Positive solutions to stationary double power nonlinear Schrödinger equations

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1 Introduction

In this note, we report a result which is obtained in a joint work with Takafumi Akahori (Shizuoka University), Slim Ibrahim (Victoria University) and Masataka Shibata (Meijo University). We consider the following semilinear elliptic equations:

$$-\Delta u + \omega u - |u|^{p-1}u - |u|^{2^*-2}u = 0 \qquad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $d \geq 3, \omega > 0, 1 < p < 2^* - 1$ and $2^* = \frac{2d}{d-2}$. The equation (1.1) derives from the following Schrödinger equations:

$$i\partial_t \psi + \Delta \psi + |\psi|^{p-1}\psi + |\psi|^{2^*-2}\psi = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^d. \tag{1.2}$$

Indeed, if we consider the so-called *standing wave solution*, which is a solution (1.2) of the form

$$\psi(t,x) = e^{i\omega t}u_\omega(x) \qquad (\omega > 0),$$

we see that u_ω satisfies (1.1).

Here, we pay our attention to positive solutions u to (1.1). From the result of Gidas, Ni and Nirenberg [7], we see that u is radially symmetric and monotone decreasing in $|x| > 0$. It follows from the result of Wei and Wu [12] that when $d = 3$ and $1 < p < 3$, (1.1) has two distinct solutions if $\omega > 0$ is sufficiently large (Unusually, positive solution to (1.1) is not unique). This result coincides with the numerical computation by Dávila, del Pino and I. Guerra [8] and Yagasaki [13]. One of the positive solutions is the ground state. We are concerned with the one, which is different from the ground state.

2 Ground State

First of all, let us explain about the ground state. If we define an action functional by

$$\mathcal{S}_\omega(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*},$$

we see that $u \in H^1(\mathbb{R}^d)$ is a solution to (1.1) if and only if u is a critical point of the functional $\mathcal{S}_\omega$. To seek a critical point of $\mathcal{S}_\omega$, we consider the following minimization problem:

$$m_\omega := \inf\left\{\mathcal{S}_\omega(u) \colon u \in H^1(\mathbb{R}^d) \setminus \{0\},\ \mathcal{N}_\omega(u) = 0\right\},$$

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where the Nehari functional $\mathcal{N}_\omega$ is defined by

$$\mathcal{N}_\omega(u) := \langle \mathcal{S}'_\omega(u), u\rangle = \|\nabla u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 - \|u\|_{L^{p+1}}^{p+1} - \|u\|_{L^{2^*}}^{2^*}.$$

We call a minimizer for m_ω a *ground state*. Zhang and Zou [14] showed that the ground state exists for any $\omega > 0$ when $d \geq 4$ and $1 < p < 2^* - 1$ or $d = 3$ and $3 < p < 2^* - 1(= 5)$. In [4] and [12], it is shown independently that there exists $\omega_c > 0$ such that the ground state exists for any $\omega \in (0, \omega_c)$ and does not exist for any $\omega \in (\omega_c, \infty)$. In addition, at $\omega = \omega_c$, the ground state exists when $1 < p < 3$ $(p \neq 3)$.

Remark 2.1. *(i) We do not know whether the ground state exists or not at $\omega = \omega_c$ when $d = 3$ and $p = 3$.*

(ii) The reason why the situation becomes different when $d = 3$ comes from an integrability of Aubin–Talenti function

$$W(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}},$$

which is a solution to the following limit equations of (1.1)*:*

$$-\Delta W = W^{2^*-1} \qquad \text{in } \mathbb{R}^d.$$

*We can easily find that $W \in L^2(\mathbb{R}^d)$ when $d \geq 5$ and $W \notin L^2(\mathbb{R}^d)$ when $d = 3, 4$ *. See [4] and [12] for about the existence and non-existence of the ground state in case of $d = 3$ and $1 < p \leq 3$ in detail.*

3 Second positive solution

Next, we seek a positive solution, which is different from the ground state. To this end, we consider the minimization problem related to conservation laws of (1.2). It is known that for any $\psi_0 \in H^1(\mathbb{R}^d)$, there exists a unique solution ψ in $C(I_{\max}; H^1(\mathbb{R}^d))$ with $\psi|_{t=0} = \psi_0$ for a maximal existence interval $I_{\max} = (-T_{\max}^-, T_{\max}^+) \subset \mathbb{R}$, and the solution ψ satisfies the following conservation laws of the mass and the energy in this order:

$$\mathcal{M}(\psi(t)) = \mathcal{M}(\psi_0), \qquad \mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0) \qquad \text{for all } t \in I_{\max},$$

where

$$\mathcal{M}(u) := \frac{1}{2}\|u\|_{L^2}^2, \qquad \mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*}.$$

If, in addition, $\psi_0 \in L^2(\mathbb{R}^d, |x|^2dx)$, then the corresponding solution ψ also belongs to $C(I_{\max}; L^2(\mathbb{R}^d, |x|^2dx))$ and satisfies the so-called virial identity:

$$\int_{\mathbb{R}^d} |x|^2|\psi(t,x)|^2dx = \int_{\mathbb{R}^d} |x|^2|\psi_0(x)|^2dx + 4t \,\mathrm{Im}\int_{\mathbb{R}^d} x\cdot\nabla\psi_0(x)\overline{\psi_0(x)}dx + 8\int_0^t\int_0^{t'} \mathcal{K}(\psi(t''))dt''dt' \qquad \text{for any } t \in I_{\max},$$

where

$$\mathcal{K}(u) := \frac{d}{d\lambda}\mathcal{S}_\omega(\lambda^{\frac{d}{2}}u(\lambda\cdot))|_{\lambda=1} = \|\nabla u\|_{L^2}^2 - \frac{d(p-1)}{2(p+1)}\|u\|_{L^{p+1}}^{p+1} - \|u\|_{L^{2^*}}^{2^*}.$$

Note that $\mathcal{M}((\lambda^{\frac{d}{2}}u(\lambda\cdot))) = \mathcal{M}(u)$ for any $\lambda > 0$. Then, for each $m > 0$, we consider the following minimization problem:

$$E(m) := \inf\left\{\mathcal{E}(u) \colon u \in H^1(\mathbb{R}^d),\ \mathcal{M}(u) = m,\ \mathcal{K}(u) = 0\right\}.$$

*When $d = 4$, $W \in L^{2+\varepsilon}(\mathbb{R}^d)$ for any $\varepsilon > 0$.

Here, the following L^2-critical exponent

$$p_* = 1 + \frac{4}{d}$$

plays an important result. To explain the exponent p_*, we consider the following nonlinear Schrödinger equations:

$$i\partial_t \psi + \Delta\psi + |\psi|^{p-1}\psi = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^d. \tag{3.1}$$

The equation (3.1) is scale invariant. More precisely, putting

$$\psi_\lambda(t,x) := \lambda^{\frac{2}{p-1}} \psi(\lambda^2 t, \lambda x) \qquad (\lambda > 0), \tag{3.2}$$

we see that if $\psi(t,x)$ satisfies (3.1), so does ψ_λ. Note that

$$\|\psi_\lambda(0,\cdot)\|_{L^2}^2 = \lambda^{\frac{4}{p-1}-d} \|\psi(0,\cdot)\|_{L^2}^2 \qquad (\lambda > 0).$$

Thus, the scaling (3.2) preserves the L^2-norm when $p = 1+4/d$. For this, the exponent $p_* = 1+4/d$ is referred to as "L^2-critical".

We now go back to our equation (1.1). From the result of Soave [10], we know the following:

Theorem 3.1 (Soave [10]). *Let $d \geq 3$ and $1 + \frac{4}{d} < p < 2^* - 1$.*

(i) For each $m > 0$, there exists a minimizer for $E(m)$, which satisfies (1.1) with $\omega = \omega(m)$ for some $\omega(m) > 0$.

(ii) $E(m)$ is non-increasing in $m > 0$.

Remark 3.1. *Soave [10], Jeanjean and Lu [9] and Wei and Wu [11] also obtained several results for the L^2-critical and L^2-subcritical case ($1 < p \leq 1 + \frac{4}{d}$).*

Theorem 3.2 (Wei and Wu [12]). *Let $d = 3$ and $1 + \frac{4}{d} < p < 3$ and $\omega(m) > 0$ be the number given in Theorem 3.1. Then, we have $\lim_{m\to 0} \omega(m) = 0$.*

We denote the minimizer for $E(m)$ by $R_{\omega(m)}$. Let us explain briefly that for sufficiently small $m > 0$, we have $R_{\omega(m)} \neq Q_{\omega(m)}$, where $Q_{\omega(m)}$ is the ground state in case of $\omega = \omega(m)$. We rescale Q_ω as follows:

$$Q_\omega(x) = \omega^{\frac{1}{p-1}} \widetilde{Q}_\omega(\omega^{\frac{1}{2}} x).$$

Then, $\widetilde{Q}_\omega$ satisfies

$$-\Delta\widetilde{Q}_\omega + \widetilde{Q}_\omega - \widetilde{Q}_\omega^p - \omega^{\frac{2^*-(p+1)}{p-1}} \widetilde{Q}_\omega^{2^*-1} = 0 \qquad \text{in } \mathbb{R}^d. \tag{3.3}$$

By a standard argument, we see that

$$\lim_{\omega\to 0} \|\widetilde{Q}_\omega - U\|_{H^1} = 0, \tag{3.4}$$

where $U \in H^1(\mathbb{R}^d)$ is the unique positive solution to the following scalar field equations:

$$-\Delta U + U - U^p = 0 \qquad \text{in } \mathbb{R}^d. \tag{3.5}$$

This yields that

$$\lim_{\omega\to 0} \omega^{-\frac{2}{p-1}+\frac{d}{2}} \|Q_\omega\|_{L^2}^2 = \lim_{\omega\to 0} \|\widetilde{Q}_\omega\|_{L^2}^2 = \|U\|_{L^2}^2.$$

Note that $-\frac{2}{p-1} + \frac{d}{2} > 0$ for $1 + \frac{4}{d} < p < 2^* - 1$. Thus, we have $\lim_{\omega\to 0} \|Q_\omega\|_{L^2}^2 = \infty$ if $1 + \frac{4}{d} < p < 2^* - 1$. From this and $\lim_{m\to 0} \omega(m) = 0$ (see Theorem 3.2), we find that $\|R_{\omega(m)}\|_{L^2} = 2m \ll \|Q_{\omega(m)}\|_{L^2}$ for sufficiently small $m > 0$, which implies that $R_{\omega(m)} \neq Q_{\omega(m)}$. Note that both $Q_{\omega(m)}$ and $R_{\omega(m)}$ are positive and satisfy (1.1) with $\omega = \omega(m)$. Thus, non-uniqueness of positive solutions holds in case of $d = 3$ and $1 < p < 3$.

Remark 3.2. *Contrary to the case where $d = 3$ and $1 < p < 3$, in [1, Proposition C.1], it is shown that (1.1) admits a unique positive solution for any $\omega > 0$ when $3 \leq d \leq 6$ and $\frac{4}{d-2} < p < 2^* - 1$.*

4 Main result

We study the uniqueness and non-degeneracy of the minimizer $R_{\omega(m)}$ for sufficiently small $m > 0$. Here, we say that $R_{\omega(m)}$ is *non-degenerate* in $H^1_{\mathrm{rad}}(\mathbb{R}^d)$ if the linearized equation

$$L_{m,+}z = 0 \qquad \text{in } \mathbb{R}^d,$$

has no non-trivial solution in $H^1_{\mathrm{rad}}(\mathbb{R}^d)$, where

$$L_{m,+} = -\Delta + \omega(m) - pR^{p-1}_{\omega(m)} - (2^* - 1)R^{2^*-2}_{\omega(m)}.$$

Theorem 4.1. *There exists $m_1 > 0$ such that $R_{\omega(m)}$ is unique up to phase transition and non-degenerate in $H^1_{rad}(\mathbb{R}^d)$ for $m \in (0, m_1)$.*

The proof of the uniqueness is based on that of Wei and Wu [11]. For the proof of the non-degeneracy of $R_{\omega(m)}$, we follow the argument of [1, 3, 5]. However, the details are different and we need to overcome several difficulties.

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