.*

5.3. **Comeager subsets in spaces of ultrametrics.** The author also have obtained comeager subsets in spaces of ultrametrics. Similarly to the Archimedean case, we also exhibit the table of $\mathcal{P}$ such that $\{ d \in \mathrm{UMet}(X; R) \mid X \text{ has } \mathcal{P} \}$. In this table, we always assume that X is R-valued ultrametrizable, i.e., $\mathrm{UMet}(X; R) \neq \emptyset$.

Table 2: Table of comeager sets in $\mathrm{UMet}(X; R)$

Reference	Assumptions on X	Property $\mathcal{P}$
[18, Proposition 6.9]	X is not discrete.	d is non-doubling.
[18, Theorem 4.15]	X is not discrete.	d is having R-rich-pseudo-cones property. Namely, the set of all pseudo-cone of (X, d) is the same to the whole of Gromov–Hausdorff space.
[18, Theorem 7.7]	X is locally non-discrete. Namely, every open subsets is non-discrete.	every open subset is (1) non-doubling, (2) having rich R-pseudo-cones property.
[19, Theorem 1.5]	X is the Cantor set.	d is non-uniformly perfect.
[25, Theorem 1.8]	X is completely metrizable	d is complete.

6. Questions

It is interesting to think $\mathrm{Met}(X)$ is always Borel in its completion $\mathrm{CPM}(X)$.

Question 6.1. Is $\mathrm{Met}(X)$ always Borel in $\mathrm{CPM}(X)$?

Related to Question 6.1, we cite a conjecture made in [24].

Conjecture 6.2. Recall that $\aleph_1$ stands for the first uncountable cardinality, and let $D_{\aleph_1}$ denote the the discrete space of cardinality of $\aleph_1$. Under this notations, the space $\mathrm{Met}(D_{\aleph_1})$ is not completely metrizable.

This conjecture is motivated by the aim to remove the assumption of the separability of X in Theorem 5.11.

Take a non-separable metrizable space X. Then X contains $D_{\aleph_1}$ as a closed subset. Thus, $\mathrm{Met}(X)$ contains $\mathrm{Met}(D_{\aleph_1})$ as a closed subset due to the main result in [24]. If Conjecture 6.2 is true, then the space $\mathrm{Met}(X)$ would not be completely metrizable. Namely, the complete metrizability of $\mathrm{Met}(X)$ would imply the separability of X. This observation is a reason why the author supports Conjecture 6.2. I am eager for someone to solve this conjecture.

We also cite more questions from [24]. A metric on a set Z is said to be *proper* if every bounded set in (Z, d) is compact. For a metrizable space X, we denote by $\mathrm{PrMet}(X)$ the set of all $d \in \mathrm{Met}(X)$ that is proper. In the paper [21], the author obtained an analogue of Hausdorff's metric extension theorem for proper metrics. It is interesting to ask whether we construct a simultaneous extension of proper metrics or not.

Question 6.3. Let X be a second-countable locally compact Hausdorff space, and A be a closed subset of X. Does there exist an extensor $F\colon \mathrm{Met}(A) \to \mathrm{Met}(X)$ satisfying the conclusions of the main result of [24]. and the additional condition that $F(\mathrm{PrMet}(A)) \subseteq \mathrm{PrMet}(X)$?

If we could obtain the sophisticated extension theorem of proper metrics, we would be able to investigate comeager subsets of spaces of proper metrics.

Question 6.4. Similarly to $\mathrm{Met}(X)$, can we investigate the topology and comeager subsets of the space $\mathrm{PrMet}(X)$ of proper metrics equipped with the supremum metric?

We are also interested in standard forms of comeager subsets.

Question 6.5. Let X be a metrizable space. For every comeager subset S of $\mathrm{Met}(X)$, does there exists a countable family $\{\mathcal{C}\}_{i\in\mathbb{Z}_{\geq 0}}$ of open covers of X such that

$$\bigcap_{i\in\mathbb{Z}_{\geq 0}} \mathrm{L}(\mathcal{C}_i) \subseteq S?$$

This question is motivated by the characterization of (co)meager sets in the Cantor set [4, Theorem 5.2].

The author believes there are intriguing relationships between the big metric spaces such as the Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$, the Urysohn universal metric space $(\mathbb{U}, \rho)$, and the hyperspace $(\mathcal{K}(X), \mathcal{HD}_d)$. We also consider their non-Archimedean analogues.

Question 6.6. Are there relationships between spaces of metrics and other big metric spaces? See Figures 2 and 3.

For the statement that the Gromov–Hausdorff space $(\mathcal{M}, \mathcal{GH})$ is isometric to the quotient metric space of hyperspace of the Urysohn universal metric space $(\mathbb{U}, \rho)$, we refer the readers to [12, Exercise (b) in the page 83] and [1, Theorem 3.4].

For the isometric equivalences between non-Archimedean Gromov-Hausdorff spaces $(\mathcal{U}_R, \mathcal{NA})$ and non-Archimedean Urysohn universal metric spaces $(\mathbb{V}_R, \sigma_R)$, see [45] and [23].

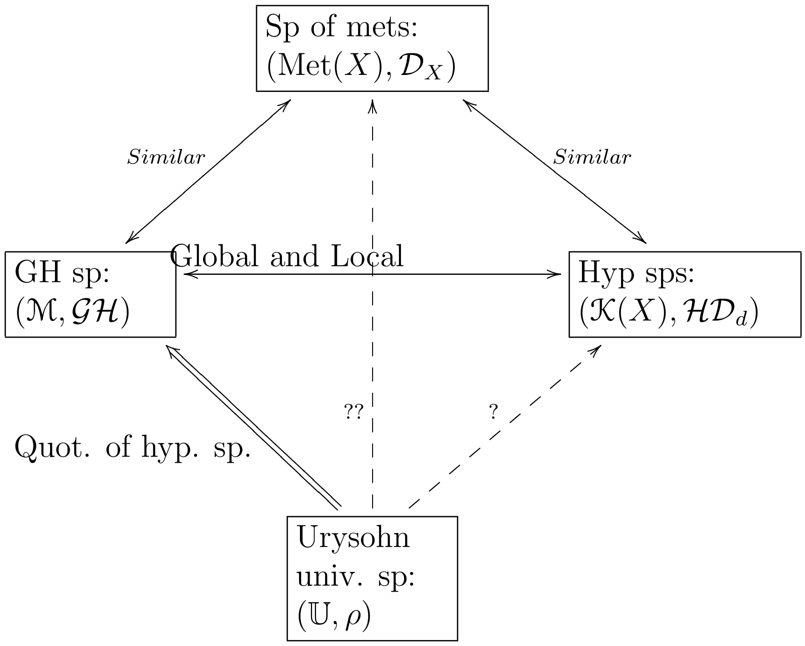

FIGURE 2. Relationships between big metrics spaces in the Archimedean world

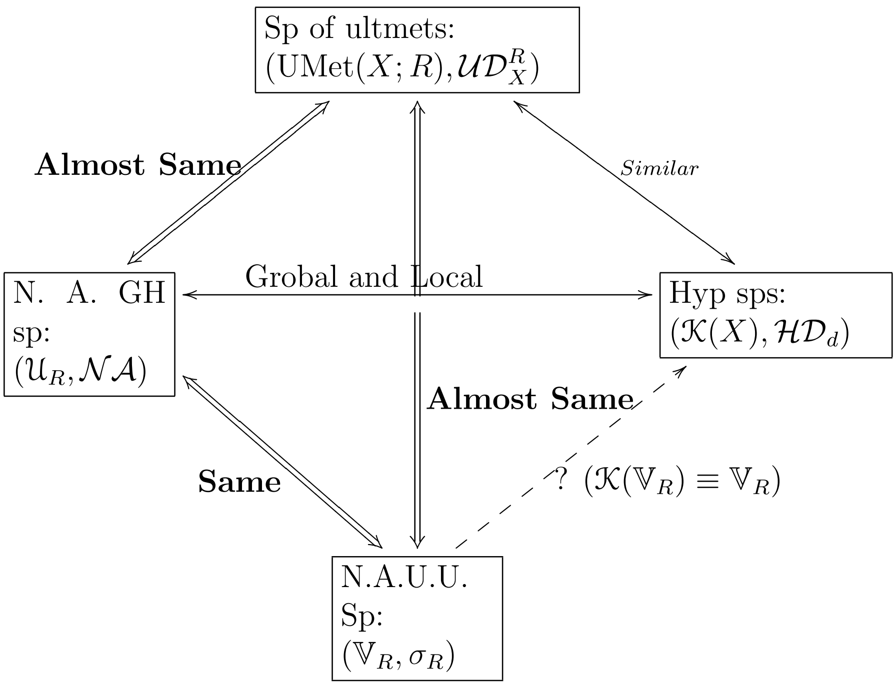

FIGURE 3. Relationships between big metrics spaces in the Non-Archimedean world

Recently, several mathematicians are researching moduli spaces of metric measure spaces. So the author wants to make a bridge between my theory and mm-spaces.

Question 6.7. Can we obtain analogues of the above results for metric measure spaces?

In the end, the author would like to give an advice on research on spaces of metrics. The author thinks that $\mathrm{Met}(X)$ is too big for geometric research. So, if the readers want to investigate $\mathrm{Met}(X)$, then it is slightly (more) reasonable to consider only spaces $\mathrm{BMet}(X)$ of bounded metrics.

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