Qualitative analysis of space-time periodic homogenization for nonlinear diffusion equations *

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1 Introduction

In this note, we consider the following nonlinear diffusion equation:

$$(P_\varepsilon) \qquad \begin{cases} \partial_t u_\varepsilon = \operatorname{div}\left(a_\varepsilon \nabla |u_\varepsilon|^{p-1}u_\varepsilon\right) & \text{in } \Omega \times I, \\ |u_\varepsilon|^{p-1}u_\varepsilon|_{\partial\Omega} = 0, \quad u_\varepsilon|_{t=0} = u_0, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $I = (0,T)$, $0 < p < +\infty$, $\varepsilon > 0$ and $u_0 \in L^{p+1}(\Omega)$. Let $\square = (0,1)^N$ and $J = (0,1)$. Let $a = a(y,s) \in [W^{1,1}(\mathbb{R}_+; L^\infty(\mathbb{R}^N))]^{N\times N}$ be an $N \times N$ symmetric matrix field satisfying $(\square \times J)$-periodicity and the uniform ellipticity, i.e., there exists $\lambda > 0$ such that $\lambda|\xi|^2 \leq a(y,s)\xi \cdot \xi \leq |\xi|^2$ for all $\xi \in \mathbb{R}^N$ and a.e. $(y,s) \in \mathbb{R}^N \times \mathbb{R}_+$. The coefficient matrix field a_ε is given as $a_\varepsilon = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$ for $x \in \Omega$ and $t \in I$.

Homogenization is known as a method of asymptotic analysis for complex structures and systems. Actually, it is often used to replace heterogeneous materials with a large number of microstructures, such as composite materials, with an equivalent homogeneous material; for instance, it is applied to models of heat conduction in composite materials. Such models are often described as linear diffusion equations (LDEs), and then their *space-time homogenization* oscillating both in space and time has been studied in various mathematical fields.

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Space-time homogenization problems for linear diffusion equations were first studied by Bensoussan, Lions and Papanicolaou in [8] based on a method of *asymptotic expansion*, and then various methods have been developed (see, e.g., [14] for *two-scale convergence theory* and [5] for *unfolding method*). Furthermore, homogenization problems for parabolic equations have been studied not only for linear ones but also for nonlinear ones. In [15, 18, 24], doubly-nonlinear parabolic equations are treated, and moreover, as for degenerate p-Laplace parabolic equations, homogenization problems involving scale parameters (e.g., $r > 0$ of $\mathrm{div}[A(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}, \nabla u_\varepsilon)]$) are discussed in [13, 25].

In this note, the critical case of *porous medium types* is treated, and in particular, it is revealed that the difference between degeneracy and singularity of diffusion is deeply related to the representation of the so-called *homogenized matrices*.

2 Main results

We first define *weak solutions* $u_\varepsilon = u_\varepsilon(x,t) : \Omega \times I \to \mathbb{R}$ of (P_ε) as follows:

Definition 2.1. *A function $u_\varepsilon = u_\varepsilon(x,t) : \Omega \times I \to \mathbb{R}$ is called a* weak solution *to (P_ε), if the following conditions are all satisfied:*

(i) $u_\varepsilon \in W^{1,2}(I; H^{-1}(\Omega)) \cap L^{p+1}(\Omega \times I)$, $|u_\varepsilon|^{p-1}u_\varepsilon \in L^2(I; H^1_0(\Omega))$ and $u_\varepsilon(t) \to u_0$ strongly in $H^{-1}(\Omega)$ as $t \to 0_+$.

(ii) It holds that

$$\langle \partial_t u_\varepsilon(t), \phi \rangle_{H^1_0(\Omega)} + \int_\Omega a_\varepsilon \nabla(|u_\varepsilon|^{p-1}u_\varepsilon)(x,t) \cdot \nabla \phi(x) \, \mathrm{d}x = 0$$

for a.e. $t \in I$ and all $\phi \in H^1_0(\Omega)$.

Remark 2.2. For $p \neq 1$, the nonlinear diffusion equation (P_ε) is called a *porous medium equation* (PME) if $1 < p < +\infty$ and a *fast diffusion equation* (FDE) if $0 < p < 1$ (see [21, 22] for details). The well-posedness for (P_ε) can be obtained by [1, 2].

Now, our main results read,

Theorem 2.3. *Let $\varepsilon_n \to 0_+$ be an arbitrary sequence in $(0, +\infty)$. In addition, suppose that $u_0 \in L^2(\Omega)$ for $p \in (0,1)$. Let u_{ε_n} be the unique weak solution to (P_{ε_n}) . Then there exist a subsequence of (ε_n) and functions*

$$u \in W^{1,2}(I; H^{-1}(\Omega)) \cap L^{p+1}(\Omega \times I) \cap C_{\mathrm{weak}}(\overline{I}; L^2(\Omega)),$$
$$z \in L^2(\Omega \times I; L^2(J; H^1_{\mathrm{per}}(\square)/\mathbb{R}))$$

such that $|u|^{p-1}u \in L^2(I; H^1_0(\Omega))$,

$$|u_{\varepsilon_n}|^{p-1}u_{\varepsilon_n} \to |u|^{p-1}u \quad \text{weakly in } \ L^2(I; H^1_0(\Omega)),$$
$$u_{\varepsilon_n} \to u \quad \text{strongly in } \ L^\rho(I; L^{p+1}(\Omega))$$

for any $\rho \in [1, +\infty)$ *and*

$$a_{\varepsilon_n} \nabla |u_{\varepsilon_n}|^{p-1}u_{\varepsilon_n}$$
$$\to j_{\mathrm{hom}} := \left\langle a(\cdot,\cdot)\left(\nabla |u|^{p-1}u + \nabla_y z\right)\right\rangle_{y,s} \quad \text{weakly in } \ [L^2(\Omega \times I)]^N.$$

Here and henceforth, $H^1_{\mathrm{per}}(\square)/\mathbb{R} := \{g \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \colon g$ *is* $\square$*-periodic and* $\langle g\rangle_y := \int_\square g(y)\,\mathrm{d}y = 0\}$, ∇_y *denotes the gradient operator with respect to* y *and* $\langle\cdot\rangle_{y,s}$ *denotes the mean over* $\square \times J$, *that is,*

$$\langle g\rangle_{y,s} = \int_0^1 \int_\square g(y,s)\,\mathrm{d}y\mathrm{d}s \quad \text{for } \ g \in L^1(\square \times J).$$

Moreover, the limit u *solves the weak form of the homogenized equation,*

$$(P_0) \qquad \begin{cases} \langle \partial_t u(t), \phi\rangle_{H^1_0(\Omega)} + \displaystyle\int_\Omega j_{\mathrm{hom}}(x,t) \cdot \nabla\phi(x)\,\mathrm{d}x = 0 \ \ \text{for } \ \phi \in H^1_0(\Omega), \\ u(\cdot, 0) = u_0 \ \ \text{in } \Omega \end{cases}$$

for a.e. $t \in I$.

Furthermore, the homogenized diffusion flux j_{hom} is characterized as follows:

Theorem 2.4. *In addition to all the assumptions of Theorem 2.3, suppose that* $u_0 \geq 0$ *for* $p \geq 2$,

$$u_0 \in L^{3-p}(\Omega) \ \text{ if } \ p \in (0,1]\,; \ \log u_0 \in L^1_{\mathrm{loc}}(\Omega) \ \text{ if } \ p = 3\,; \ u_0^{3-p} \in L^1_{\mathrm{loc}}(\Omega) \ \text{ if } \ p \in (3, +\infty).$$

Let u *be a limit of weak solutions* (u_{ε_n}) *to* (P_{ε_n}) *as a sequence* $\varepsilon_n \to 0_+$ *and let* u *be a weak solution of the homogenized equation* (P_0). *Then* $z = z(x,t,y,s)$ *appeared in Theorem 2.3 is represented as*

$$z(x,t,y,s) = \sum_{k=1}^N (\partial_{x_k} v(x,t))\Phi_k(x,t,y,s),$$

where

$$v = \begin{cases} |u|^{p-1}u & \text{if } \ p \in (0,2), \\ u^p & \text{if } \ p \in [2, +\infty) \end{cases}$$

and $\Phi_k \in L^\infty(\Omega \times I; L^2(J; H^1_{\mathrm{per}}(\square)/\mathbb{R}))$ *is the corrector characterized as follows:*

(i) *In case* $p \in (0,1]$ *(i.e., FDE and LDE),* $\Phi_k = \Phi_k(x,t,y,s)$ *solves the cell problem,*

$$\begin{cases} \frac{1}{p}|u(x,t)|^{1-p}\partial_s \Phi_k(x,t,y,s) = \mathrm{div}_y\left(a(y,s)\left[\nabla_y \Phi_k(x,t,y,s) + e_k\right]\right) & \text{in } \square \times J, \\ \Phi_k(x,t,y,0) = \Phi_k(x,t,y,1) & \text{in } \square \end{cases}$$

for each $(x,t) \in \Omega \times I$. *Here* $\{e_k\} = \{[\delta_{jk}]_{j=1,2,\dots,N}\}$ *stands for a canonical basis of* $\mathbb{R}^N$.

(ii) *In case* $p \in (1,+\infty)$ *(i.e., PME),* Φ_k *is given by*

$$\Phi_k(x,t,y,s) = \begin{cases} p|u(x,t)|^{p-1}\Psi_k(x,t,y,s) & \text{if } u(x,t) \neq 0, \\ 0 & \text{if } u(x,t) = 0, \end{cases}$$

where $\Psi_k = \Psi_k(x,t,y,s)$ *solves the cell problem,*

$$\begin{cases} \partial_s \Psi_k(x,t,y,s) = \mathrm{div}_y\left(a(y,s)\left[p|u(x,t)|^{p-1}\nabla_y \Psi_k(x,t,y,s) + e_k\right]\right) & \text{in } \square \times J, \\ \Psi_k(x,t,y,0) = \Psi_k(x,t,y,1) & \text{in } \square \end{cases}$$

for each $(x,t) \in [u \neq 0] := \{(x,t) \in \Omega \times I : u(x,t) \neq 0\}$.

Moreover, the homogenized flux $j_{\mathrm{hom}}(x,t)$ *can be written as*

$$j_{\mathrm{hom}}(x,t) = a_{\mathrm{hom}}(x,t)\nabla v(x,t),$$

where a_{hom} *is the homogenized matrix given by*

$$a_{\mathrm{hom}}(x,t)e_k = \int_0^1 \int_\square a(y,s)\left[\nabla_y \Phi_k(x,t,y,s) + e_k\right] \mathrm{d}y\mathrm{d}s. \tag{1}$$

Remark 2.5. The homogenized matrix (1) is described in terms of solutions to cell problems. For the nonlinear diffusion case $p \neq 1$, the cell problem involves the limit $u(x,t)$ of solutions, which is a function of (x,t). Thus Φ_k also depends on (x,t), and hence, so does a_{hom}. On the other hand, for the linear diffusion case $p = 1$, Φ_k is independent of (x,t). Thus a_{hom} is a constant $N \times N$ matrix. In particular, it is noteworthy that the representation of a_{hom} depends on $p \in (0,+\infty)$ since it is determined by Φ_k.

As for the qualitative properties of a_{hom}, we have

Propositon 2.6 (cf. [2, Proposition 1.8]). *Under the same assumptions as in Theorem* 2.4, *let* a_{hom} *and* $\{\Phi_k\}_{k=1,2,\dots,N}$ *be defined as in Theorem* 2.4. *Then the following* (i) *and* (ii) *hold true*:

(i) (*Improved uniform ellipticity*) *It holds that*

$$\lambda \sum_{k=1}^{N} \left(1 + \int_0^1 \|\Phi_k(x,t,\cdot,s)\|^2_{L^2(\square)}\,\mathrm{d}s\right)|\xi_k|^2$$
$$\leq a_{\mathrm{hom}}(x,t)\xi\cdot\xi \leq \sum_{k=1}^{N} \left(1 + \int_0^1 \|\Phi_k(x,t,\cdot,s)\|^2_{L^2(\square)}\,\mathrm{d}s\right)|\xi_k|^2$$

for any $\xi = [\xi_k]_{k=1,2,\dots,N} \in \mathbb{R}^N$ *and a.e.* $(x,t) \in \Omega \times I$.

(ii) (*Symmetry and asymmetry*) *The homogenized matrix* $a_{\mathrm{hom}}(x,t)$ *is not symmetric* (*respectively, symmetric*) *when* $u(x,t) \neq 0$ (*respectively,* $u(x,t) = 0$).

As mentioned in [2, 6, 12, 14, 18], the gradient ∇v_{ε_n} does not converge to ∇v strongly in $[L^2(\Omega \times I)]^N$ in general. Indeed, one can prove that

$$\nabla v_{\varepsilon_n} - \nabla v - \sum_{k=1}^{N} (\partial_{x_k} v)\nabla_y \Phi_k(x,t,\tfrac{x}{\varepsilon_n},\tfrac{t}{\varepsilon_n^2}) \to 0 \text{ strongly in } [L^2(\Omega\times I)]^N, \tag{2}$$

and hence, due to the oscillation of Φ_k, the breaking of strong compactness in $L^2(I;H^1_0(\Omega))$ is obtained. However, to guarantee strong convergence (2), we shall require regularities:$\nabla v \in [L^\sigma(\Omega\times I)]^N$ and $\nabla_y \Phi_k \in [L^\rho(\Omega\times I)]^N$ along with $\frac{1}{\sigma} + \frac{1}{\rho} = \frac{1}{2}$. Hence additional assumptions for the coefficient $a(y,s)$ and given data will also be required. This note provides a corrector result (introduced by [10]) without assumptions for the smoothness of $a(y,s)$.

Theorem 2.7. *Let* u *be a limit of weak solutions* (u_{ε_n}) *to* (P_{ε_n}) *as a sequence* $\varepsilon_n \to 0_+$ *such that* u *is a weak solution to* (P_0) *and let* Φ_k *be the corrector given by Theorem* 2.4. *Set*

$$v_{\varepsilon_n} := \begin{cases} |u_{\varepsilon_n}|^{p-1}u_{\varepsilon_n} & \text{if } p \in (0,2), \\ u_{\varepsilon_n}^p & \text{if } p \in [2,+\infty), \end{cases} \qquad v := \begin{cases} |u|^{p-1}u & \text{if } p \in (0,2), \\ u^p & \text{if } p \in [2,+\infty). \end{cases}$$

Then it holds that

$$\lim_{\varepsilon_n \to 0_+} \left\| \nabla v_{\varepsilon_n} - \nabla v - \sum_{k=1}^{N} \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v)\mathcal{U}_{\varepsilon_n}(\nabla_y \Phi_k) \right\|_{L^2(\Omega\times I)} = 0,$$

where $\mathcal{U}_{\varepsilon_n}$ *is the averaging operator* (*see Definition* 4.4 *below*).

Remark 2.8. Theorem 2.7 also implies the breaking of strong compactness in $L^2(I;H^1_0(\Omega))$ for the pressure $v_{\varepsilon_n} \in L^2(I;H^1_0(\Omega))$ since the oscillating corrector terms do not vanish as $\varepsilon_n \to 0_+$.

3 Uniform estimates and convergence

In this section, we shall derive uniform estimates for (v_ε) and $(v_\varepsilon^{1/p})$ and discuss their convergence to prove Theorems 2.3 and 2.4.

Lemma 3.1. *Let $0 < p < +\infty$. For each $\varepsilon > 0$ let $u_\varepsilon \in L^2(I; H_0^1(\Omega))$ be the unique weak solution of (P_ε) and set $v_\varepsilon = |u_\varepsilon|^{p-1}u_\varepsilon$. Then the following* (i)–(iv) *hold true:*

- (i) *(v_ε) and $(v_\varepsilon^{1/p})$ are bounded in $L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^{(p+1)/p}(\Omega))$ and $L^\infty(I; L^{p+1}(\Omega))$, respectively.*
- (ii) *$(\partial_t v_\varepsilon^{1/p})$ is bounded in $L^2(I; H^{-1}(\Omega))$.*
- (iii) *$(v_\varepsilon^{1/p})$ is bounded in $L^\infty(I; L^2(\Omega))$.*
- (iv) *$(v_\varepsilon^{1/p})$ is bounded in $L^\infty(I; L^{3-p}(\Omega)) \cap L^2(I; H_0^1(\Omega))$, provided that $p \in (0, 2)$ and $u_0 \in L^{3-p}(\Omega)$.*

Proof. See [2, Lemma 4.1]. □

As for $p \geq 2$, we have the following local uniform estimates:

Lemma 3.2. *Under the same assumptions as in Theorem 2.4, for any $\omega \Subset \Omega$, there exists a constant $C_\omega \geq 0$ such that the following holds true:*

- (i) *In case $2 \leq p < 3$,*
$$\int_0^T \|\nabla v_\varepsilon^{1/p}(t)\|_{L^2(\omega)}^2 \, \mathrm{d}t \leq C_\omega.$$
- (ii) *In case $p = 3$,*
$$\sup_{t \in \overline{I}} \left(\int_{[v_\varepsilon^{1/p}(\cdot,t) \leq 1] \cap \omega} [-\log v_\varepsilon^{1/p}(\cdot, t)] \, \mathrm{d}x \right) + \int_0^T \|\nabla v_\varepsilon^{1/p}(t)\|_{L^2(\omega)}^2 \, \mathrm{d}t \leq C_\omega,$$
provided that $\log u_0 \in L^1_{\mathrm{loc}}(\Omega)$.
- (iii) *In case $p > 3$,*
$$\sup_{t \in \overline{I}} \left(\int_\omega v_\varepsilon^{(3-p)/p}(\cdot, t) \, \mathrm{d}x \right) + \int_0^T \|\nabla v_\varepsilon^{1/p}(t)\|_{L^2(\omega)}^2 \, \mathrm{d}t \leq C_\omega,$$
provided that $u_0^{3-p} \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. See [3, Lemma 3.1]. □

By Lemmas 3.1 and 3.2, we have

Propositon 3.3. *Under the same assumptions as in Theorem* 2.3*, there exist a subsequence* (ε_n) *of* (ε) *and* $v \in L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^{(p+1)/p}(\Omega))$ *such that*

$$\begin{aligned} v_{\varepsilon_n} &\to v && \text{weakly in } L^2(I; H_0^1(\Omega)), \\ v_{\varepsilon_n}^{1/p} &\to v^{1/p} && \text{strongly in } C(\overline{I}; H^{-1}(\Omega)), \\ \partial_t v_{\varepsilon_n}^{1/p} &\to \partial_t v^{1/p} && \text{weakly in } L^2(I; H^{-1}(\Omega)), \\ v_{\varepsilon_n} &\to v && \text{strongly in } L^\rho(I; L^{(p+1)/p}(\Omega)), \\ v_{\varepsilon_n}^{1/p} &\to v^{1/p} && \text{strongly in } L^\rho(I; L^{p+1}(\Omega)), \\ \nabla v_{\varepsilon_n}^{1/p} &\to \nabla v^{1/p} && \text{weakly in } [L^2(I; L^2(\omega))]^N \end{aligned}$$

for any $\rho \in [1, +\infty)$ *and* $\omega \Subset \Omega$.

Proof. See [2, Lemma 4.3]. □

4 Space-time unfolding method

In this section, we briefly review the space-time unfolding method to characterize the limit of $a_{\varepsilon_n} \nabla v_{\varepsilon_n}$ as $\varepsilon_n \to 0_+$. The unfolding method was first introduced in [9] (see [7, 10, 11, 16] for more details), and then its space-time version was developed in [5, 20]. This method is also known as the intermediate notion between weak convergence and strong convergence, and weak and strong convergences for unfolded sequences are equivalent to weak and strong *two-scale convergence* (see, e.g., [4, 14, 17, 19, 23, 26] for more details).

Throughout this section, let $1 < q < +\infty$, when no confusion can arise. Moreover, q' denotes the Hölder conjugate of q, i.e., $1/q + 1/q' = 1$.

Definition 4.1 (cf. [5, Definition 2.1])**.** *For* $\varepsilon > 0$*, define the sets* $\hat{\Omega}_\varepsilon \subset \Omega$ *and* $\hat{I}_\varepsilon \subset I$ *by*

$$\begin{aligned} \hat{\Omega}_\varepsilon &:= \mathrm{interior}\Big(\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{\square}) \Big), \ \Xi_\varepsilon := \{\xi \in \mathbb{Z}^N : \varepsilon(\xi + \square) \subset \Omega\}, \\ \hat{I}_\varepsilon &:= \{t \in I : \varepsilon^2 \left(\lfloor \tfrac{t}{\varepsilon^2} \rfloor + 1 \right) \le T\}, \end{aligned}$$

respectively. Here $\varepsilon(\xi + \overline{\square})$ *denotes the cloesd* ε*-cell* $[0, \varepsilon]^N$ *with the origin at* $\varepsilon\xi \in \varepsilon\mathbb{Z}^N$ *and* $\lfloor \cdot \rfloor$ *denotes the floor function (i.e.,* $\lfloor \cdot \rfloor$ *denotes the integer part of* $\cdot$*). Set* $\Lambda_\varepsilon :=$ $(\Omega \times I) \setminus (\hat{\Omega}_\varepsilon \times \hat{I}_\varepsilon)$*. For* $\varepsilon > 0$*, the* space-time unfolding operator $\mathcal{T}_\varepsilon : \mathcal{M}(\Omega \times I) \to$

$\mathcal{M}(\Omega \times I \times \square \times J)$ is defined by

$$\mathcal{T}_\varepsilon(w)(x,t,y,s) = \begin{cases} w(\varepsilon\lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y, \varepsilon^2 \lfloor \frac{t}{\varepsilon^2} \rfloor + \varepsilon^2 s) & \text{for a.e. } (x,t,y,s) \in \hat{\Omega}_\varepsilon \times \hat{I}_\varepsilon \times \square \times J, \\ 0 & \text{for a.e. } (x,t,y,s) \in \Lambda_\varepsilon \times \square \times J, \end{cases}$$

for $w \in \mathcal{M}(\Omega \times I)$. Moreover, the unfolding operator (still denoted by $\mathcal{T}_\varepsilon$) can be defined analogously for $W \in [\mathcal{M}(\Omega \times I)]^N = \mathcal{M}(\Omega \times I; \mathbb{R}^N)$. Here $\mathcal{M}(A)$ stands for the set of Lebesgue measurable functions on $A \subset \mathbb{R}^{N+1}$.

As for the weak compactness of space-time unfolded sequences, we have

Propositon 4.2. *For any bounded sequence (w_ε) in $L^q(\Omega \times I)$, there exist a sequence $\varepsilon_n \to 0_+$ and a function $w \in L^q(\Omega \times I \times \square \times J)$ such that*

$$\mathcal{T}_{\varepsilon_n}(w_{\varepsilon_n}) \to w \quad \text{weakly in } L^q(\Omega \times I \times \square \times J).$$

In addition, assume that (w_ε) is bounded in $L^q(I; W^{1,q}(\Omega))$ and $w_{\varepsilon_n} \to w$ strongly in $L^q(\Omega \times I)$ for a limit $w \in L^q(I; W^{1,q}(\Omega))$. Then there exist a (not relaveled) subsequence of (ε_n) and a function $w_1 \in L^q(\Omega \times I; L^q(J; W^{1,q}_{\mathrm{per}}(\square)/\mathbb{R}))$ such that

$$\mathcal{T}_{\varepsilon_n}(\nabla w_{\varepsilon_n}) \to \nabla w + \nabla_y w_1 \quad \text{weakly in } [L^q(\Omega \times I \times \square \times J)]^N.$$

Proof. See [20, Proposition 2.9]. □

Remark 4.3. We note that $\nabla_y w_1$ vanishes in the sense of weak convergence due to the periodicity in $\square$ of $w_1 \in L^2(\Omega \times I; L^2(J; H^1_{\mathrm{per}}(\square)/\mathbb{R}))$. Thus the weak convergence of the unfolded sequence for the gradient plays a crucial role in characterizing the limit of the diffusion flux $a_\varepsilon \nabla v_\varepsilon$.

We next introduce the space-time averaging operator.

Definition 4.4. *Under the same assumption as in Definition* 4.1*, the* space-time averaging operator $\mathcal{U}_\varepsilon \colon L^q(\Omega \times I \times \square \times J) \to L^q(\Omega \times I)$ *is defined as follows*:

$$\mathcal{U}_\varepsilon(\Psi)(x,t) = \begin{cases} \displaystyle\int_0^1 \int_\square \Psi(\varepsilon\lfloor \tfrac{x}{\varepsilon} \rfloor + \varepsilon\sigma, \varepsilon^2 \lfloor \tfrac{t}{\varepsilon^2} \rfloor + \varepsilon^2 \rho, \{\tfrac{x}{\varepsilon}\}, \{\tfrac{t}{\varepsilon^2}\}) \, \mathrm{d}\sigma \mathrm{d}\rho & \text{for a.e. } (x,t) \in \hat{\Omega}_\varepsilon \times \hat{I}_\varepsilon, \\ 0 & \text{for a.e. } (x,t) \in \Lambda_\varepsilon, \end{cases}$$

for $\Psi \in L^q(\Omega \times I \times \square \times J)$. Here $\{\cdot\}$ denotes the fraction part of $\cdot$ (i.e., $\{\cdot\} := \cdot - \lfloor \cdot \rfloor$).

As for the strong convergence of unfolded sequences, we have

Propositon 4.5. *Let (w_ε) be bounded in $L^q(\Omega \times I)$ and let $w \in L^q(\Omega \times I \times \square \times J)$. Then the following* (i)–(iii) *are equivalent*:

(i) $\mathcal{T}_\varepsilon(w_\varepsilon) \to w$ *strongly in* $L^q(\Omega \times I \times \square \times J)$ *and* $\lim_{\varepsilon\to 0_+} \iint_{\Lambda_\varepsilon} |w_\varepsilon(x,t)|^q \,\mathrm{d}x\mathrm{d}t = 0$.

(ii) $w_\varepsilon - \mathcal{U}_\varepsilon(w) \to 0$ *strongly in* $L^q(\Omega \times I)$.

(iii) $\lim_{\varepsilon\to 0_+} \|w_\varepsilon\|_{L^q(\Omega\times I)} = \|w\|_{L^q(\Omega\times I\times\square\times J)}$.

Proof. See [20, Proposition 2.13]. □

5 Sketch of proof for Theorem 2.3

By employing Propositions 3.3 and 4.2, one can prove Theorem 2.3; indeed, there exist a subsequence and $z \in L^2(\Omega \times I; L^2(J; H^1_{\mathrm{per}}(\square)/\mathbb{R}))$ such that

$$\mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n}) \to \nabla v + \nabla_y z \quad \text{weakly in } [L^2(\Omega \times I \times \square \times J)]^N.$$

For any $\phi \in C^\infty_{\mathrm{c}}(\Omega)$ and $\psi \in C^\infty_{\mathrm{c}}(I)$, let $\varepsilon > 0$ be small enough such that $\phi\psi = 0$ on Λ_ε. Then we see by [11, Propositions 1.5 and 1.8] that

$$\begin{aligned}
&\int_0^T \left\langle \partial_t v(t)^{1/p}, \phi \right\rangle_{H^1_0(\Omega)} \psi(t)\,\mathrm{d}t \\
&= \lim_{\varepsilon_n\to 0_+} \int_0^T \left\langle \partial_t v_{\varepsilon_n}(t)^{1/p}, \phi \right\rangle_{H^1_0(\Omega)} \psi(t)\,\mathrm{d}t \\
&= -\lim_{\varepsilon_n\to 0_+} \int_0^T \int_\Omega a\left(\tfrac{x}{\varepsilon_n}, \tfrac{t}{\varepsilon_n^2}\right) \nabla v_{\varepsilon_n}(x,t) \cdot \nabla\phi(x)\psi(t)\,\mathrm{d}x\mathrm{d}t \\
&= -\lim_{\varepsilon_n\to 0_+} \int_0^T \int_\Omega \int_0^1 \int_\square a(y,s)\mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n})(x,t,y,s) \cdot \mathcal{T}_{\varepsilon_n}(\nabla\phi\psi)(x,t,y,s)\,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \int_\Omega \Big\langle a(y,s)\big(\nabla v(x,t) + \nabla_y z(x,t,y,s)\big) \Big\rangle_{y,s} \cdot \nabla\phi(x)\psi(t)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

which completes the proof.

6 Sketch of proof for Theorem 2.4

We only consider the case where $0 < p < 1$ for simplicity. We first note that, for any $b \in C^\infty_{\mathrm{per}}(\square)/\mathbb{R}$, there exists a unique solution $w \in C^\infty_{\mathrm{per}}(\square)/\mathbb{R}$ to

$$\Delta_y w(y) = b(y) \quad \text{in } \square. \tag{3}$$

Set $B := \nabla_y w \in [C^\infty_{\mathrm{per}}(\square)/\mathbb{R}]^N$ (i.e., $\mathrm{div}_y B(y) = b(y)$), $\Psi(x,t,y,s) = \phi(x)\psi(t)b(y)c(s)$ and $\Psi_\varepsilon(x,t) = \phi(x)\psi(t)b(\frac{x}{\varepsilon})c(\frac{t}{\varepsilon^2})$ for any $\phi \in C^\infty_{\mathrm{c}}(\Omega)$, $\psi \in C^\infty_{\mathrm{c}}(I)$, $b \in C^\infty_{\mathrm{per}}(\square)/\mathbb{R}$ and $c \in C^\infty_{\mathrm{per}}(J)$. Let $\varepsilon_n > 0$ be small enough such that $\Psi_{\varepsilon_n} = 0$ on Λ_{ε_n}. Then we observe that

$$
\begin{aligned}
&\lim_{\varepsilon_n\to 0_+}\int_0^T\int_\Omega v_{\varepsilon_n}^{1/p}(x,t)\partial_t(\varepsilon_n\Phi_{\varepsilon_n}(x,t))\,\mathrm{d}x\mathrm{d}t\\
&=\lim_{\varepsilon_n\to 0_+}\varepsilon_n\int_0^T\int_\Omega v_{\varepsilon_n}^{1/p}(x,t)\phi(x)\partial_t\psi(t)b(\tfrac{x}{\varepsilon_n})c(\tfrac{t}{\varepsilon_n^2})\,\mathrm{d}x\mathrm{d}t\\
&\quad+\lim_{\varepsilon_n\to 0_+}\int_0^T\int_\Omega v_{\varepsilon_n}^{1/p}(x,t)\phi(x)\psi(t)\nabla\cdot B(\tfrac{x}{\varepsilon_n})\partial_s c(\tfrac{t}{\varepsilon_n^2})\,\mathrm{d}x\mathrm{d}t\\
&=-\lim_{\varepsilon_n\to 0_+}\int_0^T\int_\Omega \nabla v_{\varepsilon_n}^{1/p}(x,t)\phi(x)\psi(t)\cdot B(\tfrac{x}{\varepsilon_n})\partial_s c(\tfrac{t}{\varepsilon_n^2})\,\mathrm{d}x\mathrm{d}t\\
&=-\lim_{\varepsilon_n\to 0_+}\int_0^T\int_\Omega\int_0^1\int_\square \mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n}^{1/p})(x,t,y,s)\mathcal{T}_{\varepsilon_n}(\phi\psi)(x,t,y,s)\cdot B(y)\partial_s c(s)\,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Combining Lemmas 3.1 and 3.2 with Proposition 4.2, we see that there exists $\tilde{z}\in L^2_{\mathrm{loc}}(\Omega\times I;L^2(J;H^1_{\mathrm{per}}(\square)/\mathbb{R}))$ such that

$$
\mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n}^{1/p})\to\nabla v+\nabla_y\tilde{z}\quad\text{weakly in }\ [L^2(\omega\times I\times\square\times J)]^N
$$

for any $\omega\Subset\Omega$, and hence,

$$
\begin{aligned}
&\int_0^T\int_\Omega\int_0^1\int_\square \tilde{z}(x,t,y,s)\partial_s\Psi(x,t,y,s)\,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t\\
&=\lim_{\varepsilon_n\to 0_+}\int_0^T\int_\Omega a_{\varepsilon_n}\nabla v_{\varepsilon_n}(x,t)\cdot\nabla(\varepsilon_n\Psi_{\varepsilon_n}(x,t))\,\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega\int_0^1\int_\square a(y,s)\big(\nabla v(x,t)+\nabla_y z(x,t,y,s)\big)\cdot\nabla_y\Psi(x,t,y,s)\,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Since one can prove $\tilde{z}=\frac{1}{p}|v|^{(1-p)/p}z$ as in [2, Lemma 5.4], by setting $z=\Sigma_{k=1}^N(\partial_{x_k}v)\Phi_k$, we obtain the cell problem and (1) (see [2] for details).

7 Sketch of proof for Theorem 2.7

We first observe that

$$
\begin{aligned}
&\Big\|\nabla v_{\varepsilon_n}-\nabla v-\sum_{k=1}^N\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)\mathcal{U}_{\varepsilon_n}(\nabla_y\Phi_k)\Big\|_{L^2(\Omega\times I)}\\
&\le\big\|\nabla v_{\varepsilon_n}-\mathcal{U}_{\varepsilon_n}(\nabla v)-\mathcal{U}_{\varepsilon_n}(\nabla_y z)\big\|_{L^2(\Omega\times I)}\\
&\quad+\big\|\mathcal{U}_{\varepsilon_n}(\nabla v)-\nabla v\big\|_{L^2(\Omega\times I)}+\Big\|\mathcal{U}_{\varepsilon_n}(\nabla_y z)-\sum_{k=1}^N\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)\mathcal{U}_{\varepsilon_n}(\nabla_y\Phi_k)\Big\|_{L^2(\Omega\times I)}\\
&=:I_1^{\varepsilon_n}+I_2^{\varepsilon_n}+I_3^{\varepsilon_n}.
\end{aligned}
$$

Then we shall estimate the terms $I_1^{\varepsilon_n}$, $I_2^{\varepsilon_n}$ and $I_3^{\varepsilon_n}$ below.

To prove

$$I_1^{\varepsilon_n} \to 0 \quad \text{as} \ \ \varepsilon_n \to 0_+, \tag{4}$$

we claim that

$$\lim_{\varepsilon_n \to 0_+} \|\nabla v_{\varepsilon_n}\|^2_{L^2(\Omega\times I)} = \|\nabla v_0 + \nabla_y z\|^2_{L^2(\Omega\times I\times\square\times J)}. \tag{5}$$

Indeed, noting by [2, Lemma 6.1] that

$$v_{\varepsilon_n}(t)^{1/p} \to v_0(t)^{1/p} \quad \text{weakly in } L^{p+1}(\Omega) \quad \text{for all} \ \ t \in \overline{I},$$

we see by the weak form that

$$\limsup_{\varepsilon_n \to 0_+} \int_0^T \int_\Omega a_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \,\mathrm{d}x\mathrm{d}t \le \int_0^T \int_\Omega a_{\mathrm{hom}}(x,t) \nabla v \cdot \nabla v \,\mathrm{d}x\mathrm{d}t.$$

On the other hand, by the J-periodicity of Φ_k, we have

$$\begin{aligned}
&\liminf_{\varepsilon_n \to 0_+} \int_0^T \int_\Omega a_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \,\mathrm{d}x\mathrm{d}t \\
&\ge \int_0^T \int_\Omega \int_0^1 \int_\square a(y,s)(\nabla v + \nabla_y z) \cdot (\nabla v + \nabla_y z) \,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega a_{\mathrm{hom}}(x,t) \nabla v \cdot \nabla v \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thus it follows that

$$\lim_{\varepsilon_n \to 0_+} \int_0^T \int_\Omega a_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega a_{\mathrm{hom}}(x,t) \nabla v \cdot \nabla v \,\mathrm{d}x\mathrm{d}t. \tag{6}$$

Now, let $\mathbb{I} \in \mathbb{R}^{N\times N}$ be a unit matrix and let $\gamma > 0$ be such that $(a(y,s) - \gamma\mathbb{I})\xi\cdot\xi \ge \tilde{\lambda}|\xi|^2$ for all $\xi \in \mathbb{R}^N$ and some $\tilde{\lambda} > 0$. Then we infer by the J-periodicity of Φ_k that

$$\begin{aligned}
&\liminf_{\varepsilon_n \to 0_+} \int_0^T \int_\Omega \left(a_{\varepsilon_n} - \gamma\mathbb{I}\right) \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \,\mathrm{d}x\mathrm{d}t \\
&\ge \int_0^T \int_\Omega \int_0^1 \int_\square \left(a(y,s) - \gamma\mathbb{I}\right) \left(\nabla v + \nabla_y z\right) \cdot \left(\nabla v + \nabla_y z\right) \,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega a_{\mathrm{hom}}(x,t) \nabla v \cdot \nabla v \,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \int_0^1 \int_\square \gamma\mathbb{I} \left(\nabla v + \nabla_y z\right) \cdot \left(\nabla v + \nabla_y z\right) \,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and hence, (6) ensures that

$$\limsup_{\varepsilon_n\to 0_+}\int_0^T\int_\Omega \gamma\mathbb{I}\nabla v_{\varepsilon_n}\cdot\nabla v_{\varepsilon_n}\,\mathrm{d}x\mathrm{d}t \le \int_0^T\int_\Omega\int_0^1\int_\square \gamma\mathbb{I}\left(\nabla v+\nabla_y z\right)\cdot\left(\nabla v+\nabla_y z\right)\,\mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t,$$

which together with the lower semi continuity yields (5). Thus (4) follows from the implication (iii) $\Rightarrow$ (ii) of Proposition 4.5.

We next claim that

$$I_2^{\varepsilon_n}\to 0 \quad \text{as} \ \ \varepsilon_n\to 0_+. \tag{7}$$

This also follows from the implication (iii) $\Rightarrow$ (ii) of Proposition 4.5.

We finally show that

$$I_3^{\varepsilon_n}\to 0 \quad \text{as} \ \ \varepsilon_n\to 0_+. \tag{8}$$

It suffices to prove that

$$\mathcal{U}_{\varepsilon_n}((\partial_{x_k}v)\nabla_y\Phi_k)-\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)\mathcal{U}_{\varepsilon_n}(\nabla_y\Phi_k)\to 0 \quad \text{strongly in} \ \ [L^2(\Omega\times I)]^N. \tag{9}$$

To this end, we shall use the following fact:

$$\nabla_y\Phi_k\in[L^\infty(\Omega\times I;L^2(\square\times J))]^N \tag{10}$$

(see [2, Appendix] for the proof). Since $\partial_{x_k}v$ is independent of $(y,s)\in\square\times J$, noting that, for any $(\xi,\zeta)\in\Xi_{\varepsilon_n}\times\Theta_{\varepsilon_n}:=\{\zeta\in\mathbb{N}\cup\{0\}\colon \varepsilon^2(\zeta+J)\subset I\}$, $\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)$ can be regarded as a constant in $\varepsilon_n(\xi+\square)\times\varepsilon_n^2(\zeta+J)$, we derive that

$$\begin{aligned}
&\|\mathcal{U}_{\varepsilon_n}((\partial_{x_k}v)\nabla_y\Phi_k)-\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)\mathcal{U}_{\varepsilon_n}(\nabla_y\Phi_k)\|^2_{L^2(\Omega\times I)}\\
&\quad=\sum_{\zeta\in\Theta_\varepsilon}\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon^2(\zeta+J)}\int_{\varepsilon(\xi+\square)}\left|\mathcal{U}_{\varepsilon_n}\left(\left(\partial_{x_k}v-\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)\right)\nabla_y\Phi_k\right)\right|^2\mathrm{d}x\mathrm{d}t\\
&\quad\le\|\nabla_y\Phi_k\|^2_{L^\infty(\Omega\times I;L^2(\square\times J))}\left\|\partial_{x_k}v-\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)\right\|^2_{L^2(\Omega\times I)}\to 0 \quad \text{as} \ \ \varepsilon_n\to 0_+.
\end{aligned}$$

Here we used the facts (7) and (10) in the last line (see [20] for details). Thus we have (9). Combining (4), (7) and (8), we obtain

$$\left\|\nabla v_{\varepsilon_n}-\nabla v-\sum_{k=1}^N\mathcal{U}_{\varepsilon_n}(\partial_{x_k}v)\mathcal{U}_{\varepsilon_n}(\nabla_y\Phi_k)\right\|_{L^2(\Omega\times I)}\le I_1^{\varepsilon_n}+I_2^{\varepsilon_n}+I_3^{\varepsilon_n}\to 0 \ \text{ as } \varepsilon_n\to 0_+,$$

which completes the proof.

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