Sharp interface limit for a rate function of large deviations with quasi non-linearity

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1 Introduction

This paper is based on a joint work with Kenkichi Tsunoda (Kyushu University) [10]. Our main concern is the sharp interface limit for a Glauber+Kawasaki process with speed change. For this purpose, we start by defining the late function of the large deviation principle for the process (see [10] for the details). Let $\mathbb{T}_N^d$ be the d-dimensional discrete torus with length N, that is, $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$. Here, $N \in \mathbb{N}$ is a scaling parameter which we will let infinity later. Let X_N be the configuration space $\{0,1\}^{\mathbb{T}_N^d}$ and denote its generic element as $\eta = \{\eta(x)\}_{\mathbb{T}_N^d}$. We regard a configuration $\eta \in X_N$ in the following manner: for each site $x \in \mathbb{T}_N^d$, there is a particle at x if $\eta(x) = 1$, otherwise, there is no particle at site x.

We now define the Markovian generator L_N defined as $L_N f = N^2 L_K f + \widetilde{K} L_G f$ for any function $f : X_N \to \mathbb{R}$, where L_K and L_G are operators corresponding to a "diffusion" operator and a "reaction" operator, respectively. Let $(\eta_t^N)_{t\geq 0}$ denote a Markov process generated by L_N. Let $\mathbb{T}^d$ be the d-dimensional continuum torus $(\mathbb{R}/\mathbb{Z})^d$. We define the empirical measure by

$$\pi_t^N(du) = \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta_t^N(x)\delta_{x/N}(du),$$

where δ_u stands for the Dirac measure at $u \in \mathbb{T}^d$.

The scaling limit for empirical measures is a fundamental problem in the study of interacting particle systems. For this Glauber+Kawasaki process, a large deviation principle, which determines the decay rate for the probability of an atypical event of the system, has also been studied in [9, 3, 11]. Loosely speaking, for a given density evolution $\phi : [0,T] \times \mathbb{T}^d \to [0,1]$, the probability that the empirical measure $\pi_t^N(dx)$ follows $\phi(t,x)dx$ behaves as

$$\mathbb{P}\left(\pi_\cdot^N \sim \phi(\cdot,x)dx\right) \approx \exp\{-N^d S(\phi)\},$$

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where $S(\phi)$ is given by

$$S(\phi) = \sup_{H \in C^{1,2}([0,T]\times\mathbb{T}^d)} J^H(\phi),$$

$$\begin{aligned} J^H(\phi) = & \int_{\mathbb{T}^d} \phi(T,x)H(T,x)\, dx - \int_{\mathbb{T}^d} \phi(0,x)H(0,x)\, dx \\ & - \int_0^T \int_{\mathbb{T}^d} \left\{ \phi \partial_t H + P(\phi)\Delta H + \sigma(\phi)|\nabla H|^2 \right\} dxdt \\ & - \int_0^T \int_{\mathbb{T}^d} \widetilde{K} \left\{ B(\phi)\left(e^H - 1\right) + D(\phi)\left(e^{-H} - 1\right) \right\} dxdt. \end{aligned}$$

In this paper, we assume that $P : [0,1] \to [0,\infty)$ and $B, D : [0,1] \to \mathbb{R}$ and $W : [0,1] \to \mathbb{R}$ are smooth functions satisfying the following conditions:

(A1) P satisfies $P(0) = 0$ and $P'(\rho) > 0$ for any $\rho \in [0,1]$.

(A2) $B(\rho) + D(\rho)$ is positive for any $\rho \in [0,1]$ and $B - D = -W'$.

(A3) W is a double-well potential, that is, there exist exactly three critical points $0 < \rho_- < \rho_* < \rho_+ < 1$ such that $W(\rho_\pm) < W(\rho)$ for any $\rho \neq \rho_\pm$ and $W''(\rho_\pm) > 0$.

(A4) W satisfies a P-*balance condition*, that is, it holds that

$$\int_{\rho_-}^{\rho_+} W'(\rho)P'(\rho)\, d\rho = 0.$$

We note that the conditions (A1) and (A2) are satisfied when B and D are determined from a wide class of jump rates of the Glauber dynamics. Moreover, the conditions (A3) and (A4) were introduced from the probability background as in [5].

We here note that $S(\phi)$ is non-negative and vanishes if and only if ϕ solves the reaction-diffusion equation

$$\partial_t \rho = \Delta P(\rho) + \widetilde{K}(B(\rho) - D(\rho)). \tag{1.1}$$

Letting $\varepsilon := 1/\sqrt{\widetilde{K}}$, the reaction-diffusion equation (1.1) introduces an Allen-Chan type equation

$$\partial_t \rho_\varepsilon = \Delta P(\rho_\varepsilon) + \frac{1}{\varepsilon^2}(B(\rho_\varepsilon) - D(\rho_\varepsilon)). \tag{1.2}$$

Heuristically, at each time, ρ_ε is close to a step function for sufficient small ε, and the transition layer converges to a surface Γ_t generating a mean curvature flow with a mobility constant θ determined by P, B and D as $\varepsilon \to +0$, namely, the motion of Γ_t is governed by $v_t - \theta h_t$, where v_t and h_t are the normal velocity and the mean curvature of Γ_t, respectively. In particular, the transition layer can be represented as $\rho_\varepsilon(t,x) \approx \overline{u}(d(t,x)/\varepsilon)$, where $d(t,x)$ is a signed distance function from Γ_t and $\overline{u}$ is a solution to the ordinary differential equation

$$\begin{cases} (P(\bar{u}))'' + B(\bar{u}) - D(\bar{u}) = 0 \quad \text{in } \mathbb{R}, \\ \bar{u}(\pm\infty) = \rho_\pm, \quad \bar{u}(0) = \dfrac{\rho_+ + \rho_-}{2}. \end{cases} \tag{1.3}$$

For the known convergence results, we refer to [4, 7, 1, 8, 12] for the case $P(\rho) = \rho$ and [5] for more general $P(\rho)$.

For the case $P(\rho) = \rho/2$, Bertini, Buttà and Pisante [2] characterized the functional $S_\varepsilon(\phi)$ from the perspective of the sharp interface limit by substituting a family of functions generating a transition layer around an arbitrary fixed geometric flow into the functional. Our purpose is to extend the result to a more general function P. For this purpose, for each $\varepsilon > 0$ let us define

$$S_\varepsilon(\phi) = \sup_{H\in C^{1,2}([0,T]\times\mathbb{T}^d)} J_\varepsilon^H(\phi), \tag{1.4}$$

$$\begin{aligned} J_\varepsilon^H(\phi) = &\frac{1}{\varepsilon}\int_{\mathbb{T}^d} \phi(T,x)H(T,x)\,dx - \frac{1}{\varepsilon}\int_{\mathbb{T}^d} \phi(0,x)H(0,x)\,dx \\ &- \frac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}^d} \left\{\phi\partial_t H + P(\phi)\Delta H + \sigma(\phi)|\nabla H|^2\right\}\,dxdt \\ &- \frac{1}{\varepsilon^3}\int_0^T \int_{\mathbb{T}^d} \left\{B(\phi)\left(e^H - 1\right) + D(\phi_t)\left(e^{-H} - 1\right)\right\}\,dxdt. \end{aligned} \tag{1.5}$$

and it was clarified that these conditions are needed to obtain a sharp interface limit for (1.2) leading to the motion by mean curvature.

In these settings, our goal can be stated that, restricting the form of a family of functions $\{\phi_\varepsilon\}_{\varepsilon>0}$ so that functions generating the transition layer around an arbitrary fixed geometric flow $\{\Gamma_t\}_{t\in[0,T]}$, we show a "formal" Γ-convergence from $S_\varepsilon(\phi_\varepsilon)$ to

$$S_{\mathrm{ac}}(\Gamma) = \int_0^T \int_{\Gamma_t} \frac{(v_t - \theta h_t)^2}{4\mu}\,d\mathcal{H}^{d-1}dt,$$

where v_t, h_t are respectively the normal velocity and the mean curvature of Γ_t, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and θ, μ are respectively the mobility and the transport coefficient determined by P, B and D (see (2.7) for details). To state the form of the family of functions $\{\phi_\varepsilon\}_{\varepsilon>0}$, we define a regularized version of a signed distance function from Γ_t as follows. For a family of oriented smooth hyper-surfaces $\Gamma = \{\Gamma_t\}_{t\in[0,T]}$ with $\Gamma_t = \partial\Omega_t$ for some open $\Omega_t \subset \mathbb{T}^d$ and with the finite surface area for any $t \in [0,T]$, choose $d(\cdot,t)$ as a regularized version of the signed distance from Γ_t satisfying

$$d(t,x) = \begin{cases} \mathrm{dist}(x,\Gamma_t) & \text{if } x \notin \Omega_t \text{ and } \mathrm{dist}(x,\Gamma_t) \ll 1, \\ -\mathrm{dist}(x,\Gamma_t) & \text{if } x \in \Omega_t \text{ and } \mathrm{dist}(x,\Gamma_t) \ll 1. \end{cases} \tag{1.6}$$

Then, the main result in this paper is stated as follows.

Theorem 1.1. *Assume the properties (A1)–(A4) hold. Let $\Gamma = \{\Gamma_t\}_{t\in[0,T]}$ be a family of oriented smooth hyper-surfaces with $\Gamma_t = \partial\Omega_t$ for some open $\Omega_t \subset \mathbb{T}^d$ and with the finite surface area for any $t \in [0,T]$. Let also $\bar{u}$ be the unique smooth solution to* (1.3). *For smooth functions $Q : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ and $R_\varepsilon : [0,T] \times \mathbb{T}^d \to \mathbb{R}$, define the function $\phi_\varepsilon : [0,T] \times \mathbb{T}^d \to [0,1]$ by*

$$\phi_\varepsilon(t,x) = \bar{u}\left(\frac{d(t,x)}{\varepsilon} + \varepsilon Q\left(t,x,\frac{d(t,x)}{\varepsilon}\right)\right) + \varepsilon R_\varepsilon(t,x). \tag{1.7}$$

Then we have the following.

1. *If Q and R_ε satisfy*

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}}\left(\frac{|\partial_t Q|}{1+|\xi|}+\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{|\partial_\xi^i\nabla^j Q|}{1+|\xi|}\right)<\infty, \tag{1.8}$$

$$\lim_{\varepsilon\to 0}\left(\sup_{(t,x)\in[0,T]\times\mathbb{T}^d}(|R_\varepsilon|+|\partial_t R_\varepsilon|+|\nabla R_\varepsilon|+|\nabla^2 R_\varepsilon|)\right)=0, \tag{1.9}$$

then

$$\liminf_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon)\geq S_{\mathrm{ac}}(\Gamma).$$

2. *There exists $\widehat{Q}$ such that, choosing $Q=\widehat{Q}$ and $R_\varepsilon=0$, it holds that*

$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon)=S_{\mathrm{ac}}(\Gamma).$$

2 Outline of the proof

We here discuss the outline of the proof in the case when $R_\varepsilon\equiv 0$ for simplicity. We denote by $H_{\max,\varepsilon}$ the maximizer (depending on ϕ) of the maximum problem (1.4), which satisfies the Euler-Lagrange equation

$$\partial_t\phi+\nabla\cdot[2\sigma(\phi)\nabla H_{\max,\varepsilon}]=\Delta P(\phi)+\frac{B(\phi)e^{H_{\max,\varepsilon}}-D(\phi)e^{-H_{\max,\varepsilon}}}{\varepsilon^2}. \tag{2.1}$$

To compute the limit of $S_\varepsilon(\phi_\varepsilon)$ as $\varepsilon\to 0$, our first purpose is to calculate the power series expansion of $S_\varepsilon(\phi_\varepsilon)$ in ε, namely, to decompose $S_\varepsilon(\phi_\varepsilon)$ as the following form:

$$S_\varepsilon(\phi_\varepsilon)=\sum_{k\in\mathbb{Z}}\varepsilon^k\int_0^T\int_{\mathbb{T}^d}\phi_Q^k\left(t,x,\frac{d(t,x)}{\varepsilon}\right)\,dxdt, \tag{2.2}$$

where ϕ_Q^k is a function depending on Q. A key tool to obtain this kind of decomposition of $S_\varepsilon(\phi_\varepsilon)$ is the decomposition of the maximizer $H_{\max,\varepsilon}$ (depending on ϕ_ε) as

$$H_{\max,\varepsilon}(t,x)=\varepsilon\widehat{H}_1(t,x,d(t,x)/\varepsilon)+\varepsilon^2\widehat{K}_\varepsilon(t,x), \tag{2.3}$$

where $\widehat{H}_1$ is a unique solution to a linearized problem of (4.1) and is determined by the function Q appeared in the choice of ϕ_ε. We then apply the Taylor expansion for the integrands of $S_\varepsilon(\phi_\varepsilon)$ to conclude that, concerning the form (2.2); (i) $S_\varepsilon(\phi_\varepsilon)$ consists of terms with the coefficient ε^k with $k\geq -1$; (ii) as $\varepsilon\to 0$, the term with coefficient ε^{-1} is of constant order and converges to the iterated integral of $\phi_Q^{-1}(t,x,\xi)$ along $t\in[0,T]$, $x\in\Gamma_t$ and $\xi\in\mathbb{R}$; (iii) the other terms vanish as $\varepsilon\to 0$. The conditions (ii) and (iii) follows from the condition (i) by applying the following proposition:

Proposition 2.1. *Let $\Gamma=\{\Gamma_t\}_{t\in[0,T]}$ be a family of oriented smooth hyper-surfaces with $\Gamma_t=\partial\Omega_t$ for some open $\Omega_t\subset\mathbb{T}^d$. Assume Γ_t has a finite surface area for any $t\in[0,T]$. Denote by $d(x,t)$ be a regularized version of the signed distance from Γ_t satisfying* (1.6)*. Let $\gamma'>0$ be an arbitrary positive constant. Then, the following statements hold:*

(1) Let $\widetilde{\mathcal{R}}_\varepsilon : [0,T]\times\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying

$$\overline{\lim_{\varepsilon\to 0}} \sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}} e^{\gamma'|\xi|}|\widetilde{\mathcal{R}}_\varepsilon(t,x,\xi)| = 0$$

Then, it holds that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{T}^d}\widetilde{\mathcal{R}}_\varepsilon(t,x,d(t,x)/\varepsilon)\,dxdt = 0.$$

(2) Let $A : [0,T]\times\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}} e^{\gamma'|\xi|}|A(t,x,\xi)| < \infty \tag{2.4}$$

Then, it holds that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{T}^d}A(t,x,d(t,x)/\varepsilon)\,dxdt = \int_0^T\int_{\Gamma_t}\int_{\mathbb{R}}A(t,x,\xi)\,d\xi d\mathcal{H}^{d-1}(dx)dt. \tag{2.5}$$

Brief proof of Proposition 2.1. We give a brief proof for the case (2). For each t, we divide the integral domain $\mathbb{T}^d$ by

$$D_1(t) := \{x\in\mathbb{T}^d : |d(t,x)|\le\kappa\} \quad\text{and}\quad D_2(t) := \{x\in\mathbb{T}^d : |d(t,x)|>\kappa\}$$

for a sufficiently small constant $\kappa > 0$. The conditions (2.4) yields

$$\left|\frac{1}{\varepsilon}\int_0^T\int_{D_2(t)}A(t,x,d(t,x)/\varepsilon)\,dxdt\right| \le \frac{C}{\varepsilon}e^{-\gamma'\kappa/\varepsilon}\to 0 \quad\text{as}\quad \varepsilon\to 0,$$

where C is a constant independent of ε. The remained integral can be calculated as

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^T\int_{D_1(t)}A(t,x,d(t,x)/\varepsilon)\,dxdt\\
&=\frac{1}{\varepsilon}\int_0^T\int_{-\kappa}^{\kappa}\int_{\{x:d(t,x)=s\}}A(t,x,s/\varepsilon)\,\mathcal{H}^{d-1}(dx)dsdt\\
&=\frac{1}{\varepsilon}\int_0^T\int_{-\kappa}^{\kappa}\int_{\Gamma_t}A(t,y+sn_t(y),s/\varepsilon)|\det(\nabla_{\Gamma_t}\mathrm{Id}(y)+s\nabla_{\Gamma_t}n_t(y))|\,\mathcal{H}^{d-1}(dy)dsdt\\
&=\int_0^T\int_{-\kappa/\varepsilon}^{\kappa/\varepsilon}\int_{\Gamma_t}A(t,y+\varepsilon\widetilde{s}n_t(y),\widetilde{s})|\det(\nabla_{\Gamma_t}\mathrm{Id}(y)+\varepsilon\widetilde{s}\nabla_{\Gamma_t}n_t(y))|\,\mathcal{H}^{d-1}(dy)d\widetilde{s}dt,
\end{aligned}$$

where y is a point on Γ_t, n_t is a unit normal vector of Γ_t, ∇_{Γ_t} is the divergence operator on Γ_t and Id is the identity map on $\mathbb{T}^d$. We note that the co-area formula (see [6, Theorem 3.10] for example) have been used at the first equality and $|\det(\nabla_{\Gamma_t}\mathrm{Id}(y)+\varepsilon\widetilde{s}\nabla_{\Gamma_t}n_t(y))|\,\mathcal{H}^{d-1}(dy)$ describes the surface area element of the surface $\{x : d(t,x)=\varepsilon\widetilde{s}\}$. The above calculations yields (2.5) by letting $\varepsilon\to 0$. □

We now return to the consideration of the limit for the power series expansion (2.2). Since the maximizer $H_{\max,\varepsilon}$ is uniquely determined depending on ϕ_ε and the form of ϕ_ε is restricted as in (1.7) (with $R_\varepsilon \equiv 0$), the limit of $\varepsilon^{-1}\iint \phi_Q^{-1} dxdt$ can be represented as a functional of Q. The minimizing problem of the functional with respect to Q is solvable, which shows that the minimum value is $S_{\mathrm{as}}(\Gamma)$ and $\widehat{Q}$ in Theorem 1.1 can be chosen as the minimizer. In this paper, the following sections will include notes not written in the original paper [10], as well as the mathematical structure that yields the propositions in each step of the proof of the main theorem described at the beginning of this section.

Remark 2.2. *In order to apply Proposition 2.1, we have to prove that the function $\widehat{H}_1(t,x,\xi)$ obtained in the decomposition* (2.3) *and the minimizer $\widehat{Q}$ respectively satisfy the exponential decay estimate with respect to ξ as in* (2.4) *and the decay estimate with respect to ξ as in* (1.8)*. Although similar estimates were discussed in the case $P(\rho) = \rho/2$ (see [2]), in our problem, the inability to write $\widehat{H}_1$ and $\widehat{Q}$ in the form of variable separations necessitated a slight re-consideration of the estimates in the previous study. In the previous problem, $\widehat{H}_1$ and $\widehat{Q}$ are separable as $\widehat{H}_1(t,x,\xi) = A(t,x)h(\xi)$ and $\widehat{Q}(t,x,\xi) = B(t,x)Q^*(\xi)$. In this paper, we omit the details of the arguments on the above estimates in our problem.*

To discuss the power series expansion (2.2) and the minimizing problem of $\widehat{S}(Q)$ in more detail, we introduce several notions and known theorems are listed. We first discuss on the ODE (1.3). A standard theory as in [13, Lemma 2.6.1] can be applied to obtain the following properties:

Lemma 2.3 (Application of [13, Lemma 2.6.1])**.** *Assume that smooth functions $P : [0,1] \to [0,\infty)$, $B, D : [0,1] \to \mathbb{R}$ and $W : [0,1] \to \mathbb{R}$ satisfy the properties* (A1)–(A4)*. Then,* (1.3) *admits a unique smooth solution. Furthermore, there exist $\gamma > 0$ and $C > 0$ such that*

$$\bar{u}'(\xi) > 0 \quad \text{for } \xi \in \mathbb{R}, \qquad |\bar{u}'(\xi)| + |\bar{u}''(\xi)| + |\bar{u}'''(\xi)| \le Ce^{-\gamma|\xi|} \quad \text{for } \xi \in \mathbb{R}.$$

The exponential decay of $\bar{u}$ is key estimate to apply Proposition 2.1. In the following arguments, we also use the composition function of P and $\overline{u}$ which is denoted by $\overline{v} := P(\overline{u})$. Let the linear operator $L_{\bar{u}} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$L_{\bar{u}}\psi(\xi) = [2\sigma(\bar{u}(\xi))\psi'(\xi)]' - [B(\bar{u}(\xi)) + D(\bar{u}(\xi))]\,\psi(\xi) \tag{2.6}$$

for $\psi \in H^2(\mathbb{R})$. Let ν be the constant defined by

$$\nu := \langle \overline{v}', (-L_{\bar{u}})\overline{v}' \rangle_{L^2}/2,$$

where $\langle \cdot,\cdot \rangle_{L^2}$ denotes the standard L^2-norm on $\mathbb{R}$. We also define the constants θ_1, θ_2 by

$$\theta_1 = \int_{\rho_-}^{\rho_+} \sqrt{2\widetilde{W}(\rho)}\, d\rho, \quad \theta_2 = \int_{\rho_-}^{\rho_+} P'(\rho)\sqrt{2\widetilde{W}(\rho)}\, d\rho,$$

where the function $\widetilde{W}$ is defined as

$$\widetilde{W}(\rho) = \int_{\rho_-}^{\rho} W'(\widetilde{\rho})P'(\widetilde{\rho})\, d\widetilde{\rho}.$$

Note that it holds that $\langle \bar{u}', \bar{v}' \rangle_{L^2} = \theta_1, \langle \bar{v}', \bar{v}' \rangle_{L^2} = \theta_2$. Then, the mobility μ and the transport coefficient θ can be chosen as

$$\mu := \nu/\theta_1^2, \quad \theta := \theta_2/\theta_1, \tag{2.7}$$

respectively.

3 Decomposition of maximizer $H_{\max,\varepsilon}$

The decomposition (2.3) can be obtained by applying the Taylor expansion for each term in (4.1). For simplicity, let $d_\varepsilon := d(t,x)/\varepsilon$ here. For example, a simple calculation yields by using the form of ϕ_ε in (1.7) (with $R_\varepsilon \equiv 0$)

$$\partial_t \phi_\varepsilon = \bar{u}'(d_\varepsilon + \varepsilon Q(t,x,d_\varepsilon)) \left(\frac{\partial_t d(t,x)}{\varepsilon} + \varepsilon \partial_t Q(t,x,d_\varepsilon) + \partial_\xi Q(t,x,d_\varepsilon) \partial_t d(t,x) \right)$$

and the Taylor expansion (for $\bar{u}'(d_\varepsilon + \varepsilon Q(t,x,d_\varepsilon))$ at the point d_ε) implies

$$\bar{u}'(d_\varepsilon + \varepsilon Q(t,x,d_\varepsilon)) = \bar{u}'(d_\varepsilon) + \varepsilon \bar{u}''(d_\varepsilon + \varepsilon\theta Q(t,x,d_\varepsilon)) Q(t,x,d_\varepsilon),$$

where $\theta \in (0,1)$ is a constant, which give us the quantity

$$\partial_t \phi_\varepsilon(t,x) = \bar{u}'(d_\varepsilon) \frac{\partial_t d(t,x)}{\varepsilon} + \mathcal{R}_\varepsilon(t,x,d_\varepsilon),$$

where the remainder $\mathcal{R}_\varepsilon : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\limsup_{\varepsilon \to 0} \sup_{(t,x,\xi) \in [0,T] \times \mathbb{T}^d \times \mathbb{R}} e^{\gamma|\xi|/2} |\mathcal{R}_\varepsilon(t,x,\xi)| < \infty. \tag{3.1}$$

This estimate follows from the exponential decay estimate of $\bar{u}$. By applying a similar argument for the remained terms in (4.1), we obtain

$$\begin{aligned}
&\bar{u}'(d_\varepsilon) \frac{\partial_t d}{\varepsilon} + \frac{2}{\varepsilon^2} \Big((\sigma \circ \bar{u})'(d_\varepsilon) \nabla H_{\max,\varepsilon} + (\sigma \circ \bar{u})(d_\varepsilon) \Delta H_{\max,\varepsilon} \Big) \\
&\approx \frac{1}{\varepsilon^2} \Big((P \circ \bar{u})''(d_\varepsilon) + (B \circ \bar{u})(d_\varepsilon) - (D \circ \bar{u})(d_\varepsilon) \Big) \\
&\quad + \frac{1}{\varepsilon} \Big((P \circ \bar{u})'''(d_\varepsilon) + (B \circ \bar{u})'(d_\varepsilon) - (D \circ \bar{u})'(d_\varepsilon) \Big) \\
&\quad + \frac{1}{\varepsilon} \Big((P \circ \bar{u})' \Delta d + 2(P \circ \bar{u})'(d_\varepsilon) \partial_\xi Q + (P \circ \bar{u})' \partial_\xi^2 Q \Big) \\
&\quad + \frac{1}{\varepsilon^2} \Big((B \circ \bar{u})(d_\varepsilon) + (D_\circ \bar{u})(d_\varepsilon) \Big) H_{\max,\varepsilon}.
\end{aligned} \tag{3.2}$$

Due to the ODE (1.3), the second line and third line vanish, which yields that $H_{\max,\varepsilon}$ converges 0 as $\varepsilon \to 0$ with the order at least $O(\varepsilon)$ so that the orders with respect to ε on the both sides in (3.2) are balanced. Therefore, $H_{\max,\varepsilon}$ should be decomposable as in (2.3) and $\widehat{H}_1$ should satisfies

$$L_{\bar{u}} \widehat{H}_1(t,x,\xi) = \bar{v}'(\xi) \Delta d(t,x) + 2\bar{v}''(\xi) \partial_\xi Q(t,x,\xi) + \bar{v}'(\xi) \partial_\xi^2 Q(t,x,\xi) - \bar{u}'(\xi) \partial_t d(t,x). \tag{3.3}$$

As a result, the following proposition holds:

Proposition 3.1. *Let $Q : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying* (1.8). *Define $\phi_\varepsilon : [0,T] \times \mathbb{T}^d \to [0,1]$ by* (1.7) *with $R_\varepsilon = 0$. Let $H_{\max,\varepsilon} : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ and $\widehat{H}_1 : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ be the solution of* (4.1) *and* (3.3)*, respectively. Define the function $\widehat{K}_\varepsilon : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ through the decomposition*

$$H_{\max,\varepsilon}(t,x) = \varepsilon \widehat{H}_1(t,x,d_\varepsilon) + \varepsilon^2 \widehat{K}_\varepsilon(t,x) \quad \textit{for} \quad (t,x) \in [0,T] \times \mathbb{T}^d.$$

Then, there exists $0 < \widetilde{\gamma} < \gamma$ such that

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}} e^{\widetilde{\gamma}|\xi|} \sum_{i=0}^{2} \sum_{j=0}^{2} |\partial_\xi^i \nabla^j \widehat{H}_1| < \infty, \tag{3.4}$$

$$\overline{\lim_{\varepsilon\to 0}} \left\{ \sup_{(t,x)\in[0,T]\times\mathbb{T}^d} |\widehat{K}_\varepsilon| + \varepsilon |\nabla \widehat{K}_\varepsilon| \right\} < \infty.$$

4 The power series expansion of $S_\varepsilon(\phi_\varepsilon)$

We next discuss the power series expansion of $S_\varepsilon(\phi_\varepsilon)$ as in (2.2). Since $H_{\max,\varepsilon}$ is the maximizer for the maximum problem as in (1.4), integrating by parts for (1.5) and substituting the Euler-Lagrange equation into it yields

$$\begin{aligned} S_\varepsilon(\phi_\varepsilon) = &\frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}^d} \sigma(\phi_\varepsilon) |\nabla H_{\max,\varepsilon}|^2 \, dxdt \\ &+ \frac{1}{\varepsilon^3} \int_0^T \int_{\mathbb{T}^d} B(\phi_\varepsilon) \left(1 - e^{H_{\max,\varepsilon}} + H_{\max,\varepsilon} e^{H_{\max,\varepsilon}}\right) \, dxdt \\ &+ \frac{1}{\varepsilon^3} \int_0^T \int_{\mathbb{T}^d} D(\phi_\varepsilon) \left(1 - e^{-H_{\max,\varepsilon}} - H_{\max,\varepsilon} e^{-H_{\max,\varepsilon}}\right) \, dxdt. \end{aligned} \tag{4.1}$$

Furthermore, due to the decomposition (2.3), we have by applying the Taylor expansion (as to obtain (3.2))

$$\begin{aligned} S_\varepsilon(\phi_\varepsilon) = &\frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}^d} \sigma(\bar{u}(d_\varepsilon))(\partial_\xi \widehat{H}(t,x,d_\varepsilon))^2 + \frac{B(\bar{u}(d_\varepsilon)) + D(\bar{u}(d_\varepsilon))}{2} (\widehat{H}_1(t,x,d_\varepsilon))^2 \, dxdt \\ &+ \int_0^T \int_{\mathbb{T}^d} \widehat{R}_\varepsilon(t,x,d_\varepsilon) \, dxdt, \end{aligned}$$

where $\widehat{R}_\varepsilon$ is a remainder satisfying (3.1). Therefore, letting $\varepsilon \to 0$, we obtain by applying Proposition 2.1

$$\begin{aligned} &\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) \\ &= \int_0^T \int_{\Gamma_t} \int_{\mathbb{R}} \sigma(\bar{u}(\xi))(\partial_\xi \widehat{H}(t,x,\xi))^2 + \frac{B(\bar{u}(\xi)) + D(\bar{u}(\xi))}{2} (\widehat{H}_1(t,x,\xi))^2 \, d\xi d\mathcal{H}^{n-1}(x) dt \end{aligned}$$

Recalling the definition of $L_{\bar{u}}$ in (2.6), since H_1 is the solution of (3.3), the limit can be re-written as

$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) = \int_0^T \int_{\Gamma_t} \int_{\mathbb{R}} F_Q (-L_{\bar{u}})^{-1} F_Q \, d\xi d\mathcal{H}^{n-1}(x) dt,$$

where $F_Q : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ is a function defined (depending on Q) as

$$F_Q(t,x,\xi) := \bar{u}'(\xi)\partial_t d(t,x) - \bar{v}'(\xi)\Delta d(t,x) - 2\bar{v}''(\xi)\partial_\xi Q(t,x,\xi) - \bar{v}'(\xi)\partial_\xi^2 Q(t,x,\xi).$$

Thus, it is sufficient to prove that

$$\inf_Q \int_0^T \int_{\Gamma_t} \int_{\mathbb{R}} F_Q(-L_{\bar{u}})^{-1} F_Q \, d\xi d\mathcal{H}^{n-1}(x) dt = S_{\mathrm{ac}}(\Gamma) \tag{4.2}$$

and the minimum is achieved when $Q = \widehat{Q}$ to prove the second claim in Theorem 1.1.

Remark 4.1. *In the first claim in Theorem 1.1, the vanishing property $R_\varepsilon \equiv 0$ is not assumed, and thus the decomposition of $H_{\max,\varepsilon}$ as in* (2.3) *is not applicable according to the assumption in Proposition 3.1. However, due to the definition of S_ε as in* (1.4)*, we have*

$$S_\varepsilon(\phi_\varepsilon) \geq J_\varepsilon^{\varepsilon \widehat{H}_1}(\phi_\varepsilon),$$

where $\widehat{H}_1$ is the solution of (3.3) *(which is defined depending on Q). Although the Euler-Lagrange equation cannot be applied as when $R_\varepsilon \equiv 0$, the limit of the functional $J_\varepsilon^{\varepsilon \widehat{H}_1}(\phi_\varepsilon)$ can be calculated by using the Taylor expansion and the estimate of H_1 in* (3.4) *as*

$$J_\varepsilon^{\varepsilon \widehat{H}_1}(\phi_\varepsilon) = \int_0^T \int_{\Gamma_t} \int_{\mathbb{R}} F_Q(-L_{\bar{u}})^{-1} F_Q \, d\xi d\mathcal{H}^{n-1}(x) dt.$$

The above explanation also explains why only the lower semi-continuity, not the full-convergence, can be shown when $R_\varepsilon \not\equiv 0$.

5 Minimizing problem

In this section, for each fixed point $(t,x) \in [0,T] \times \mathbb{T}^d$, we discuss the minimizing problem

$$\inf_{\bar{Q}} \int_{\mathbb{R}} F_{\bar{Q}}(-L_{\bar{u}})^{-1} F_{\bar{Q}} \, d\xi,$$

where $\bar{Q} : [0,T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$\sup_{(t,x,\xi) \in [0,T] \times \mathbb{T}^d \times \mathbb{R}} \frac{|\bar{Q}| + |\partial_\xi \bar{Q}| + |\partial_\xi^2 \bar{Q}|^2}{1 + |\xi|} < \infty. \tag{5.1}$$

Our purpose is to prove the following proposition:

Proposition 5.1. *Let $\bar{Q} : [0,T] \times \mathbb{T}^d \times \mathbb{R}$ be a smooth function satisfying* (5.1)*. Then, it holds that*

$$\int_{\mathbb{R}} F_{\bar{Q}}(-L_{\bar{u}})^{-1} F_{\bar{Q}} \, d\xi \geq \frac{(\partial_t d - \theta \Delta d)^2}{2\mu} \quad \textit{for} \ (t,x) \in [0,T] \times \mathbb{T}^d.$$

Furthermore, a minimizer $\bar{Q}_{\min}$ is given by

$$\begin{aligned}&\bar{Q}_{\min}(t,x,\xi)\\ &= \int_0^{\xi} \frac{1}{(\bar{v}')^2(\widetilde{\xi})} \int_{-\infty}^{\widetilde{\xi}} \left(\bar{u}'(\widehat{\xi})\partial_t d(t,x) - \bar{v}'(\widehat{\xi})\Delta d(t,x) - \frac{\lambda(t,x)}{2} L_{\bar{u}} \bar{v}'(\widehat{\xi}) \right) \bar{v}'(\widehat{\xi})\, d\widehat{\xi} d\widetilde{\xi},\end{aligned} \tag{5.2}$$

where $\lambda : [0,T] \times \mathbb{T}^d$ is a smooth function defined as

$$\lambda(t,x) = \frac{2(\|\bar{v}'\|_{L^2}^2 \Delta d(t,x) - \langle \bar{u}', \bar{v}'\rangle_{L^2} \partial_t d(t,x))}{\langle -L_{\bar{u}} \bar{v}', \bar{v}'\rangle_{L^2}}, \tag{5.3}$$

and $\bar{Q}_{\min}$ satisfies (1.8) *replaced Q by $\bar{Q}_{\min}$.*

Brief proof of Proposition 5.1. We use $\bar{Q}'$ instead of $\partial_\xi \bar{Q}(t,x,\xi)$ and omit the variables t,x for simplicity. Noticing $2\bar{v}''\bar{Q}' + \bar{v}'\bar{Q}''$ is perpendicular with $\bar{v}'$ in $L^2(\mathbb{R})$, we can re-formulate the minimizing problem as

$$\begin{aligned}&\inf \left\{ \int_{\mathbb{R}} F_{\bar{Q}} (-L_{\bar{u}})^{-1} F_{\bar{Q}} \, d\xi : \bar{Q} \text{ satisfies (5.1)} \right\} \\ &\geq \inf \left\{ \langle \bar{u}'\partial_t d - \bar{v}'\Delta d - \psi, (-L_{\bar{u}})^{-1} (\bar{u}'\partial_t d - \bar{v}'\Delta d - \psi)\rangle_{L^2} : \psi \in L^2(\mathbb{R}) \text{ s.t. } \psi \perp \bar{v}' \right\},\end{aligned}$$

where we denote $\psi \perp \phi$ for $\psi, \phi \in L^2(\mathbb{R})$ if $\langle \psi, \phi\rangle_{L^2} = 0$. We note that the equality holds if a minimizer $\psi_{\min}$ for the latter minimizing problem exists and a solution $\bar{Q}_{\min}$ to

$$2\bar{v}''\bar{Q}'_{\min} + \bar{v}'\bar{Q}''_{\min} = \psi_{\min} \tag{5.4}$$

satisfies (5.1); hence, it is sufficient to solve the solution $\bar{Q}_{\min}$ and prove that $\bar{Q}_{\min}$ satisfies the stronger estimate (1.8) than (5.1).

We thus define functional

$$G(\psi) := \langle \bar{u}'\partial_t d - \bar{v}'\Delta d - \psi, (-L_{\bar{u}})^{-1} (\bar{u}'\partial_t d - \bar{v}'\Delta d - \psi)\rangle_{L^2} \quad \text{for } \psi \in L^2(\mathbb{R})$$

and consider the minimizing problem

$$\inf_{\psi \in L^2 : \psi \perp \bar{v}'} G(\psi). \tag{5.5}$$

Applying the method of Lagrange multiplier, we see that a minimizer $\psi_{\min} \in L^2(\mathbb{R})$ of (5.5) satisfies

$$\langle \phi, (-L_{\bar{u}})^{-1} (\bar{u}'\partial_t d - \bar{v}'\Delta d - \psi_{\min})\rangle_{L^2} + \langle \bar{u}'\partial_t d - \bar{v}'\Delta d - \psi_{\min}, (-L_{\bar{u}})^{-1}\phi\rangle_{L^2} = \lambda \langle \bar{v}', \phi\rangle_{L^2}$$

for any $\phi \in L^2(\mathbb{R})$, where λ is the Lagrange multiplier, if the minimizer exists. Since $L_{\bar{u}}$ is self-adjoint on $L^2(\mathbb{R})$, it is equivalent to

$$\psi_{\min} = \bar{u}'\partial_t d - \bar{v}'\Delta d - \frac{\lambda}{2} L_{\bar{u}} \bar{v}'.$$

Therefore, the orthogonal condition $\psi_{\min} \perp \bar{v}'$ shows that λ is given by (5.3) if the minimizer $\psi_{\min}$ exists. We next prove that $\psi_{\min}$ is a minimizer of (5.5). For this purpose, note that λ is chosen so that $\psi_{\min} \perp \bar{v}'$ holds. Therefore it is enough to prove $G(\psi_{\min}+\psi) \geq G(\psi_{\min})$ for any function $\psi \in L^2(\mathbb{R})$ with $\psi \perp \bar{v}'$. By direct calculations, we have

$$G(\psi_{\min}) = \frac{\lambda^2}{4}\langle -L_{\bar{u}}\bar{v}', \bar{v}'\rangle_{L^2} = \frac{(\partial_t d\langle \bar{u}', \bar{v}'\rangle_{L^2} - \Delta d\|\bar{v}'\|_{L^2}^2)^2}{\langle -L_{\bar{u}}\bar{v}', \bar{v}'\rangle_{L^2}} = \frac{(\partial_t d - \theta\Delta d)^2}{2\mu}.$$

On the other hand, since $L_{\bar{u}}$ is self-adjoint on $L^2(\mathbb{R})$ and $\psi \perp \bar{v}'$ holds, we obtain

$$\begin{aligned} G(\psi_{\min}+\psi) &= \frac{\lambda^2}{4}\langle -L_{\bar{u}}\bar{v}', \bar{v}'\rangle_{L^2} + \frac{\lambda}{2}\left(\langle L_{\bar{u}}\bar{v}', (-L_{\bar{u}})^{-1}\psi\rangle_{L^2} - \langle \psi, \bar{v}'\rangle_{L^2}\right) + \langle \psi, (-L_{\bar{u}})^{-1}\psi\rangle_{L^2} \\ &= G(\psi_{\min}) + \langle \psi, (-L_{\bar{u}})^{-1}\psi\rangle_{L^2}. \end{aligned}$$

Letting $\phi := (-L_{\bar{u}})^{-1}\psi$, we see

$$\langle \psi, (-L_{\bar{u}})^{-1}\psi\rangle_{L^2} = \int_{\mathbb{R}} 2\sigma(\bar{u})(\phi')^2 + [B(\bar{u}) + D(\bar{u})]\phi^2 \, d\xi \geq 0,$$

which yields

$$G(\widetilde{\psi}) \geq G(\psi_{\min}) = \frac{(\partial_t d - \theta\Delta d)^2}{2\mu} \quad \text{for } \widetilde{\psi} \in L^2(\mathbb{R}) \text{ with } \widetilde{\psi} \perp \bar{v}'.$$

Therefore, $\psi_{\min}$ is a minimizer of the minimizing problem (5.5).

Multiplying $\bar{v}'$ by the both sides of and integrating it, we have

$$(\bar{v}')^2(\xi)\bar{Q}'_{\min}(t,x,\xi) = \int_{-\infty}^{\xi} \left(\bar{u}'(\widehat{\xi})\partial_t d(t,x) - \bar{v}'(\widehat{\xi})\Delta d(t,x) - \frac{\lambda(t,x)}{2}L_{\bar{u}}\bar{v}'(\widehat{\xi})\right)\bar{v}'(\widehat{\xi})\, d\widehat{\xi},$$

which yields (5.2). We here omit the arguments on the estimate (1.8). □

Due to the Proposition 5.1, we can prove the second claim in Theorem 1.1 by choosing $\widehat{Q} = \bar{Q}_{\min}$.

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