Equivariant Irregular Riemann–Hilbert Correspondence

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Abstract

We propose definitions of an equivariant category of algebraic $\mathbb{C}$-constructible enhanced subanalytic sheaves and an equivariant algebraic enhanced de Rham functor. Moreover, we prove that its functor induces an equivalence of categories between the equivariant derived category of holonomic $\mathcal{D}$-modules and the equivariant category of algebraic $\mathbb{C}$-constructible enhanced subanalytic sheaves. Furthermore, as a small application of this equivalence, we give an approach to the proof of the well-known fact about equivariant algebraic coherent $\mathcal{D}$-modules.

1 Introduction

The original Riemann–Hilbert problem asks for the existence of a linear ordinary differential equation with regular singularities and a given monodromy on a curve.

In [Del], P. Deligne formulated it as a correspondence between meromorphic connections on a complex manifold X with regular singularities along a hypersurface Y and local systems on $X \setminus Y$.

Moreover, in [Kas84], M. Kashiwara extended Deligne's correspondence as an equivalence of categories between the triangulated category of regular holonomic $\mathcal{D}$-modules and that of $\mathbb{C}$-constructible sheaves which is called the regular Riemann–Hilbert correspondence:

$$\mathrm{DR}_X \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X),\ \mathcal{M} \mapsto \Omega_X \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M},$$

where DR_X is the de Rham functor and Ω_X is the de Rham complex.

After the appearance of the regular Riemann–Hilbert correspondence, A. Beilinson and J. Bernstein developed systematically a theory of regular holonomic $\mathcal{D}$-modules on smooth algebraic varieties over $\mathbb{C}$ and obtained an algebraic version of the regular Riemann–Hilbert correspondence which is called the algebraic regular Riemann–Hilbert correspondence. See [Be, Bor] and also [Sai] for the details.

Note that by using the equivariant version of the algebraic regular Riemann–Hilbert correspondence, Brylinski–Kashiwara[BK] and Beilinson–Bernstein[BB] independently solved the Kazhdan–Lusztig conjecture, which was of course a great breakthrough in representation theory.

In this paper, we propose an equivariant version of the algebraic irregular Riemann–Hilbert correspondence. These results are based on a joint work with Taito Tauchi of Aoyama Gakuin University [IT24]. The main definitions are Definition 7.1, 7.8 and the main theorem are Theorem 6.2, 7.7, 7.10 . Moreover, the application of these results is the proof of Fact 7.11.

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2 Irregular Riemann–Hilbert Correspondence

The problem of extending the regular Riemann–Hilbert correspondence to cover the case of holonomic $\mathcal{D}$-modules with irregular singularities had been open for 30 years.

2.1 Irregular Riemann–Hilbert Correspondence and Enhanced Ind-Sheaves

After a groundbreaking development in the theory of irregular meromorphic connections by K. S. Kedlaya [Ked10, Ked11] and T. Mochizuki [Moc09, Moc11], A. D'Agnolo and M. Kashiwara established the Riemann–Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules in [DK16] as follows.

We denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ the derived category of $\mathcal{D}_X$-modules and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X), \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ the triangulated category of coherent, holonomic $\mathcal{D}_X$-modules, respectively. Let us denote by Ω_X^{E} the enhanced de Rham complex which is defined in [DK16, Def. 8.2.1] and consider the functor from $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ to the triangulated category $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ of enhanced ind-sheaves on X which is called the enhanced de Rham functor:

$$\mathrm{DR}_X^{\mathrm{E}}(\mathcal{M})\colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X),\ \mathcal{M} \mapsto \Omega_X^{\mathrm{E}} \otimes^{\mathbf{L}}_{\pi^{-1}\beta_X\mathcal{D}_X} \pi^{-1}\beta_X\mathcal{M}.$$

See [DK16, Def. 9.1.1] for the details. We denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ the full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ consisting of $\mathbb{R}$-constructible enhanced ind-sheaves on X. See [DK16, Def. 4.9.2] for the definition of it. Then we have:

Theorem 2.1 ([DK16, Thm. 9.5.3, Prop. 9.5.4])**.** The enhanced de Rham functor induces a fully faithful functor:

$$\mathrm{DR}_X^{\mathrm{E}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) & \xhookrightarrow{\mathrm{DR}_X^{\mathrm{E}}} & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X) \\
\cup & & \uparrow{\scriptstyle e_X \circ \iota_X} \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}_X]{\sim} & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X),
\end{array}$$

where $e_X \circ \iota_X \colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ is the natural embedding functor (see [KS01, § 4.1] for the definition of ι_X and [DK16, Prop. 4.7.15] for the definition of e_X).

Furthermore, T. Mochizuki proved that the essential image of the fully faithful functor $\mathrm{DR}_X^{\mathrm{E}} \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ can be characterized by the curve test [Moc22]. See also [Kuwa21, Thm. 8.6] for T. Kuwagaki's another approach to the irregular Riemann–Hilbert correspondence.

On the other hand, in [Ito20], the author defined $\mathbb{C}$-constructibility for enhanced ind-sheaves on X and proved the following. We denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ the full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ consisting of $\mathbb{C}$-constructible enhanced ind-sheaves on X. See [Ito20, Def 3.19] for the definition of $\mathbb{C}$-constructible enhanced ind-sheaves.

Theorem 2.2 ([Ito20, Thm. 3.26], [Ito23, Prop. 3.1][1])**.** The enhanced de Rham functor $\mathrm{DR}_X^{\mathrm{E}}$ induces an equivalence of categories:

$$\mathrm{DR}_X^{\mathrm{E}} \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X)$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow[\sim]{\mathrm{DR}_X^{\mathrm{E}}} & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) \\
\cup & & \uparrow{\scriptstyle e_X \circ \iota_X} \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}_X]{\sim} & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X).
\end{array}$$

Remark that the author reproved the regular Riemann–Hilbert correspondence of Kashiwara by using the irregular Riemann–Hilbert correspondence (Theorem 2.2) in [Ito23].

2.2 Algebraic Irregular Riemann–Hilbert Correspondence

The author proved an algebraic version of Theorem 2.2 in [Ito21] as below.

Let X be a smooth algebraic variety over $\mathbb{C}$ and denote by $\widetilde{X}$ a smooth completion of X (i.e., $\widetilde{X}$ is a smooth complete algebraic variety over $\mathbb{C}$ which contains X as an open subvariety and $\widetilde{X} \setminus X$ is a normal crossing divisor of $\widetilde{X}$). We denote by $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_\infty})$ the triangulated category of enhanced ind-sheaves on a bordered space $X^{\mathrm{an}}_\infty = (X^{\mathrm{an}}, \widetilde{X}^{\mathrm{an}})$. Here, X^{an} (resp. $\widetilde{X}^{\mathrm{an}}$) is the underlying complex manifold of X (resp. $\widetilde{X}$). See [DK16, Def. 3.2.1] for the definition of bordered spaces, [DK21, § 2.6] and [KS16, Def. 2.12] for the details of enhanced ind-sheaves on bordered spaces.

Let us define the algebraic enhanced de Rham functor by

$$\mathrm{DR}^{\mathrm{E}}_{X_\infty} \colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty}),\ \mathcal{M} \mapsto \mathbf{E}(j_{X^{\mathrm{an}}_\infty})^{-1}\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}^{\mathrm{an}}}\left((\mathbf{D}j_{X*}\mathcal{M})^{\mathrm{an}}\right).$$

where $\mathbf{E}(j_{X^{\mathrm{an}}_\infty})^{-1} \colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{X}^{\mathrm{an}}}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_\infty})$ is the inverse image functor of the morphism $j_{X^{\mathrm{an}}_\infty} \colon X^{\mathrm{an}}_\infty \to \widetilde{X}^{\mathrm{an}}$ of bordered spaces which is induced by the open embedding $j^{\mathrm{an}}_X \colon X^{\mathrm{an}} \hookrightarrow$

[1] In [Ito20], although Theorem 3.26 and Proposition 3.1 were stated by using the enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}_X$, we can obtain a similar statement by using the enhanced de Rham functor $\mathrm{DR}^{\mathrm{E}}_X$.

$\widetilde{X}^{\rm an}$, $\mathbf{D}j_{X*}\colon \mathbf{D}^{\rm b}(\mathcal{D}_X) \to \mathbf{D}^{\rm b}(\mathcal{D}_{\widetilde{X}})$ is the direct image functor of the open embedding $j_X\colon X \hookrightarrow \widetilde{X}$ and $(\cdot)^{\rm an}\colon \mathbf{D}^{\rm b}(\mathcal{D}_{\widetilde{X}}) \to \mathbf{D}^{\rm b}(\mathcal{D}_{\widetilde{X}^{\rm an}})$ is the analytification functor.

We denote by $\mathbf{E}^{\rm b}_{\mathbb{C}\text{-}c}({\rm I}\mathbb{C}_{X_\infty})$ the full triangulated subcategory of $\mathbf{E}^{\rm b}({\rm I}\mathbb{C}_{X_\infty})$ consisting of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves on $X^{\rm an}_\infty$. See [Ito21, Defs. 3.1, 3.10] for the definition of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves.

Theorem 2.3 ([Ito21, Thm. 3.11, Prop. 3.14][2])**.** The algebraic enhanced de Rham functor ${\rm DR}^{\rm E}_{X_\infty}$ induces an equivalence of categories:

$$
{\rm DR}^{\rm E}_{X_\infty}\colon \mathbf{D}^{\rm b}_{\rm hol}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{E}^{\rm b}_{\mathbb{C}\text{-}c}({\rm I}\mathbb{C}_{X_\infty})
$$

and the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbf{D}^{\rm b}_{\rm hol}(\mathcal{D}_X) & \xrightarrow[\sim]{{\rm DR}^{\rm E}_{X_\infty}} & \mathbf{E}^{\rm b}_{\mathbb{C}\text{-}c}({\rm I}\mathbb{C}_{X_\infty}) \\
\cup & & \uparrow{\scriptstyle e_{X^{\rm an}_\infty}\circ\iota_{X^{\rm an}_\infty}} \\
\mathbf{D}^{\rm b}_{\rm rh}(\mathcal{D}_X) & \xrightarrow[{\rm DR}_X]{\sim} & \mathbf{D}^{\rm b}_{\mathbb{C}\text{-}c}(\mathbb{C}_X),
\end{array}
$$

where the second horizontal arrow is the algebraic regular Riemann–Hilbert correspondence and $e_{X^{\rm an}_\infty}\circ\iota_{X^{\rm an}_\infty}\colon \mathbf{D}^{\rm b}(\mathbb{C}_{X^{\rm an}}) \to \mathbf{E}^{\rm b}({\rm I}\mathbb{C}_{X^{\rm an}_\infty})$ is the natural embedding functor (see [DK16, The paragraph before Notation 3.2.11] for the definition of $\iota_{X^{\rm an}_\infty}$, [DK21, The paragraph before Notation 2.3] and [KS16, Def. 2.19] for the definition of $e_{X^{\rm an}_\infty}$).

Remark that the author reproved the algebraic regular Riemann–Hilbert correspondence by using the algebraic irregular Riemann–Hilbert correspondence (Theorem 2.3) in [Ito23].

3 Irregular Riemann–Hilbert Correspondence and Enhanced Subanalytic Sheaves

3.1 Subanalytic Sheaves

Let us briefly recall some basic notions and results of subanalytic sheaves on real analytic bordered spaces. References are made to [Kas16, §§3.4–3.7][3] and also [Ito24a, § 3.1]. See also [KS01, §6], [Pre08] for the notion of subanalytic sheaves on real analytic manifolds.

Let $M_\infty = (M, \check{M})$ be a real analytic bordered space. See [Ito24a, § 2.4] for the definition of it. We denote by $\mathbf{D}^{\rm b}(\mathbb{C}^{\rm sub}_{M_\infty})$ the derived category of the abelian category ${\rm Mod}(\mathbb{C}^{\rm sub}_{M_\infty})$ of subanalytic sheaves on M_∞. Note that there exists a natural left exact embedding $\rho_{M_\infty*}\colon {\rm Mod}(\mathbb{C}_M) \to {\rm Mod}(\mathbb{C}^{\rm sub}_{M_\infty})$. It has an exact left adjoint $\rho^{-1}_{M_\infty}$, that has in turn an exact fully faithful left adjoint functor $\rho_{M_\infty!}$. Moreover, the restriction $\rho^{\mathbb{R}\text{-}c}_{M_\infty*}$ of $\rho_{M_\infty*}$ to the category ${\rm Mod}_{\mathbb{R}\text{-}c}(\mathbb{C}_{M_\infty})$ of $\mathbb{R}$-constructible sheaves on M_∞ is exact.

[2]In [Ito21], although Theorem 3.11 and Proposition 3.14 were stated by using the enhanced solution functor ${\rm Sol}^{\rm E}_{X_\infty}$, we can obtain a similar statement by using the enhanced de Rham functor ${\rm DR}^{\rm E}_{X_\infty}$.

[3]In [Kas16], M. Kashiwara introduced the notion of subanalytic sheaves on subanalytic bordered spaces. However, in [Ito24a], the author only consider them on real analytic bordered spaces.

3.2 Main Result of [Kas16]

At the 16th Takagi lecture, M. Kashiwara explained a similar result of Theorem. 2.1 by using subanalytic sheaves instead of enhanced ind-sheaves as below.

Let X be a complex manifold and denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty})$ the derived category of subanalytic sheaves on a real analytic bordered space $X \times \mathbb{R}_\infty$. Here $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ and $\overline{\mathbb{R}}$ is the 2-point compactification of $\mathbb{R}$. Let us also denote by $\pi\colon X \times \mathbb{R}_\infty \to X$ the first projection.

Theorem 3.1 ([Kas16, Thm. 6.3])**.** There exists a fully faithful functor:

$$\mathrm{DR}^{\mathsf{T}}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}).$$

See [Kas16, § 5.4] for the definition of $\mathrm{DR}^{\mathsf{T}}_X$.

3.3 Enhanced Subanalytic Sheaves

Let us briefly recall some basic notions of enhanced subanalytic sheaves on bordered spaces and results on those. References are made to [Ito24a, § 3.3].

Let $M_\infty = (M, \check{M})$ be a real analytic bordered space. We denote by $\mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ the triangulated category of enhanced subanalytic sheaves on a bordered space M_∞. Note that there exists a fully faithful functor

$$\mathbf{R}^{\mathrm{E,sub}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}).$$

Note also that $\mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ has a standard t-structure $\big(\mathbf{E}^{\leq 0}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}), \mathbf{E}^{\geq 0}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})\big)$. We set

$$\mathbf{E}^{0}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) := \mathbf{E}^{\leq 0}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \cap \mathbf{E}^{\geq 0}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}).$$

For a morphism $f\colon M_\infty \to N_\infty$ of bordered spaces, we have the Grothendieck six operations $\overset{+}{\otimes}$, $\mathbf{R}\mathcal{I}hom^{+,\mathrm{sub}}, \mathbf{E}f^{-1}$, $\mathbf{E}f_*$, $\mathbf{E}f^!$, $\mathbf{E}f_{!!}$ for enhanced subanalytic sheaves on bordered spaces. Note that these functors have many properties as similar to classical sheaves. We shall skip the explanation of it. Moreover, we have a natural embedding

$$e^{\mathrm{sub}}_{M_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}),$$

see [Ito24a, Prop. 3.21] for the details.

We denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ the full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ consisting of $\mathbb{R}$-constructible enhanced subanalytic sheaves. See [Ito24a, Def. 3.19] for the definition of it. Then we have

Theorem 3.2 ([Ito24a, Thms. 3.15, 3.20])**.** There exists a fully faithful functor

$$I^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$$

which has $J^{\mathrm{E}}_{M_\infty}\colon \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty}) \to \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ as the right adjoint functor. Moreover, there exists an equivalence of categories:

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \underset{J^{\mathrm{E}}_{M_\infty}}{\overset{I^{\mathrm{E}}_{M_\infty}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{M_\infty}).$$

See also [Ito24a, Prop. 3.16] for the compatibilities between $I^{\mathrm{E}}_{M_\infty}, J^{\mathrm{E}}_{M_\infty}$ and the Grothendieck six operations.

3.4 Irregular Riemann–Hilbert Correspondence and Enhanced Subanalytic Sheaves for the Analytic Case

Let X be a complex manifold and set $\Omega_X^{\mathrm{E,sub}} := \pi^{-1}\rho_{X!}\Omega_X \otimes^{\mathbf{L}}_{\pi^{-1}\rho_{X!}\mathcal{O}_X} \mathcal{O}_X^{\mathrm{E,sub}}$, see [Ito24a, Def. 3.36] for the definition of $\mathcal{O}_X^{\mathrm{E,sub}}$. Then we obtain a functor

$$\mathrm{DR}_X^{\mathrm{E,sub}}\colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{E}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}}),\ \mathcal{M} \mapsto \pi^{-1}\rho_{X!}\mathcal{M} \otimes^{\mathbf{L}}_{\pi^{-1}\rho_{X!}\mathcal{O}_X} \Omega_X^{\mathrm{E,sub}}.$$

Note that for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ one has $\mathrm{DR}_X^{\mathrm{E,sub}}(\mathcal{M}) \simeq \mathbf{Q}_X^{\mathrm{sub}}\big(\mathrm{DR}_X^{\mathsf{T}}(\mathcal{M})\big)[1]$. Moreover, we have:

Theorem 3.3 ([Ito24a, Thms. 3.38, 3.39[4]])**.** The functor $\mathrm{DR}_X^{\mathrm{E,sub}}$ induces a fully faithful functor

$$\mathrm{DR}_X^{\mathrm{E,sub}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$$

and the following diagram is commutative:

$$\begin{array}{ccccc}
 & & & & \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}) \\
 & \xrightarrow{\mathrm{DR}_X^{\mathsf{T},\mathrm{sub}}(\cdot)[1]} & & & \uparrow \mathbf{R}_X^{\mathrm{E,sub}} \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}_X^{\mathrm{E,sub}}]{} & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}}) & \subset & \mathbf{E}^{\mathrm{b}}(\mathbb{C}_X^{\mathrm{sub}}) \\
 & \searrow^{\mathrm{DR}_X^{\mathrm{E}}} & I_X^{\mathrm{E}} \downarrow \wr & & \downarrow I_X^{\mathrm{E}} \\
 & & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X) & \subset & \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X).
\end{array}$$

Moreover, the author defined $\mathbb{C}$-constructibility for enhanced subanalytic sheaves, and prove that there exists an equivalence of categories between the triangulated category $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$ of $\mathbb{C}$-constructible enhanced subanalytic sheaves and that of holonomic $\mathcal{D}_X$-modules. See [Ito24b, Def. 4.16] for the definition of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$.

Theorem 3.4 ([Ito24b, Thm. 4.19, 4.22, Prop. 4.21][5])**.** There exists an equivalence of categories:

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}}) \underset{J_X^{\mathrm{E}}}{\overset{I_X^{\mathrm{E}}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X).$$

Moreover, the functor $\mathrm{DR}_X^{\mathrm{E,sub}}$ induces an equivalence of categories:

$$\mathrm{DR}_X^{\mathrm{E,sub}}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}})$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow[\sim]{\mathrm{DR}_X^{\mathrm{E,sub}}} & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\mathrm{sub}}) \\
\cup & & \uparrow e_X^{\mathrm{sub}}\circ \mathbf{R}\rho_{X*} \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}_X]{\sim} & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X).
\end{array}$$

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[4]In [Ito24a], although Theorems 3.38 and 3,39 were stated by using the enhanced solution functor $\mathrm{Sol}_X^{\mathrm{E,sub}}$, we can obtain a similar statement by using the enhanced de Rham functor $\mathrm{DR}_X^{\mathrm{E,sub}}$.

[5]In [Ito24b], although Theorem 4.22 was stated by using the enhanced solution functor $\mathrm{Sol}_X^{\mathrm{E,sub}}$, we can obtain a similar statement by using the enhanced de Rham functor $\mathrm{DR}_X^{\mathrm{E,sub}}$.

One can summarize the above results about the analytic irregular Riemann–Hilbert correspondence in the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & \mathbf{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X\times\mathbb{R}_\infty}) \\
 & \nearrow^{\mathrm{DR}^{\mathrm{T,sub}}_X(\cdot)[1]} & & & & & \uparrow \mathbf{R}^{\mathrm{E,sub}}_X \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}^{\mathrm{E,sub}}_X]{\sim} & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X) & \subset & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_X) & \subset & \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X) \\
 & \searrow^{\sim}_{\mathrm{DR}^{\mathrm{E}}_X} & I^{\mathrm{E}}_X \downarrow \wr & & I^{\mathrm{E}}_X \downarrow \wr & & I^{\mathrm{E}}_X \downarrow \\
 & & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_X) & \subset & \mathbf{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X) & \subset & \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X).
\end{array}$$

3.5 Irregular Riemann–Hilbert Correspondence and Enhanced Subanalytic Sheaves for the Algebraic Case

In [Ito24b], the author proved an algebraic version of Theorem 3.4.

Let X be a smooth algebraic variety over $\mathbb{C}$ again, and denote by $\widetilde{X}$ a smooth completion of X. Then we set

$$\mathrm{DR}^{\mathrm{E,sub}}_{X_\infty}\colon \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_\infty}),\ \mathcal{M} \mapsto \mathbf{E}(j_{X^{\mathrm{an}}_\infty})^{-1}\mathrm{DR}^{\mathrm{E,sub}}_{\widetilde{X}^{\mathrm{an}}}\left((\mathbf{D}j_{X*}\mathcal{M})^{\mathrm{an}}\right),$$

where $\mathbf{E}(j_{X^{\mathrm{an}}_\infty})^{-1}\colon \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\widetilde{X}^{\mathrm{an}}}) \to \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_\infty})$ is the inverse image functor of the morphism $j_{X^{\mathrm{an}}_\infty}\colon X^{\mathrm{an}}_\infty \to \widetilde{X}^{\mathrm{an}}$ of bordered spaces. Let us denote by $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$ the full triangulated subcategory of $\mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_\infty})$ consisting of algebraic $\mathbb{C}$-constructible enhanced subanalytic sheaves on a real analytic bordered space $X^{\mathrm{an}}_\infty = (X^{\mathrm{an}}, \widetilde{X}^{\mathrm{an}})$. See [Ito24b, Defs. 5.5, 5.9] for the definition of algebraic $\mathbb{C}$-constructible enhanced subanalytic sheaves. Then we have:

Theorem 3.5 ([Ito24b, Thms. 5.8, 5.10, Prop. 5.12][6])**.** There exists an equivalence of categories:

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_{X_\infty}) \underset{J^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}}{\overset{I^{\mathrm{E}}_{X^{\mathrm{an}}_\infty}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{X_\infty}).$$

Moreover, there exists an equivalence of categories:

$$\mathrm{DR}^{\mathrm{E,sub}}_{X_\infty}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow[\sim]{\mathrm{DR}^{\mathrm{E,sub}}_{X_\infty}} & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_{X_\infty}) \\
\cup & & \uparrow e^{\mathrm{sub}}_{X^{\mathrm{an}}_\infty}\circ \mathbf{R}\rho_{X^{\mathrm{an}}_\infty *} \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}_X]{\sim} & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X).
\end{array}$$

[6]In [Ito24b], although Theorem 5.8 was stated by using the enhanced solution functor $\mathrm{Sol}^{\mathrm{E,sub}}_{X_\infty}$, we can obtain a similar statement by using the enhanced de Rham functor $\mathrm{DR}^{\mathrm{E,sub}}_{X_\infty}$.

4 Equivariant Derived Category

We shall briefly recall equivariant derived categories.

4.1 Equivariant Derived Category of Sheaves

Let us briefly recall the definition of the equivariant derived category of sheaves.

A topological space is called good if it is Hausdorff, locally compact, countable at infinity, and has finite flabby dimension. We denote by gTop the category of good topological spaces. Moreover, let us denote by gTop^G the category of G-spaces for a group object G of gTop and by $\mathrm{gTop}^G_{/M}$ the slice category of a G-space M in gTop^G.

In this paper, we say that an action of G on a G-space R is free if the quotient map $R \to \overline{R} := R/G$ makes into a principal G-bundle. A resolution R of M is an object of $\mathrm{gTop}^G_{/M}$ with free action. Let us denote by $\mathrm{Res}^G_X \subset \mathrm{gTop}^G_{/M}$ the full subcategory of resolutions of X satisfying $\overline{R} := R/G$ is good. For a morphism $f\colon M \to N$ of good topological spaces, we denote by $\overline{f}\colon \overline{M} \to \overline{N}$ the morphism induced by f.

Let us set the category $\mathcal{I}$ as below:

- An object is finite sequences $I = (i_1, \ldots, i_k) \in (\mathbb{Z}_{\geq 0})^k$ for some $k \in \mathbb{N}$.
- For two objects $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_l)$, there exists unique arrow $I \to J$ if and only if J is a subsequence of I which means that there exists a sequence $1 \leq n_1 < \cdots < n_l \leq k$ such that $j_m = i_{n_m}$ for any $m \in \mathbb{N}_{\leq l}$.

Let $n \in \mathbb{N}$. We say that a morphism $f\colon M \to N$ of good topological spaces is n-acyclic if it satisfies the following two conditions:

$(\mathrm{Ac1})_n$ For any $F \in \mathrm{Mod}(\mathbb{C}_N)$, the adjoint pair $(f^{-1}, \mathbf{R}f_*)$ induces an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_N)$:

$$F \xrightarrow{\sim} \tau^{\leq n}\mathbf{R}f_*f^{-1}F,$$

where $\mathrm{Mod}(\mathbb{C}_N)$ is the abelian category of sheaves on N and $\tau^{\leq n}$ is the truncation functor.

$(\mathrm{Ac2})_n$ For any good topological spaces N' and any morphism $g\colon N' \to N$ of good topological spaces, the base change $f'\colon M' := M \times_N N' \to N'$ satisfies the property $(\mathrm{Ac1})_n$.

We also say that a morphism $f\colon M \to N$ of good topological spaces is n-preacyclic if it satisfies the property $(\mathrm{Ac1})_n$. Moreover, we say that a good topological spaces A is n-acyclic if A satisfies the following two conditions:

$(\mathrm{A1})_n$ The unique morphism $a\colon A \to \{\mathrm{pt}\}$ from A to the one-point topological space is n-preacyclic.

$(\mathrm{A2})_n$ A is non empty , connected and locally connected.

Note that $a\colon A \to \{\mathrm{pt}\}$ is n-acyclic if A is n-acyclic. See [BL, Criterion in Sect. 1.9.4] for the details.

We say that a group object G of gTop has enough acyclic objects if for any $n \in \mathbb{N}$ there exists $A_n \in \mathrm{gTop}^G_{\{\mathrm{pt}\}}$ which is n-acyclic. Then we set $A_0 := G$ and set

$$M_I := A_{i_1} \times \cdots A_{i_k} \times M$$

for $I = (i_1, \ldots, i_k) \in (\mathbb{Z}_{\geq 0})^k$. Moreover, if $\mathrm{Hom}_{\mathcal{I}}(I, J) \neq \emptyset$ we write

$$P^M_{IJ} \colon M_I \to M_J, \qquad \overline{P}^M_{IJ} \colon \overline{M}_I \to \overline{M}_J$$

for the projection and the morphism induced by P^M_{IJ}, respectively. We sometimes abbreviate $\overline{P}^M_{IJ}$ to $\overline{P}_{IJ}$. Then, the equivariant derived category $\mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_M)$ of sheaves of $\mathbb{C}$-vector spaces is defined as below:

- An object is a pair
  $$F := (\{F_I\}_{I \in \mathcal{I}}, \{\varphi_{IJ}\}_{I,J \in \mathcal{I}}),$$
  where $F_I \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\overline{M}_I})$ and $\varphi_{IJ} \colon \overline{P}^{-1}_{IJ} F_J \xrightarrow{\sim} F_I$ such that
  $$\varphi_{II} = \mathrm{id}_{F_I}$$
  and the following diagram is commutative:
  $$\begin{array}{ccc} \overline{P}^{-1}_{IJ}(\overline{P}^{-1}_{JK} F_K) & \xrightarrow{\overline{P}^{-1}_{IJ}(\varphi_{JK})} & \overline{P}^{-1}_{IJ} F_J \\ \Big\| \wr & & \Big\downarrow \varphi_{IJ} \\ \overline{P}^{-1}_{IK} F_K & \xrightarrow[\varphi_{IK}]{} & F_I. \end{array}$$

- A morphism $\alpha \colon F \to F'$ is a set $\alpha = \{\alpha_I\}_{I \in \mathcal{I}}$ of morphisms $\alpha_I \colon F_I \to F'_I$ such that the following diagram is commutative:
  $$\begin{array}{ccc} \overline{P}^{-1}_{IJ} F_J & \xrightarrow{\varphi_{IJ}} & F_I \\ {\scriptstyle \overline{P}^{-1}_{IJ}(\alpha_J)} \Big\downarrow & & \Big\downarrow \alpha_I \\ \overline{P}^{-1}_{IJ} F'_J & \xrightarrow[\varphi'_{IJ}]{} & F'_I. \end{array}$$

Note that this definition is equivalent to that of [BL] because the category $\mathcal{I}$ and the functor $\mathcal{I} \ni I \to X_I \in \mathrm{Res}^G_X$ satisfy the conditions of [BL, Prop. in Sect. 2.4.4]. Note also that the definition of $\mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ does not depend on the choice of $\{A_n\}_{n \in \mathbb{N}}$.

4.2 Equivariant Derived Category of $\mathcal{D}$-Modules

Let us briefly recall the definition of the equivariant derived category of $\mathcal{D}$-modules. Reference are made to [Be], [Bor] and [HTT] for algebraic $\mathcal{D}$-modules.

In this paper, algebraic varieties are all quasi-projective. For $n \in \mathbb{N}$, we say that a morphism $f \colon X \to Y$ of algebraic varieties is $\mathcal{D}$-n-acyclic if it satisfies the following two conditions:

(DAc1)$_n$ For any $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Y)$, the adjoint pair $(\mathbf{D}f^*, \mathbf{D}f_*[d_x - d_Y])$ induces an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$:

$$\mathcal{M} \xrightarrow{\sim} \tau^{\leq n}\mathbf{D}f_*\mathbf{D}f^*\mathcal{M}[d_Y - d_X],$$

where $\mathrm{Mod}(\mathcal{D}_Y)$ is the abelian category of $\mathcal{D}_Y$-modules, $\tau^{\leq n}$ is the truncation functor and d_X (resp. d_Y) is the complex dimension of X (resp. Y).

(DAc2)$_n$ For any algebraic variety Y' and any morphism $g\colon Y' \to Y$ of algebraic varieties such that $X' := X \times_Y Y'$ is an algebraic variety, the base change $f'\colon X' \to Y'$ satisfies the property (DAc1)$_n$.

Note that for an algebraic variety X the unique morphism $a\colon X \to \{\mathrm{pt}\}$ is $\mathcal{D}$-n-acyclic if the underlying complex manifold X^{an} of X is n-acyclic.

Let G be a smooth algebraic group. We say that G has enough $\mathcal{D}$-acyclic ojects if for any $n \in \mathbb{N}$ there exists a smooth algebraic G-variety A_n with free G-action such that the unique morphism $a\colon A_n \to \{\mathrm{pt}\}$ is $\mathcal{D}$-n-acyclic, and $(A_n \times X)/G$ is an algebraic variety for any algebraic variety X. Then, the equivariant derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathbb{D}_X)$ of coherent $\mathcal{D}_X$-modules on a smooth G-variety X is defined as below:

- An object is a pair

  $$\mathcal{M} := (\{\mathcal{M}_I\}_{I\in\mathcal{I}}, \{\varphi_{IJ}\}_{I,J\in\mathcal{I}}),$$

  where $\mathcal{M}_I \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\overline{X}_I})$ and $\varphi_{IJ}\colon \mathbf{D}\overline{P}^*_{IJ}\mathcal{M}_J \xrightarrow{\sim} \mathcal{M}_I$ such that

  $$\varphi_{II} = \mathrm{id}_{\mathcal{M}_I}$$

  and the following diagram is commutative:

  $$\begin{array}{ccc} \mathbf{D}\overline{P}^*_{IJ}(\mathbf{D}\overline{P}^*_{JK}\mathcal{M}_K) & \xrightarrow{\mathbf{D}\overline{P}^*_{IJ}(\varphi_{JK})} & \mathbf{D}\overline{P}^*_{IJ}\mathcal{M}_J \\ \big\| \wr & & \big\downarrow \varphi_{IJ} \\ \mathbf{D}\overline{P}^*_{IK}\mathcal{M}_K & \xrightarrow[\varphi_{IK}]{} & \mathcal{M}_I. \end{array}$$

- A morphism $\alpha\colon \mathcal{M} \to \mathcal{M}'$ is a set $\alpha = \{\alpha_I\}_{I\in\mathcal{I}}$ of morphisms $\alpha_I\colon \mathcal{M}_I \to \mathcal{M}'_I$ such that the following diagram is commutative:

  $$\begin{array}{ccc} \mathbf{D}\overline{P}^*_{IJ}\mathcal{M}_J & \xrightarrow{\varphi_{IJ}} & \mathcal{M}_I \\ {\scriptstyle \mathbf{D}\overline{P}^*_{IJ}(\alpha_J)}\big\downarrow & & \big\downarrow \alpha_I \\ \mathbf{D}\overline{P}^*_{IJ}\mathcal{M}'_J & \xrightarrow[\varphi'_{IJ}]{} & \mathcal{M}'_I. \end{array}$$

4.3 Equivariant Regular Riemann-Hilbert Correspondence

Let us recall the equivariant version of the algebraic regular Riemann–Hilbert correspondence.

Let X be an algebraic variety and denote by X^{an} the underlying complex manifold of X.

We say that a smooth algebraic group G has enough compatible $\mathcal{D}$-acyclic objects if for any $n \in \mathbb{N}$ there exists a smooth algebraic G-variety A_n with free G-action such that the underlying complex manifold A_n^{an} of A_n is n-acyclic, and that $(A_n \times X)/G$ is an algebraic variety for any smooth algebraic G-variety X. Note that any linear algebraic group has enough compatible $\mathcal{D}$-acyclic objects. See [IT24, Lem. 6.7] for the details.

Let G be a linear algebraic group and $\{A_n\}_{n\in\mathbb{N}}$ a sequence which is satisfies the conditions in the definition of enough compatible $\mathcal{D}$-acyclic objects as above. Then for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X)$ we set

$$\mathrm{DR}_X(\mathcal{M}) := \big(\{\mathrm{DR}_{\overline{X}_I}(\mathcal{M}_I)\}_{I\in\mathcal{I}}, \{\mathrm{DR}_{\overline{X}_I}(\varphi_{IJ})\}_{I,J\in\mathcal{I}}\big)$$

and we have a functor $\mathrm{DR}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_X)$. We set the full subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X)$ by

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{rh},G}(\mathcal{D}_X) := \{\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X) \mid \mathcal{M}_I \in \mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_{\overline{X}_I}) \text{ for any } I \in \mathcal{I}\}$$

and set the full subcategory of $\mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_{X^{\mathrm{an}}})$ by

$$\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}_X) := \{\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_{X^{\mathrm{an}}}) \mid \mathcal{M}_I \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_{\overline{X}_I}) \text{ for any } I \in \mathcal{I}\}.$$

Then we have:

Theorem 4.1 ([Kas08, Thm. 4.6.2])**.** The functor $\mathrm{DR}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ induces an equivalence of categories:

$$\mathrm{DR}_X\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{rh},G}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}_X).$$

This equivalence of categories is called the equivariant regular Riemann–Hilbert correspondence.

5 Equivariant Category of Enhanced Ind-Sheaves on Bordered Spaces of Compact Type

Let us briefly recall some basic notions and results of [IT24].

We say that a bordered space $M_\infty := (M, \check{M})$ is compact type if $\check{M}$ is compact. See [DK16, Def. 3.2.1] for the definition of bordered spaces.

Note that tanks to the Alexandroff compactification, for any good topological space M we can take a compact good topological spaces $\check{M}$ such that $(M, \check{M})$ is a bordered space of compact type. Moreover, for any two compact good topological spaces $\check{M}, M^*$ which are contain M as open subset, $(M, \check{M})$ is isomorphic to (M, M^*) as bordered spaces. See [IT24, Def. 2.9] for the details. Hence, for a good topological space M, we can denote by M_{cp} such a bordered space of compact type.

Note also that any morphism $f\colon M \to N$ of good topological spaces induces a morphism $f_{\mathrm{cp}}\colon M_{\mathrm{cp}} \to N_{\mathrm{cp}}$ of bordered spaces of compact type. See [IT24, Lem. 2.10] for the details.

Let us set

$$\mathbf{E}^0(\mathrm{I}\mathbb{C}_{M_\infty}) := \mathbf{E}^{\leq 0}(\mathrm{I}\mathbb{C}_{M_\infty}) \cap \mathbf{E}^{\geq 0}(\mathrm{I}\mathbb{C}_{M_\infty}),$$

where a pair $\big(\mathbf{E}^{\leq 0}(\mathrm{I}\mathbb{C}_{M_\infty}), \mathbf{E}^{\geq 0}(\mathrm{I}\mathbb{C}_{M_\infty})\big)$ is a t-structure on the triangulated category $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_\infty})$ of enhanced ind-sheaves on M_∞. See [DK19, Prop. 2.6.2] for the details.

Definition 5.1. Let $n \in \mathbb{N}$. We say that a morphism $f\colon M \to N$ of good topological spaces is enhanced n-acyclic of compact type if it satisfies the following two conditions:

$(\mathrm{cEAc1})_n$ For any $K \in \mathbf{E}^0(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})$, the adjoint pair $(\mathbf{E}f_{\mathrm{cp}}^{-1}, \mathbf{E}f_{\mathrm{cp}*})$ induces an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})$:

$$K \xrightarrow{\sim} \tau^{\leq n}\mathbf{E}f_{\mathrm{cp}*}\mathbf{E}f_{\mathrm{cp}}^{-1}K,$$

where $\tau^{\leq n}$ is the truncation functor.

$(\mathrm{cEAc2})_n$ For any good topological space N' and any morphism $g\colon N' \to N$ of good topological spaces, the base change $f'\colon M' \to N'$ satisfies the property $(\mathrm{cEAc1})_n$.

We also say that $f\colon M \to N$ is enhanced n-preacyclic of compact type if it satisfies the condition $(\mathrm{cEAc1})_n$.

The following lemma mean that if a group object $G \in \mathsf{gTop}$ has enough acyclic objects, it has also "enough enhanced acyclic objects of compact type".

Lemma 5.2. If a good topological spaces A is $(n+1)$-acyclic, then $a\colon A \to \{\mathrm{pt}\}$ is enhanced n-acyclic of compact type.

Definition 5.3. Let M be a good topological space and G be a group object of gTop having enough acyclic objects and take a sequence $\{A_n\}_{n\in\mathbb{Z}_{\geq 1}} \subset \mathsf{Res}^G_{\{\mathrm{pt}\}}$ of n-acyclic objects. An equivariant category $\mathbf{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}})$ of enhanced ind-sheaves on M_{cp} is defined as below:

- An object is a pair

  $$L := (\{L_I\}_{I\in\mathcal{I}}, \{\varphi_{IJ}\}_{I,J\in\mathcal{I}}),$$

  where $L_I \in \mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\overline{M}_I)_{\mathrm{cp}}})$ and $\varphi_{IJ}\colon \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L_J \xrightarrow{\sim} L_I$ such that

  $$\varphi_{II} = \mathrm{id}_{L_I}$$

  and the following diagram is commutative:

  $$\begin{array}{ccc}
  \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}(\mathbf{E}(\overline{P}_{JK})^{-1}_{\mathrm{cp}}L_K) & \xrightarrow{\mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}(\varphi_{JK})} & \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L_J \\
  \big\| \wr & & \big\downarrow \varphi_{IJ} \\
  \mathbf{E}(\overline{P}_{IK})^{-1}_{\mathrm{cp}}L_K & \xrightarrow[\varphi_{IK}]{} & L_I.
  \end{array}$$

- A morphism $\alpha\colon L \to L'$ is a set $\alpha = \{\alpha_I\}_{I\in\mathcal{I}}$ of morphisms $\alpha_I\colon L_I \to L'_I$ such that the following diagram is commutative:

  $$\begin{array}{ccc}
  \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L_J & \xrightarrow{\varphi_{IK}} & L_I \\
  {\scriptstyle \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}(\alpha_J)}\big\downarrow & & \big\downarrow \alpha_I \\
  \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L'_J & \xrightarrow[\varphi'_{IK}]{} & L'_I.
  \end{array}$$

Note that the category $\mathbf{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}})$ becomes a triangulated category. See [IT24, Lem. 4.7] for the details. Note also that there exists a fully faithful functor

$$e_{M_{\mathrm{cp}}} \circ \iota_{M_{\mathrm{cp}}}\colon \mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_M) \to \mathbf{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}}).$$

6 Equivariant Irregular Riemann–Hilbert Correspondence and Enhanced Ind-Sheaves

Let G be a linear algebraic group. Recall that G has enough compatible $\mathcal{D}$-acyclic objects. See [IT24, Lem. 6.7] for the details. We take a sequence $\{A_n\}_{n\in\mathbb{N}}$ which is satisfies the conditions in the definition of enough compatible $\mathcal{D}$-acyclic objects.

Let X be a smooth algebraic G-variety and $\widetilde{X}$ a completion of X.

Definition 6.1. For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X)$, we set

$$\mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G}(\mathcal{M}) := \Big(\{\mathrm{DR}^{\mathrm{E}}_{(\overline{X}_I)_{\mathrm{cp}}}(\mathcal{M}_I)\}_{I\in\mathcal{I}}, \{\mathrm{DR}^{\mathrm{E}}_{(\overline{X}_I)_{\mathrm{cp}}}(\varphi_{IJ})\}_{I,J\in\mathcal{I}}\Big)$$

and define a functor

$$\mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G} \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}_{G}(\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}}).$$

We set the full subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X)$ by

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{hol},G}(\mathcal{D}_X) := \{\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X) \mid \mathcal{M}_I \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{\overline{X}_I}) \text{ for any } I \in \mathcal{I}\}$$

and set the full subcategory of $\mathbf{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}})$ by

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) := \{\mathcal{M} \in \mathbf{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}}) \mid \mathcal{M}_I \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathrm{I}\mathbb{C}_{(\overline{X}_I)_{\mathrm{cp}}}) \text{ for any } I \in \mathcal{I}\}.$$

See [Ito21, Defs. 3.1, 3.10] for the definition of algebraic $\mathbb{C}$-constructible enhanced ind-sheaves. Then we have:

Theorem 6.2. The functor $\mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G}$ induces an equivalence of categories:

$$\mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G} \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol},G}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol},G}(\mathcal{D}_X) & \xrightarrow[\sim]{\mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G}} & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \\
\cup & & \uparrow{\scriptstyle e_{X^{\mathrm{an}}_{\mathrm{cp}}}\circ\iota_{X^{\mathrm{an}}_{\mathrm{cp}}}} \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh},G}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}_{X,G}]{\sim} & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}_X).
\end{array}$$

Proof. This assertion follows from Theorem 2.3. □

7 Main Definitions and Results

Let us explain an equivariant category of $\mathbb{C}$-constructible enhanced subanalytic sheaves and prove that an equivariant version of the algebraic irregular Riemann–Hilbert correspondence. The main definitions are Definition 7.1, 7.8 and the main theorem are Theorem 7.7, 7.10.

7.1 Equivariant Category of Enhanced Subanalytic Sheaves on Bordered Spaces of Compact Type

Let G be an algebraic group having enough acyclic objects and take a sequence $\{A_n\}_{n\in\mathbb{N}}$ of acyclic objects, and X a smooth algebraic G-variety over $\mathbb{C}$.

Definition 7.1. An equivariant category $\mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}})$ of enhanced subanalytic sheaves on $X^{\mathrm{an}}_{\mathrm{cp}}$ is defined as below:

- An object is a pair
$$L := (\{L_I\}_{I\in\mathcal{I}}, \{\varphi_{IJ}\}_{I,J\in\mathcal{I}}),$$
where $L_I \in \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{(\overline{X}_I)_{\mathrm{cp}}})$ and $\varphi_{IJ}\colon \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L_J \xrightarrow{\sim} L_I$ such that $\varphi_{II} = \mathrm{id}_{L_I}$ and the following diagram is commutative:
$$\begin{array}{ccc} \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}(\mathbf{E}(\overline{P}_{JK})^{-1}_{\mathrm{cp}}L_K) & \xrightarrow{\mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}(\varphi_{JK})} & \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L_J \\ \wr\Big| & & \Big\downarrow{\scriptstyle \varphi_{IJ}} \\ \mathbf{E}(\overline{P}_{IK})^{-1}_{\mathrm{cp}}L_K & \xrightarrow[\varphi_{IK}]{} & L_I. \end{array}$$

- A morphism $\alpha\colon L \to L'$ is a set $\alpha = \{\alpha_I\}_{I\in\mathcal{I}}$ of morphisms $\alpha_I\colon L_I \to L'_I$ such that the following diagram is commutative:
$$\begin{array}{ccc} \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L_J & \xrightarrow{\varphi_{IK}} & L_I \\ {\scriptstyle \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}(\alpha_J)}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_I} \\ \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L'_J & \xrightarrow[\varphi'_{IK}]{} & L'_I. \end{array}$$

Proposition 7.2. For any $F = (\{F_I\}_{I\in\mathcal{I}}, \{\varphi_{IJ}\}_{I,J\in\mathcal{I}}) \in \mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_{X^{\mathrm{an}}})$, we set
$$e^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}} \circ \mathbf{R}\rho_{X^{\mathrm{an}}_{\mathrm{cp}}*}(F) := \Big(\{e^{\mathrm{sub}}_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}}(\mathbf{R}\rho_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}*}(F_I))\}_{I\in\mathcal{I}}, \{e^{\mathrm{sub}}_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}}(\mathbf{R}\rho_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}*}(\varphi_{IJ}))\}_{I,J\in\mathcal{I}}\Big).$$
Then we have a functor
$$e^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}} \circ \mathbf{R}\rho_{X^{\mathrm{an}}_{\mathrm{cp}}*}\colon \mathbf{D}^{\mathrm{b}}_G(\mathbb{C}_{X^{\mathrm{an}}}) \to \mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}),\ F \mapsto e^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}} \circ \mathbf{R}\rho_{X^{\mathrm{an}}_{\mathrm{cp}}*}(F).$$

Proof. This assertion follows from [Ito24a, Prop. 3.22]. □

Let us explain a relation between the equivariant category of enhanced ind-sheaves on $X^{\mathrm{an}}_{\mathrm{cp}}$ and that of enhanced subanalytic sheaves on $X^{\mathrm{an}}_{\mathrm{cp}}$.

Proposition 7.3. For any $L = (\{L_I\}_{I\in\mathcal{I}}, \{\varphi_{IJ}\}_{I,J\in\mathcal{I}}) \in \mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}})$, we set
$$I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}(L) := \Big(\{I^{\mathrm{E}}_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}}(L_I)\}_{I\in\mathcal{I}}, \{I^{\mathrm{E}}_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}}(\varphi_{IJ})\}_{I,J\in\mathcal{I}}\Big).$$
Then we have a fully faithful functor:
$$I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}\colon \mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}) \to \mathbf{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}}),\ L \mapsto I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}(L).$$

Proof. By [Ito24a, Prop. 3.16 (2)(i)], we obtain a functor $I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}\colon \mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}) \to \mathbf{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}}),\ L \mapsto I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}(L)$ and it is fully faithful by Theorem 3.2 ([Ito24a, Thm. 3.15]). □

Let us define a $\mathbb{C}$-constructibility for an object of $\mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}})$ as below.

Definition 7.4. We set the full subcategory of $\mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}})$ by

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}}) := \{\mathcal{M} \in \mathbf{E}^{\mathrm{b}}_G(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}) \mid \mathcal{M}_I \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^{\mathrm{sub}}_{(\overline{X}_I)_{\mathrm{cp}}}) \text{ for any } I \in \mathcal{I}\}.$$

See [Ito24b, Defs. 5.5, 5.9] for the definition of algebraic $\mathbb{C}$-constructible enhanced subanalytic sheaves.

Proposition 7.5. The functor $e^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}} \circ \mathbf{R}\rho_{X^{\mathrm{an}}_{\mathrm{cp}}*}$ induces a fully faithful functor:

$$e^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}} \circ \mathbf{R}\rho_{X^{\mathrm{an}}_{\mathrm{cp}}*}\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}_{X^{\mathrm{an}}}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}).$$

Proof. This assertion follows from [Ito24b, Prop. 5.12]. □

Moreover we have the following functor.

Lemma 7.6. For any $L = (\{L_I\}_{I\in\mathcal{I}}, \{\varphi_{IJ}\}_{I,J\in\mathcal{I}}) \in \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}})$, we obtain an object of $\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}})$:

$$J^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}(L) := \left(\{J^{\mathrm{E}}_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}}(L_I)\}_{I\in\mathcal{I}}, \{J^{\mathrm{E}}_{(\overline{X}^{\mathrm{an}}_I)_{\mathrm{cp}}}(\varphi_{IJ})\}_{I,J\in\mathcal{I}}\right).$$

Moreover, we have a functor

$$J^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}\colon \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}}) \to \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}),\ L \mapsto J^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}(L).$$

Proof. These assertions follow from [Ito24a, Prop. 3.16 (4)(i)] and [Ito24b, Thm. 5.10]. □

Then we have the following results which is the main theorem of this paper.

Theorem 7.7. There exists an equivalence of categories:

$$\mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}) \underset{J^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}}{\overset{I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}}{\rightleftarrows}} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}}).$$

Proof. This assertion follows from [Ito24b, Thm. 5.10] and Lemma 7.6. □

7.2 Equivariant Irregular Riemann–Hilbert Correspondence and Enhanced Subanalytic Sheaves

Let G be a linear algebraic group with a sequence $\{A_n\}_{n\in\mathbb{N}}$ which is satisfies the conditions in the definition of enough compatible $\mathcal{D}$-acyclic objects (see §4.2 for the details), and X a smooth algebraic G-variety over $\mathbb{C}$.

Definition 7.8. For any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X)$, we set

$$\mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}(\mathcal{M}) := \Big(\{\mathrm{DR}^{\mathrm{E,sub}}_{(\overline{X}_I)_{\mathrm{cp}}}(\mathcal{M}_I)\}_{I\in\mathcal{I}}, \{\mathrm{DR}^{\mathrm{E,sub}}_{(\overline{X}_I)_{\mathrm{cp}}}(\varphi_{IJ})\}_{I,J\in\mathcal{I}}\Big).$$

Proposition 7.9. We have a functor

$$\mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}_{G}(\mathbb{C}_{X^{\mathrm{an}}_{\mathrm{cp}}}),\ \mathcal{M} \mapsto \mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}(\mathcal{M}).$$

Proof. It is enough to show that

$$\mathrm{DR}^{\mathrm{E,sub}}_{(\overline{X}_I)_{\mathrm{cp}}}(\mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}L_J) \simeq \mathbf{E}(\overline{P}_{IJ})^{-1}_{\mathrm{cp}}(\mathrm{DR}^{\mathrm{E,sub}}_{(\overline{X}_J)_{\mathrm{cp}}}(L_J))$$

for any $I, J \in \mathcal{I}$ which are satisfy $\mathrm{Hom}_{\mathcal{I}}(I,J) \neq \emptyset$ and any $L_J \in \mathbf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{(\overline{X}_J)_{\mathrm{cp}}})$. This assertion follows from [IT24, Lem. 6.3] and [Ito24a, Thm. 3.15, Prop. 3.16 (2)(i)]. □

We call this functor $\mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}$ the equivariant algebraic enhanced de Rham functor. Then we have the following results which is the main theorem of this paper.

Theorem 7.10. The functor $\mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}$ induces an equivalence of categories:

$$\mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{hol},G}(\mathcal{D}_X) \xrightarrow{\sim} \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}})$$

and the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol},G}(\mathcal{D}_X) & \xrightarrow[\sim]{\mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G}} & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}}) \\
\cup & & \uparrow {\scriptstyle e^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}\circ \mathbf{R}\rho_{X^{\mathrm{an}}_{\mathrm{cp}}*}} \\
\mathbf{D}^{\mathrm{b}}_{\mathrm{rh},G}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}_{X,G}]{\sim} & \mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}_X).
\end{array}$$

Proof. This assertion follows from Theorem 3.5. □

Remark that the following diagram is commutative:

$$\begin{array}{ccccc}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol},G}(\mathcal{D}_X) & \xrightarrow[\mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}]{\sim} & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}}) & \subset & \mathbf{E}^{\mathrm{b}}_{G}(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}}) \\
& {\scriptstyle \mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G}}\searrow^{\sim} & \downarrow {\scriptstyle I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}}\ \wr & & \downarrow {\scriptstyle I^{\mathrm{E}}_{X^{\mathrm{an}}_{\mathrm{cp}},G}} \\
& & \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) & \subset & \mathbf{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}).
\end{array}$$

As an application of this theorem, we prove the following well-known fact.

Fact 7.11 ([Kas89, Thm. 9.3.1], [HTT, Thm. 11.6.1])**.** If the number of G-orbits on X is finite, any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X)$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh},G}(\mathcal{D}_X)$.

Proof. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},G}(\mathcal{D}_X)$. Then by the induction of the number of orbits, we obtain

$$\mathrm{DR}^{\mathrm{E,sub}}_{X_{\mathrm{cp}},G}(\mathcal{M}) \in e^{\mathrm{sub}}_{X^{\mathrm{an}}_{\mathrm{cp}}}\left(\mathbf{R}\rho_{X^{\mathrm{an}}_{\mathrm{cp}}*}\left(\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}_X)\right)\right) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}^{\mathrm{sub}}_{X_{\mathrm{cp}}})$$

and hence

$$\mathrm{DR}^{\mathrm{E}}_{X_{\mathrm{cp}},G}(\mathcal{M}) \in e_{X^{\mathrm{an}}_{\mathrm{cp}}}\left(\iota_{X^{\mathrm{an}}_{\mathrm{cp}}}\left(\mathbf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathbb{C}_X)\right)\right) \cap \mathbf{E}^{\mathrm{b}}_{\mathbb{C}\text{-}c,G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}).$$

This means that $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{rh},G}(\mathcal{D}_X)$ by using [Ito23, Lem. 3.10]. □

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