Boundedness of bundle diffeomorphism groups over a circle — Quasimorphisms induced from the rotation angle

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1 Backgroud

The algebraic properties of diffeomorphism groups of manifolds and their subgroups are studied by many people. As a classical result, many of these groups are known to be perfect and some of them are even simple. In more quantitative consideration of these properties, boundedness of diffeomorphism groups (uniform perfectness, uniform simplicity, evaluation of commutator length, etc) are studied by D. Burago, S. Ivanov and L. Polterovich (2008) in dimension 3 ([6]) and T. Tsuboi (2008 - 2012) ([19, 20, 21]) in dimension $\neq 2, 4$. After these works, many authors have studied boundedness of various kind of automorphism groups of manifolds with a structure. (groups of equivariant diffeomorphisms, leaf preserving diffeomorphisms, etc).

In this expository article we are concerned with diffeomorphism groups of manifold pairs (K. Abe and K. Fukui [1, 3, 4, 5]) and bundle diffeomorphism groups (K. Fukui and T. Yagasaki [12]). Our aim is a comprehensive study of boundedness of these groups ([12, 13, 14]). In a consideration of these groups, the one dimensional cases, (i.e., bundle diffeomorphism groups over a circle and diffeomorphism groups of manifold pairs (M, L) with L being a finite disjoint union of circles) are studied commonly using quasimorphisms induced from the rotation angle on circles. In each case, the quasimorphism induces an invariant, which detects boundedness of the group. This exposition will focus on this common strategy.

Throughout this article we work in the C^∞ category, though some conclusions also hold in the C^r category ($r \in \mathbb{Z}_{\geq 0}$). We refer to [8] for generalities on quasimorphisms.

2 Quasimorphisms induced from the rotation angle on a circle

In order to introduce the rotation angle on a topological circle S^1 we need to fix an orientation and an arc-length measure on S^1. If we normalize as the total length of $S^1 = 1$, then these data are equivalent to a choice of a universal covering $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \approx S^1$ up to the choice of a base point in S^1. Below we fix a universal covering $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \approx S^1$.

Every path $c : I \to S^1$ admits a lift $\widetilde{c} : I \to \mathbb{R}$. The rotation angle of c is defined by $\lambda(c) := \widetilde{c}(1) - \widetilde{c}(0)$. This quantity is independent of the choice of the lift $\widetilde{c}$. Note that $c_1 \simeq_* c_2$, then $\lambda(c_1) = \lambda(c_2)$ and that if c is a loop (a closed path), then $\lambda(c) = \deg c$.

We fix a distinguish point p. For any isotopy $F : S^1 \times I \to S^1$ the rotation angle of F (with respect to p) is defined by $\mu(F) \equiv \mu_p(F) := \lambda(F(p, *))$. Then the map

$\mu : \mathrm{Isot}(S^1)_0 \to \mathbb{R} : F \mapsto \mu(F)$

is a quasimorphism of defect 1 and it restricts to a surjective group homomorphism

$\mu| : \mathrm{Isot}(S^1)_{\mathrm{id},\mathrm{id}} \to \mathbb{Z}$.

Since $\mu(F) = \mu(G)$ if $F \simeq_* G$, the map μ reduces to the map on the universal covering of $\mathrm{Diff}(S^1)_0$

$\widetilde{\mu} : \widetilde{\mathrm{Diff}}(S^1)_0 \to \mathbb{R} : \widetilde{\mu}([F]) = \mu(F)$,

which is also a quasimorphism of defect 1 and it restricts to a surjective group isomorphism

$\widetilde{\mu}| = \deg : \pi_1(\mathrm{Diff}(S^1)_0) \cong \mathbb{Z}$.

As explained in the subsequent sections, the quasimorphisms μ and $\widetilde{\mu}$ induce the associated quasimorphisms on the following groups (cf. [5, 12]).

(1) for a fiber bundle $\pi : M \to S^1$
  the group $\mathrm{Isot}_\pi(M)_0$ of bundle isotopies of π and
  the universal covering group $\widetilde{\mathrm{Diff}}_\pi(M)_0$
  of the group $\mathrm{Diff}_\pi(M)_0$ of bundle diffeomorphisms of π

(2) for a manifold pair (M, L) with L a finite disjoint union of circles in M
  the group $\mathrm{Isot}(M, L)_0$ of isotopies of (M, L) and
  the universal covering group $\widetilde{\mathrm{Diff}}(M, L)_0$
  of the group $\mathrm{Diff}(M, L)_0$ of diffeomorphisms of (M, L)

3 Boundedness of $\mathrm{Diff}_\pi(M)_0$ for a fiber bundle $\pi : M \to S^1$

Suppose $\pi : M \to S^1$ is a fiber bundle with fiber N and structure group $\Gamma < \mathrm{Diff}(N)$. We refer to [12] for notations and resutls in this section.

3.1 Quasimorphism $\widetilde{\nu}$ on $\widetilde{\mathrm{Diff}}_\pi(M)_0$

Each $f \in \mathrm{Diff}_\pi(M)$ determines a diffeomorphism $\underline{f} \in \mathrm{Diff}(S^1)_0$. This correspondence yields the surjective group homomorphisms

$P : \mathrm{Diff}_\pi(M)_0 \longrightarrow \mathrm{Diff}(S^1)_0, \ \ P(f) = \underline{f}$ and

$P_I : \mathrm{Isot}_\pi(M)_0 \longrightarrow \mathrm{Isot}(S^1)_0, \ \ P_I(F) = \underline{F} := (\underline{F_t})_{t \in I}$.

If $F \simeq_* G$ in $\mathrm{Isot}_\pi(M)_0$, then $\underline{F} \simeq_* \underline{G}$ in $\mathrm{Isot}(S^1)_0$. Hence we also have the surjective group homomorphism

$\widetilde{P} : \widetilde{\mathrm{Diff}}_\pi(M)_0 \longrightarrow \widetilde{\mathrm{Diff}}(S^1)_0, \ \ \widetilde{P}([F]) = [\underline{F}]$.

These induce the quasimorphisms of defect 1

$\nu := \mu \circ P_I : \mathrm{Isot}_\pi(M)_0 \longrightarrow \mathbb{R}$ and $\widetilde{\nu} := \widetilde{\mu} \circ \widetilde{P} : \widetilde{\mathrm{Diff}}_\pi(M)_0 \longrightarrow \mathbb{R}$.

These are included in the following diagram :

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Isot}_\pi(M)_{\mathrm{id}_M,\mathrm{id}_M} & \subset & \mathrm{Isot}_\pi(M)_0 & \xrightarrow{R} & \mathrm{Diff}_\pi(M)_0 & \longrightarrow & 1 \\
 & & \downarrow \ \searrow \nu| & & \downarrow \ \searrow \nu & & \| & & \\
1 & \longrightarrow & \pi_1\mathrm{Diff}_\pi(M)_0 & \subset & \widetilde{\mathrm{Diff}}_\pi(M)_0 & \xrightarrow{\widetilde{R}} & \mathrm{Diff}_\pi(M)_0 & \longrightarrow & 1 \\
 & & \widetilde{\nu}| \downarrow \ \swarrow & & \widetilde{\nu} \downarrow \ \swarrow & & \downarrow \widehat{\nu} & & \\
0 & \longrightarrow & k\mathbb{Z} & \subset & \mathbb{R} & \longrightarrow & \mathbb{R}/k\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

Here, $\widetilde{R}([F]) = F_1$, $\widehat{\nu}(F_1) = [\nu(F)]$ $(F \in \mathrm{Isot}_\pi(M)_0)$ and we have $\mathbb{Z} > \mathrm{Im}\,\nu| = k\mathbb{Z}$ for a unique $k \in \mathbb{Z}_{\geq 0}$. This integer $k = k(\pi) \in \mathbb{Z}_{\geq 0}$ can be used to detect the boundedness of the group $\mathrm{Diff}_\pi(M)_0$. We do not take the homogenization of the quasimorphism ν, since we are concerned with the values $\nu(F)$ and $\widehat{\nu}(f)$ themselves to deduce some evaluations of the commutator length $cl f$ etc.

3.2 Application of the function $\widehat{\nu}$ to the boundedness of the group $\mathrm{Diff}_\pi(M)_0$

We are concerned with the estimate of conjugation invariant norms cl, clb_π and ζ_g ($g \in \mathrm{Diff}_\pi(M)_0 - \mathrm{Ker}\,P$) and the diameters of $\mathrm{Diff}_\pi(M)_0$ with respect to these norms. Here, clb_π is the commutator length with support in arcs in S^1 and ζ_g is the conjugation generated norm with respect to g. Note that $cl \leq clb_\pi$ by their definition.

Consider the following basic condition on (N, Γ).

($*$) $\mathrm{Diff}_{pr,c}(\mathbb{R} \times N)_0$ is perfect for the product (N, Γ)-bundle $pr : \mathbb{R} \times N \longrightarrow \mathbb{R}$.

At the moment the condition ($*$) is known to hold in the following cases

(i) a principal G bundle with G a compact Lie group ([2])

(ii) N is a closed manifold and $\Gamma = \mathrm{Diff}(N)$ ([12, 17, 18])

Proposition 3.1. Suppose $f \in \mathrm{Diff}_\pi(M)_0$ and $\widehat{\nu}(f) = [s] \in \mathbb{R}/k\mathbb{Z}$, where $s \in \left(-\frac{k}{2}, \frac{k}{2}\right]$ in the case $k \in \mathbb{Z}_{\geq 1}$. Then, we have the following estimates.

(1) (i) $cl\, f \geq \dfrac{1}{4}(\lfloor |s| \rfloor + 2)$ (ii) $\zeta_g(f) \leq 4\, clb_\pi\, f$ for any $g \in \mathrm{Diff}_{\pi,c}(M)_0 - \mathrm{Ker}\,P$

(2) $clb_\pi f \leq 2\lfloor |s| \rfloor + 3$ if (N, Γ) satisfies the condition ($*$).

Here, due to our convention, $cl\, f = \infty$ for $f \notin \left[\mathrm{Diff}^r_\pi(M)_0, \mathrm{Diff}^r_\pi(M)_0\right]$ and $cld\, \mathrm{Diff}^r_\pi(M)_0 = \infty$ when $\mathrm{Diff}^r_\pi(M)_0$ is not perfect. Similar conventions are applied to clb_π and ζ_g.

Corollary 3.1. If (N, Γ) satisfies the condition ($*$), then the group $\mathrm{Diff}_\pi(M)_0$ is simple relative to $\mathrm{Ker}\,P$.

The boundedness of the group $\mathrm{Diff}_\pi(M)_0$ is detected by the invariant $k = k(\pi, r)$.

[**1**] The case that $k = 0$:

In this case, the map $\widehat{\nu} : \mathrm{Diff}_\pi(M)_0 \to \mathbb{R}$ itself is a surjective quasimorphism of defect 1 and it restricts to a surjective group homomorphism $\widehat{\nu} : \mathrm{Ker}\,P \longrightarrow \mathbb{Z}$. Hence we have the following conclusion.

Theorem 3.1. If $k = 0$, then $\mathrm{Diff}_\pi(M)_0$ is unbounded and not uniformly perfect.

[2] The case that $k \geq 1$:

Theorem 3.2. Suppose (N, Γ) satisfies the condition $(*)$. Then the following holds.

(1) $\frac{1}{8}(k+2) \leq cld\,\mathrm{Diff}_\pi(M)_0 \leq clb_\pi d\,\mathrm{Diff}_\pi(M)_0 \leq k+3$.

(2) $\mathrm{Diff}_\pi(M)_0$ is uniformly simple rel. $\mathrm{Ker}\,P$, and so it is bounded and uniformly perfect.

Any (N, Γ) bundle π over S^1 is represented as a mapping torus M_φ associated to some attaching map $\varphi \in \Gamma$. We can describe $k(\pi)$ in term of the attaching map φ and construct some explicit examples of (un)bounded bundle diffeomorphism groups over S^1.

4 Boundedness of $\mathrm{Diff}(M, L)_0$ for a finite disjoint union L of circles in M

4.1 Previous results

In [1, 3, 4, 5] K. Abe and K. Fukui studied the uniform perfectness of the group $\mathrm{Diff}(M, N)_0$ of diffeomorphisms of a manifold pair (M, N). Suppose M is a connected closed manifold of dim ≥ 2 and N is a proper submanifold of M of dim ≥ 1. Let N_i $(i \in [m])$ denote the connected components of N.

The group $\mathrm{Diff}(M, N)_0$ is not simple. In fact, it admits the restriction maps

$$P : \mathrm{Diff}(M, N)_0 \longrightarrow \mathrm{Diff}(N)_0 \quad \text{and} \quad P_i : \mathrm{Diff}(M, N)_0 \longrightarrow \mathrm{Diff}(N_i)_0 \ \ (i \in [m]).$$

These are surjective group homomorphisms and $\mathrm{Diff}(M, N)_0$ includes the normal subgroups $\mathrm{Ker}\,P$ and $\mathrm{Ker}\,P_i$ $(i \in [m])$.

They showed that the group $\mathrm{Diff}(M, N)_0$ is perfect and obtained the following results for the uniform perfectness.

Theorem 4.1.

(1) If $\mathrm{Diff}(N)_0$ and $\mathrm{Diff}_c(M - N)_0$ are uniformly perfect and $|\pi_0 \mathrm{Ker}\,P| < \infty$, then $\mathrm{Diff}(M, N)_0$ is uniformly perfect.

(2) In the case $\dim N = 1$ (i.e, N is a finite disjoint union of circles), if $|\pi_0 \mathrm{Ker}\,P| = \infty$, then $\mathrm{Diff}(M, N)_0$ admits a unbounded quasimorphism, so that it is not uniformly perfect.

Their proof of Theorem 4.1 (2) was based on a quasimorphism on $\mathrm{Diff}(M, N)_0$ induced from the quasimorphism $\mu : \mathrm{Isot}(S^1)_0 \to \mathbb{R}$ in Section 2. They used the criterion on $|\pi_0 \mathrm{Ker}\,P|$ in Theorem 4.1 to show that for a knot K of the 3-sphere S^3 the group $\mathrm{Diff}(S^3, K)_0$ is uniformly perfect if and only if K is a torus knot.

Our aim is a more comprehensive study of the boundedness of the group $\mathrm{Diff}(M, N)_0$ (effective evaluations of the norms cl, clb and ζ_g, the uniform relative simplicity, etc) (cf. [14]). We can show that the fragmentation lemma and the simplicity of $\mathrm{Diff}(M, N)_0$ relative to $\mathcal{S} := \bigcup_{i \in [m]} \mathrm{Ker}\,P_i$ due to the standard arguments.

The group $\mathrm{Diff}(M,N)_0$ includes the normal subgroup $\mathcal{G} := \mathrm{Diff}_c(M-N)_0$. Since the manifold $M-L$ is the interior of a compact manifold with nonempty boundary, we can apply our results in [11] for the boundedness of the group $\mathcal{G}$. Then, for each $f \in \mathrm{Diff}(M,N)_0$ it is natural to seek a factorization $f = gh$ such that $g \in \mathcal{G}$ and $\mathrm{supp}\, h$ is contained in a small neighborhood of N and deduce some estimates on $cl\, f$, $clb\, f$ and $\zeta_g(f)$.

Below we focus on the case where $\dim N = 1$ and only discuss the boundedness of $\mathrm{Diff}(M,N)_0$ module $\mathcal{G}$ to simplify the explanation.

4.2 Quasimorphism $\widetilde{\nu}$ on $\widetilde{\mathrm{Diff}}(M,L)_0$

We consider a pair (M,L) such that M is a connected closed manifold of dim ≥ 2 and L is a finite disjoint union of circles L_i $(i \in [m])$ in M. For each $i \in [m]$ we have

the restriction maps $\quad P_i : \mathrm{Diff}(M,L)_0 \longrightarrow \mathrm{Diff}(L_i)_0, \quad P_{I,i} : \mathrm{Isot}(M,L)_0 \longrightarrow \mathrm{Isot}(L_i)_0,$

the induced map $\quad \widetilde{P}_i : \widetilde{\mathrm{Diff}}(M,L)_0 \longrightarrow \widetilde{\mathrm{Diff}}(L_i)_0 : \ \widetilde{P}_i([F]) = [F|_{L_i \times I}].$

These are surjective group homomorphisms and induce the quasimorphisms of defect 1,

$$\nu_i := \mu \circ P_{I,i} : \mathrm{Isot}(M,L)_0 \longrightarrow \mathbb{R} \quad \text{and} \quad \widetilde{\nu}_i := \widetilde{\mu} \circ \widetilde{P}_i : \widetilde{\mathrm{Diff}}(M,L)_0 \longrightarrow \mathbb{R}.$$

Finally we obtain the vector-valued quasimorphisms

$$\nu = (\nu_i)_{i \in [m]} : \mathrm{Isot}(M,L)_0 \longrightarrow \mathbb{R}^m \quad \text{and} \quad \widetilde{\nu} = (\widetilde{\nu}_i)_{i \in [m]} : \widetilde{\mathrm{Diff}}(M,L)_0 \longrightarrow \mathbb{R}^m.$$

These are included in the following diagram :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Isot}(M,L)_{\mathrm{id}_M,\mathrm{id}_M} & \subset & \mathrm{Isot}(M,L)_0 & \xrightarrow{R} & \mathrm{Diff}(M,L)_0 & \longrightarrow & 1 \\
 & & \downarrow \quad \searrow{\scriptstyle \nu|} & & \downarrow \quad \searrow{\scriptstyle \nu} & & \| & & \\
1 & \longrightarrow & \pi_1\mathrm{Diff}(M,L)_0 & \subset & \widetilde{\mathrm{Diff}}(M,L)_0 & \xrightarrow{\widetilde{R}} & \mathrm{Diff}(M,L)_0 & \longrightarrow & 1 \\
 & & {\scriptstyle \widetilde{\nu}|}\downarrow & & {\scriptstyle \widetilde{\nu}}\downarrow & & \downarrow{\scriptstyle \widehat{\nu}} & & \\
0 & \longrightarrow & A & \subset & \mathbb{R}^m & \longrightarrow & \mathbb{R}^m/A & \longrightarrow & 0
\end{array}$$

where $\widetilde{R}([F]) = F_1$, $\widehat{\nu}(F_1) = [\nu(F)]$ $(F \in \mathrm{Isot}(M,L)_0)$ and $A := \mathrm{Im}\, \nu| < \mathbb{Z}^m$. Here, we note that the quasimorphism ν has no information for the subgroup $\mathcal{G} \equiv \mathrm{Diff}_c(M-L)_0$.

Since $\pi_0 \mathrm{Ker}\, P \cong \mathbb{Z}^m/A$, from Theorem 4.1 it follows that the group $\mathrm{Diff}(M,L)_0$ is uniformly perfect modulo $\mathcal{G}$ if and only if $\mathrm{rank}\, A = m$. If $\mathrm{rank}\, A < m$, the group $\mathrm{Diff}(M,L)_0$ admits a quasimorphism onto $\mathbb{R}$, so that $\mathrm{Diff}(M,L)_0$ is unbounded.

4.3 Application of the function $\widehat{\nu}$ to the boundedness of the group $\mathrm{Diff}(M,L)_0$

Due to Epstein ([7]) we need an effective evaluation of the norm clb on $\mathrm{Diff}(M,L)_0$ modulo $\mathcal{G}$ to detect the uniform simplicity of $\mathrm{Diff}(M,L)_0$ relative to $\mathcal{S}$ (that is, uniform boundedness of the norms ζ_g $(g \in \mathrm{Diff}(M,L)_0 - \mathcal{S})$). This also yields an evaluation of $cld\, \mathrm{Diff}(M,L)_0$ modulo $\mathcal{G}$. We can deduce some estimate of $clb\, f$ $(f \in \mathrm{Diff}(M,L)_0)$ in term of the affine lattice $\widehat{\nu}(f) = \nu(F) + A \subset \mathbb{R}^m$ $(F \in \mathrm{Isot}(M,L)_{\mathrm{id},f})$.

In the case that $\operatorname{rank} A = m$, the quotient group $\mathbb{Z}^m/A$ is a finite abelian group and we can introduce the quantity : $\quad k := \max_{i\in[m]} k_i \in \mathbb{Z}_{\geq 1}$,
where e_i $(i \in [m])$ is the standard basis of $\mathbb{R}^m$ and $k_i := \operatorname{ord}[e_i] \in \mathbb{Z}_{\geq 1}$ in $\mathbb{Z}^m/A$ $(i \in [m])$.

Theorem 4.2. ([14]) If $\operatorname{rank} A = m$, then

(1) $clb\,(f \bmod \mathcal{G}) \leq 2\lfloor k/2 \rfloor + 3 \quad (f \in \operatorname{Diff}(M,N)_0)$.

(2) $cld\,\operatorname{Diff}(M,N)_0 \leq 2\lfloor k/2 \rfloor + 3 + cld\,\mathcal{G}$ and $clbd\,\operatorname{Diff}(M,N)_0 \leq 2\lfloor k/2 \rfloor + 3 + clbd\,\mathcal{G}$.

(3) $\operatorname{Diff}(M,N)_0$ is bounded and uniformly simple relative to $\mathcal{S} = \bigcup_{i\in[m]} \operatorname{Ker} P_i$, when $\dim M \neq 2, 4$. Moreover, $cld\,\operatorname{Diff}(M,N)_0 \leq 2\lfloor k/2 \rfloor + 7$, when $\dim M$ is odd.

In the case $m = 1$ (i.e., L consists of a circle K), we have $A = k\mathbb{Z}$ for a unique $k \in \mathbb{Z}_{\geq 0}$. Then, $\operatorname{rank} A < m$ if and only if $k = 0$. In [1] it is shown that for a knot K in the 3-sphere S^3, $\operatorname{Diff}^\infty(S^3, K)_0$ is uniformly perfect if and only if K is a torus knot. We can extend this example to the following form.

Example 4.1. Suppose M is a closed connected C^∞ n-manifold $(n \geq 2)$ and K is a circle in M.

(1) Suppose M admits a smooth S^1 action ϱ such that $K = S^1 \cdot p$ for some point p of K and the orbit map $\varrho_p : S^1 \to K$, $\varrho_p(z) = z \cdot p$, has degree ℓ (up to $\pm$). Then, $\ell \in k\mathbb{Z}$ and $k|\ell$. In fact, the S^1 action ϱ induces an isotopy $F \in \operatorname{Isot}(M,K)_{\mathrm{id,id}}$: $F(x,t) = e^{2\pi i t} \cdot x$. It follows that $k\mathbb{Z} \ni \nu(F) = \deg F_p = \deg \varrho_p = \ell$.

   (i) In the case M is a Seifert fibered 3-manifold, if K a regular fiber, then $k = 1$ and if K is a (p,q) multiple fiber, then $k|p$.

   (ii) In particular, if K is a torus knot in S^3, then $k = 1$, since K is a regular fiber of a standard Seifert fibering of S^3 with two multiple fibers.

(2) $k = 0$ in the following cases :

   (†) (a) $\pi_1(M)$ has a trivial center and (b) $\pi_1(K) \to \pi_1(M)$ is injective.

   (‡) $n = 3$

      (a) $\pi_1(M - K)$ has a trivial center and

      (b) $\pi_1(D - K) \to \pi_1(M - K)$ is injective for a tubular neighborhood D of K in M.

   (i) Any non-torus knot K in S^3 satisfies the condition (‡).

   (ii) The following example (M,K) satisfies the condition (†) for $n \geq 4$. Suppose G is a finitely presented group such that $Z(G) = 1$ and G includes an element a of infinite order (for example, $G = \mathbb{Z} * H$, H is a nontrivial group). Since $n \geq 4$, there exists a closed connected C^∞ n-manifold M with $\pi_1(M) \cong G$. Take a circle K in M which represents the element a in $\pi_1(M)$, so that the inclusion $i : K \subset M$ induces an isomorphism $i_* : \pi_1(K) \cong \langle a \rangle < \pi_1(M)$.

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