Large time behavior of Navier-Stokes flows perturbed from Riemann data in 1D

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1 Introduction

This survey paper presents recent results on the large-time behavior of the one-dimensional barotropic compressible Navier-Stokes equations. In particular, we summarize the two results from [11] and [4], establishing the time asymptotic behaviors of the Navier-Stokes equations, whose initial data is perturbed from a Riemann data. Precisely, in [11], the large-time behavior of the composite waves of shock and rarefaction is investigated, while the case of composite waves of two shocks is studied in [4]. The main tool used in these literatures is the so-called "method of a-contraction with shift", which is recently developed by the second author and Vasseur [7, 24]. The purpose of the paper is to introduce state-of-the-art results on these problems and to explain the key idea of the method of a-contraction with shift.

We consider the following one-dimensional compressible barotropic Navier-Stokes system, which is described by the Lagrangian mass coordinates:

$$\begin{aligned} &v_t - u_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \\ &u_t + p(v)_x = \left(\mu \frac{u_x}{v}\right)_x, \end{aligned} \tag{1.1}$$

subject to the initial data

$$(v(t,x), u(t,x))|_{t=0} = (v_0(x), u_0(x)), \quad x \in \mathbb{R}.$$

Here, the unknown functions $v = v(t,x) > 0$ and $u = u(t,x)$ represent the specific volume and the velocity of the fluid, respectively. The pressure function $p = p(v)$ is given by the γ-law as $p(v) = bv^{-\gamma}$, with $b > 0$ and $\gamma > 1$. Finally, a positive constant $\mu > 0$ denotes the viscosity coefficient of the fluid. For convenience, we normalize the coefficients as $b = 1$ and $\mu = 1$. We consider the initial data (v_0, u_0) of the system (1.1) which connects the prescribed far-field constant states:

$$\lim_{x\to\pm\infty} (v_0(x), u_0(x)) = (v_\pm, u_\pm). \tag{1.2}$$

A heuristic argument (see e.g. [15]) deduces that the large-time behavior of solutions (v, u) to the Navier-Stokes equations (1.1) has a close relationship with the Riemann problem of the associated Euler equations:

$$\begin{aligned} &v_t - u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ &u_t + p(v)_x = 0, \end{aligned} \tag{1.3}$$

subject to the Riemann initial data

$$(v(t,x), u(t,x))|_{t=0} = \begin{cases} (v_-, u_-), & x < 0, \\ (v_+, u_+), & x > 0. \end{cases} \tag{1.4}$$

Riemann Problem for the Euler equations

We first discuss the Riemann problem for the inviscid model (1.3)–(1.4). The Euler equations (1.3) can be rewritten in the form of hyperbolic system of conservation laws:

$$U_t + f(U)_x = 0,$$

where $f(v, u) = (-u, p(v))$ is the flux function in conserved variables $U = (v, u)$. This system is strictly hyperbolic, since the derivative of flux function f,

$$Df = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

is diagonalizable and has two distinct real eigenvalues, $\lambda_1(v) = -\sqrt{-p'(v)} < 0$ and $\lambda_2(v) = \sqrt{-p'(v)} > 0$, and corresponding right eigenvectors, $r_1(v) = (1, -\lambda_1(v))$ and $r_2(v) = (1, -\lambda_2(v))$. Since $p'(v) < 0$ and $p''(v) > 0$, both characteristic fields are genuinely nonlinear :

$$D\lambda_1 \cdot r_1 = \frac{1}{2}(-p'(v))^{-1/2} p''(v) > 0, \quad D\lambda_2 \cdot r_2 = -\frac{1}{2}(-p'(v))^{-1/2} p''(v) < 0.$$

Thus, the Riemann problem (1.3)–(1.4) has four families of simple wave solutions: 1-rarefaction wave, 2-rarefaction wave, 1-shock wave, and 2-shock wave. Each simple wave is defined by its associated curve, and the Riemann solution is determined by a combination of at most two simple waves. In addition, each curve is defined in the appropriate neighborhood of a given constant left state (v_L, u_L) or right state (v_R, u_R). Given the right state $(v_R, u_R) \in \mathbb{R}_+ \times \mathbb{R}$, we present the construction of each simple wave as follows.

We first review the rarefaction waves. The 1-rarefaction curve $R_1(v_R, u_R)$ is defined by the integral curve of the right eigenvector r_1 which passes through (v_R, u_R) and satisfies $\lambda_1(v) < \lambda_1(v_R)$, and the 2-rarefaction curve $R_2(v_R, u_R)$ is defined in the same way by using λ_2 :

$$R_i(v_R, u_R) := \left\{ (v, u) \mid u = u_R - \int_{v_R}^{v} \lambda_i(s)\, ds, \quad \lambda_i(v) < \lambda_i(v_R) \right\} \quad (i = 1, 2). \tag{1.5}$$

When the initial data (1.4) of the Riemann problem is given by $(v_-, u_-) = (v_L, u_L), (v_+, u_+) = (v_R, u_R)$, such that $(v_-, u_-) \in R_1(v_+, u_+)$, a solution to (1.3)–(1.4) is given by the 1-rarefaction wave (v_1^r, u_1^r) defined as

$$(v_1^r, u_1^r)(x/t) = \begin{cases} (v_-, u_-), & x < \lambda_1(v_-)t, \\ \begin{cases} v_1^r = \lambda_1^{-1}(x/t), \\ u_1^r = u_+ - \int_{v_+}^{v_1^r} \lambda_1(s)\, ds \end{cases} & \lambda_1(v_-)t \le x \le \lambda_1(v_+)t, \\ (v_+, u_+), & x > \lambda_1(v_+)t. \end{cases} \tag{1.6}$$

Moreover, the 1-Riemann invariant z_1 is given by $z_1(v, u) = u - \int^v \lambda_1(s)\, ds$ and it satisfies $z_1(v_1^r(x/t), u_1^r(x/t)) = z_1(v_\pm, u_\pm)$. This implies that the Riemann invariant is constant along the rarefaction curve R_1. The case of 2-rarefaction wave (v_2^r, u_2^r) is defined in the same way by using λ_2 in (1.6).

On the other hand, the shock curve $S(v_R, u_R)$ passing through (v_R, u_R) is defined by using the Rankine-Hugoniot condition and the Lax-entropy condition. In fact, the shock curve $S(v_R, u_R)$ is an one-parameter family of solution (v, u) with the shock speed σ to the Rankine-Hugoniot condition: there exists σ such that

$$\begin{cases} -\sigma(v_R - v) - (u_R - u) = 0, \\ -\sigma(u_R - u) + (p(v_R) - p(v)) = 0. \end{cases} \tag{1.7}$$

This condition defines two shock speeds $\sigma = \sigma_i$ $(i = 1, 2)$, which are given as

$$\sigma_1 = -\sqrt{-\frac{p(v_R) - p(v)}{v_R - v}}, \quad \sigma_2 = \sqrt{-\frac{p(v_R) - p(v)}{v_R - v}}.$$

Furthermore, under the Rankine-Hugoniot condition, Lax's entropy condition is assumed to be mathematically and physically reasonable:

$$v > v_R,\ u > u_R,\ (i = 1); \qquad v < v_R,\ u > u_R,\ (i = 2). \tag{1.8}$$

Therefore, for each $i = 1, 2$, the i-shock curve $S_i(v_L, u_L)$ is defined by

$$S_i(v_R, u_R) := \{ (v, u) \mid u = u_R + \sigma_i(v_R - v),\ \lambda_i(v) > \lambda_i(v_R) \}, \quad (i = 1, 2). \tag{1.9}$$

Then, for $(v_-, u_-) \in S_i(v_+, u_+)$, where the initial data (1.4) is prescribed with $(v_-, u_-) = (v_L, u_L), (v_+, u_+) = (v_R, u_R)$, the traveling wave solution (v_i^s, u_i^s) to (1.3), called the i-shock wave, is defined as

$$(v_i^s, u_i^s)(x, t) = \begin{cases} (v_-, u_-), & x < \sigma_i t, \\ (v_+, u_+), & x > \sigma_i t, \end{cases} \quad (i = 1, 2). \tag{1.10}$$

Generally, for any given end states $(v_-, u_-), (v_+, u_+) \in \mathbb{R}_+ \times \mathbb{R}$, there exists a unique intermediate state $(v_m, u_m) \in \mathbb{R}_+ \times \mathbb{R}$ such that (v_m, u_m) is on the curve $R_2(v_+, u_+)$ or $S_2(v_+, u_+)$ from (v_+, u_+), and (v_-, u_-) is on the curve $R_1(v_m, u_m)$ or $S_1(v_m, u_m)$ from

(v_m, u_m). In this case, the Riemann solution $(\overline{v}, \overline{u})$ to (1.3)–(1.4) is given by the composition of the two associated waves

$$(\overline{v}, \overline{u})(t, x) = (\overline{v}_1, \overline{u}_1)(t, x) + (\overline{v}_2, \overline{u}_2)(t, x) - (v_m, u_m),$$

where $(\overline{v}_i, \overline{u}_i)$ are either i-rarefaction or i-shock, depending on the far-field values $(v_\pm, u_\pm)$. Of course, if the two end states are connected by a single rarefaction or shock curve, the Riemann solution degenerates to a single self-similar solution. For example it is the 1-rarefaction or 1-shock if $(v_-, u_-) = (v_m, u_m)$ and it is the 2-rarefaction or 2-shock if $(v_m, u_m) = (v_+, u_+)$.

Time-asymptotic behavior of the Navier-Stokes system

We now briefly discuss the previous results on the time-asymptotic behavior of the viscous system (1.1) with (1.2). The time-asymptotic behavior depends on the associated Riemann solution to the inviscid model (1.3)–(1.4). When the Riemann data (1.4) generates a i-shock, then solutions of (1.1)-(1.2), as time goes to infinity, would tend to the viscous counterpart for (1.1), called viscous shock, as the traveling wave solution $(\widetilde{v}_i, \widetilde{u}_i)$ satisfying the following ODE:

$$\begin{cases} -\sigma_i(\widetilde{v}_i)' - (\widetilde{u}_i)' = 0, \\ -\sigma_i(\widetilde{u}_i)' + p(\widetilde{v}_i)' = \left(\frac{(\widetilde{u}_i)'}{\widetilde{v}_i}\right)', \\ (\widetilde{v}_1, \widetilde{u}_1)(-\infty) = (v_-, u_-), \quad (\widetilde{v}_1, \widetilde{u}_1)(+\infty) = (v_m, u_m), \\ (\widetilde{v}_2, \widetilde{u}_2)(-\infty) = (v_m, u_m), \quad (\widetilde{v}_2, \widetilde{u}_2)(+\infty) = (v_+, u_+). \end{cases} \tag{1.11}$$

The time-asymptotic behavior towards the viscous shock wave is first studied by Matsumura and Nishihara [16] and the same result is shown by Goodman [3], where the general system with artificial diffusion. These results are based on the celebrated anti-derivative method, together with the zero mass condition, but later on, this zero mass condition is removed by introducing the constant shift [12, 14, 22]. Masica and Zumbrum introduced the spectral stability of viscous shock under the spectral condition, which is a slightly weaker condition that the zero mass condition. Finally, the case of degenerate viscosity [19] and the general class of viscosity [1] are also established.

On the other hand, the time-asymptotic stability of rarefaction wave is developed by using a completely different approach, based on the energy method. The stability of rarefaction wave for the Navier-Stokes equations was first developed by Matsumura and Nishihara [17, 18] and the result is generalized to the Navier-Stokes-Fourier system [13, 20].

All of the mentioned literature above do not consider the generic composition waves. Indeed, the stability analysis of the composite wave has been barely studied. In [5], the stability of the composite wave of two viscous shocks for the Navier-Stokes-Fourier system is considered, and it is mentioned in [15] that the same result can be obtained for the case of Navier-Stokes system (1.1). However, in this literature, the strength of two viscous shocks should have the same order of smallness. On top of that, the time-asymptotic stability of the composite of shock and rarefaction waves is a challenging problem [16, 17], since the two traditional approaches to obtaining the stability of shock and rarefaction waves are incompatible with each other.

Main results and outline of the paper

The main goal of the paper is to present the results for time-asymptotic behavior of the composition of shock and rarefaction waves, and the composition of two shocks, which are the main results of the author's recent papers [11] and [4] respectively. The precise statements of the stability results read as follows.

Theorem 1.1 (Stability for composition of shock and rarefaction)**.** *For a given constant state $(v_+, u_+) \in \mathbb{R}_+ \times \mathbb{R}$, there exist positive constants δ_0, ε_0 such that the following holds true.*
For any $(v_m, u_m) \in S_2(v_+, u_+)$ and $(v_-, u_-) \in R_1(v_m, u_m)$ such that

$$|v_+ - v_m| + |v_m - v_-| < \delta_0, \tag{1.12}$$

let $(v^r, u^r)(\frac{x}{t})$ be the 1-rarefaction solution to (1.3) with end states (v_-, u_-) and (v_m, u_m), and $(\widetilde{v}^S, \widetilde{u}^S)(x - \sigma_2 t)$ the 2-viscous shock solution of (1.11) with end states (v_m, u_m) and (v_+, u_+). Let (v_0, u_0) be any initial data such that

$$\sum_{\pm} \left(\|v_0 - v_\pm\|_{L^2(\mathbb{R}_\pm)} + \|u_0 - u_\pm\|_{L^2(\mathbb{R}_\pm)} \right) + \|\partial_x v_0\|_{L^2(\mathbb{R})} + \|\partial_x u_0\|_{L^2(\mathbb{R})} < \varepsilon_0,$$

where $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$. Then, the compressible Navier-Stokes system (1.1)–(1.2) admits a unique global-in-time solution (v, u) in the following sense: there exists an absolutely continuous shift function $X(t)$ such that

$$\begin{aligned}
&v(t,x) - \left(v^r\left(\frac{x}{t}\right) + \widetilde{v}^S\left(x - \sigma_2 t - X(t)\right) - v_m \right) \in C(0, +\infty; H^1(\mathbb{R})),\\
&u(t,x) - \left(u^r\left(\frac{x}{t}\right) + \widetilde{u}^S(x - \sigma_2 t - X(t)) - u_m \right) \in C(0, +\infty; H^1(\mathbb{R})),\\
&u_{xx}(t,x) - \widetilde{u}^S_{xx}(x - \sigma_2 t - X(t)) \in L^2(0, +\infty; L^2(\mathbb{R})).
\end{aligned}$$

Moreover, we have the large-time behavior:

$$\begin{aligned}
&\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| v(t,x) - \left(v^r\left(\frac{x}{t}\right) + \widetilde{v}^S(x - \sigma_2 t - X(t)) - v_m \right) \right| = 0,\\
&\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| u(t,x) - \left(u^r\left(\frac{x}{t}\right) + \widetilde{u}^S(x - \sigma_2 t - X(t)) - u_m \right) \right| = 0,
\end{aligned}$$

where

$$\lim_{t\to+\infty} |\dot{X}(t)| = 0. \tag{1.13}$$

Theorem 1.2 (Stability for composition of two shocks)**.** *For a given constant state $(v_+, u_+) \in \mathbb{R}_+ \times \mathbb{R}$, there exist positive constants δ_0, ε_0 such that the following holds.*
For any constant states (v_m, u_m) and (v_-, u_-) satisfying (4.1) with

$$|v_+ - v_m| + |v_m - v_-| < \delta_0, \tag{1.14}$$

let $(\widetilde{v}_i, \widetilde{u}_i)(x - \sigma_i t)$ *be the i-viscous shock wave satisfying* (1.11)*. In addition, let* (v_0, u_0) *be any initial data satisfying*

$$\sum_{\pm} \big(\|v_0 - v_\pm\|_{L^2(\mathbb{R}_\pm)} + \|u_0 - u_\pm\|_{L^2(\mathbb{R}_\pm)} \big) + \|\partial_x v_0\|_{L^2(\mathbb{R})} + \|\partial_x u_0\|_{L^2(\mathbb{R})} < \varepsilon_0.$$

Then, the compressible Navier-Stokes system (1.1)*–*(1.2) *admits a unique global-in-time solution* (v, u) *in the following sense: there exist absolutely continuous shift functions* $X_1(t), X_2(t)$ *such that*

$$v(t,x) - (\widetilde{v}_1(x - \sigma_1 t - X_1(t)) + \widetilde{v}_2(x - \sigma_2 t - X_2(t)) - v_m) \in C(0, +\infty; H^1(\mathbb{R})),$$
$$u(t,x) - (\widetilde{u}_1(x - \sigma_1 t - X_2(t)) + \widetilde{u}_2(x - \sigma_2 t - X_2(t)) - u_m) \in C(0, +\infty; H^1(\mathbb{R})).$$

Moreover, we have the large-time behavior:

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \Big| v(t,x) - \big(\widetilde{v}_1(x - \sigma_1 t - X_1(t)) + \widetilde{v}_2(x - \sigma_2 t - X_2(t)) - v_m\big) \Big| = 0,$$
$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \Big| u(t,x) - \big(\widetilde{u}_1(x - \sigma_1 t - X_1(t)) + \widetilde{u}_2(x - \sigma_2 t - X_2(t)) - u_m\big) \Big| = 0,$$

where

$$\lim_{t\to+\infty} |\dot{X}_i(t)| = 0, \quad \textit{for} \quad i = 1, 2. \tag{1.15}$$

Especially, the shifts are well-separated in the following sense:

$$X_1(t) + \sigma_1 t \le \frac{\sigma_1}{2} t < 0 < \frac{\sigma_2}{2} t \le X_2(t) + \sigma_2 t, \quad t > 0. \tag{1.16}$$

The remaining part of the paper is organized as follows. In Section 2, we discuss the key ideas of the proof of the main theorem, in particular, introduce the method of a-contraction with shift and explain the background behind it. Then, we present the outline of the proof of Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4. Finally, Section 5 is devoted to short concluding remarks.

2 Main ideas and the method of a-contraction with shifts

As we mentioned in the introduction, the main tool to obtain the desired nonlinear stability is the method of a-contraction with shifts, which was introduced in [7, 24] to study the stability of extremal shocks for the hyperbolic system of conservation laws such as the Euler system (1.3). We also refer to [2, 6, 8, 9, 10, 25] for the literature in which the method of a-contraction with shift has been used to diverse models.

To illustrate a key idea of the method for the viscous system (1.1), consider the entropy η of the Euler system (1.3) defined as

$$\eta(U) := \frac{u^2}{2} + Q(v), \quad Q(v) := \frac{1}{(\gamma - 1)v^{\gamma-1}},$$

where $U = (v, u)$. Then, the relative entropy $\eta(U|\overline{U})$ between two states U and $\overline{U}$ is defined as

$$\eta(U|\overline{U}) := \eta(U) - \eta(\overline{U}) - D\eta(\overline{U})(U - \overline{U}) = \frac{|u - \overline{u}|^2}{2} + Q(v|\overline{v}),$$

where $Q(v|\overline{v}) := Q(v) - Q(\overline{v}) - Q'(\overline{v})(v - \overline{v})$. Since $Q(v)$ is a strictly convex function in v, $Q(\cdot|\cdot)$ is locally quadratic in the sense that for any $0 < a < b$, there exists $C > 0$ such that

$$C^{-1}|v_1 - v_2|^2 \leq Q(v_1|v_2) \leq C|v_1 - v_2|^2, \qquad \forall v_1, v_2 \in [a, b].$$

From the definition the relative entropy $\eta(U|\overline{U})$ is nonnegative and it vanishes if and only if $U = \overline{U}$. Therefore, the relative entropy can be understood as a pseudo-metric between the conserved quantities U and $\overline{U}$. The relative entropy was used to derive the contraction property of the rarefaction wave. Indeed, if $\overline{U}$ is a rarefaction wave, and U is any weak entropic solution to (1.3), then it can be shown in [23] that

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(U|\overline{U})\, dx \leq 0.$$

However, this nice contraction property does not hold when $\overline{U}$ is a shock. A similar type of contraction property for the shock can be recovered only after adding weight and shift to the relative entropy. When $\overline{U}(x - \sigma t)$ is a single viscous shock, it was proved in [10] that there exist a monotonic function $a = a(x)$ and a shift $X(t)$ such that the weighted relative entropy with the shift is not increasing in time:

$$\frac{d}{dt}\int_{\mathbb{R}} a(x - \sigma t - X(t))\eta(U(t, x)|\overline{U}(x - \sigma t - X(t)))\, dx \leq 0. \tag{2.1}$$

The method of a-contraction with shift is to derive a similar contraction property to the given system, by defining appropriate weight function a and the shift X. In the present paper, we will explain how the method of a-contraction with shift can be applied to the system of Navier-Stokes equations, for the case of composite waves of shock-rarefaction and shock-shock. In what follows, we provide a sketch of the process to obtain the a-contraction with shift. For a precise design of the weight function and shift for each case we consider, we refer to Section 3 and Section 4 respectively.

First of all, by a standard computation based on the relative entropy method, the left-hand side of (2.1) can be decomposed into three parts (as in Lemma 3.1 for example):

$$\text{LHS} = X'(t)Y(U) + \mathcal{J}^{\text{bad}}(U) - \mathcal{J}^{\text{good}}(U),$$

where $\mathcal{J}^{\text{bad}}(U)$ and $\mathcal{J}^{\text{good}}(U)$ consist of all bad terms and all good terms respectively, where the "good term" means that it has a definite positive sign. To make the right-hand side non-positive, we might use the typical energy method for parabolic equations. However, since the barotropic Naiver-Stokes system has the diffusion in one variable only (more precisely, in the u variable for (1.1)), the weight function a would be found to provide an additional good term in terms of the v variable, by which the bad terms could be represented only by the u variables. Indeed, since σ is a non-zero constant, constructing a monotone function a satisfying $\sigma a' > 0$, we have a good term

$$-\sigma \int_{\mathbb{R}} a'(x - \sigma t - X(t))\eta(U(t,x)|\overline{U}(x - \sigma t - X(t)))\, dx.$$

In fact, the weight function a will be defined by the first component $\overline{v}$ of the viscous shock such that a' localizes the perturbation in space as done by $\overline{v}'$, and the image of a is a bounded open interval. Using the above term, we maximize, over all $v > 0$, the worst hyperbolic terms related to the v variable, from which the remaining bad terms are related to the u variable only, and localized by a' or $\overline{v}'$. To absorb the remaining bad terms by the diffusion term, we may use the following Poincaré-type inequality.

Lemma 2.1. *[10, Lemma 2.9] For any $f : [0,1] \to \mathbb{R}$ satisfying $\int_0^1 y(1-y)|f'|^2 dy < \infty$,*

$$\int_0^1 \left| f - \int_0^1 f dy \right|^2 dy \le \frac{1}{2} \int_0^1 y(1-y)|f'|^2 dy. \tag{2.2}$$

However, to apply Lemma 2.1, we may need the other good term as an average of linear perturbation on u-variable, which together with the bad terms would give a variance of the perturbation that could be absorbed by the diffusion as in the Poincaré-type inequality. Here, the desired good term would be extracted from the shift part $X'(t)Y(U)$ by a sophisticated construction of the shift $X(t)$. This gives the desired contraction estimate.

We now provide the detailed construction of the weight function a and the shift X to obtain the desired estimates for each case of shock-rarefaction and shock-shock composition. We also briefly explain how the desired estimates yield the time-asymptotic behavior of the Navier-Stokes system, in the main theorems.

Notation. For a function $x \mapsto F(x)$, we use the notation $F^X(x) := F(x - X(t))$.

3 Proof of Theorem 1.1

In this section, we provide the a priori estimates for the composition waves of shock and rarefaction, and the sketch of proof for the time asymptotic stability of it. Without loss of generality, we consider the end states $(v_\pm, u_\pm)$ such that there exists a unique state (v_m, u_m) which is connected with (v_-, u_-) by 1-rarefaction wave and with (v_+, u_+) by 2-shock curve. That is, there exists a unique (v_m, u_m) such that

$$\begin{aligned}
&u_- = u_m - \int_{v_m}^{v_-} \lambda_1(s)\, ds, \quad \lambda_1(v_-) < \lambda_1(v_m), \quad v_- < v_m, \quad u_- < u_m;\\
&\begin{cases} -\sigma_2(v_+ - v_m) - (u_+ - u_m) = 0, \\ -\sigma_2(u_+ - u_m) + (p(v_+) - p(v_m)) = 0, \end{cases} \qquad \sigma_2 := \sqrt{-\frac{p(v_+) - p(v_m)}{v_+ - v_m}},\\
&v_m < v_+, \quad u_m > u_+.
\end{aligned} \tag{3.1}$$

Then, the Euler equations (1.3) with (1.4)–(3.1) admit a unique self-similar solution, the so-called Riemann solution $(\bar{v}, \bar{u})$, represented by the composition $(\bar{v}, \bar{u}) = (v_1^r, u_1^r) + (v_2^s, u_2^s) - (v_m, u_m)$ of 1-rarefaction wave (v_1^r, u_1^r) and 2-shock wave (v_2^s, u_2^s) defined as (see e.g. [21])

$$
(v_1^r, u_1^r)(x/t) = \begin{cases} (v_-, u_-), & x < \lambda_1(v_-)t, \\ \begin{cases} v_1^r = \lambda_1^{-1}(x/t), \\ u_1^r = u_m - \int_{v_m}^{v_1^r} \lambda_1(s)\, ds \end{cases} & \lambda_1(v_-)t \le x \le \lambda_1(v_m)t, \\ (v_m, u_m), & x > \lambda_1(v_m)t. \end{cases}
$$

$$
(v_2^s, u_2^s)(t,x) = \begin{cases} (v_m, u_m), & x < \sigma_2 t, \\ (v_+, u_+), & x > \sigma_2 t. \end{cases}
$$

For convenience, we use the notation $\sigma := \sigma_2$. We rewrite the Navier-Stokes system (1.1) using the variable associated to the speed of shock $\xi = x - \sigma t$:

$$
\begin{aligned}
&v_t - \sigma v_\xi - u_\xi = 0, \\
&u_t - \sigma u_\xi + p(v)_\xi = \left(\frac{u_\xi}{v}\right)_\xi .
\end{aligned} \tag{3.2}
$$

In order to compare the solution to (3.2) with the composition of the rarefaction and shock, we consider the smooth approximate rarefaction wave $(\widetilde{v}^R, \widetilde{u}^R)$ defined as

$$
(\widetilde{v}^R, \widetilde{u}^R)(t,x) := \left(\lambda_1^{-1}(w(1+t,x)), u_m - \int_{v_m}^{\widetilde{v}^R(t,x)} \lambda_1(s)\, ds \right),
$$

where $w(t,x)$ is the smooth solution to the Burgers equation $w_t + w w_x = 0$ subject to the initial data

$$
w(0,x) := \frac{w_m + w_-}{2} + \frac{w_m - w_-}{2} \tanh x,
$$

and $(v_-, v_m) = (\lambda_1^{-1}(w_-), \lambda_1^{-1}(w_m))$. Then, it is well-known in [17] that the approximated rarefaction wave is asymptotically the same as the original rarefaction wave:

$$
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |(\widetilde{v}^R, \widetilde{u}^R)(t,x) - (v^r, u^r)(x/t)| = 0. \tag{3.3}
$$

Thanks to the estimate (3.3), it suffices to derive the stability estimate for the approximated rarefaction $(\widetilde{v}^R, \widetilde{u}^R)$, instead of the exact rarefaction (v^r, u^r). To this end, we define the superposition wave of the approximated rarefaction wave and the viscous shock wave shifted by $X(t)$ as

$$
(\widetilde{v}, \widetilde{u})(t,\xi) := (\widetilde{v}^R(t, \xi + \sigma t) + (\widetilde{v}^S)^X(\xi) - v_m, \widetilde{u}^R(t, \xi + \sigma t) + (\widetilde{u}^S)^X(\xi) - u_m). \tag{3.4}
$$

The key step for the proof of Theorem 1.1 is to show the a priori estimates as follows.

Proposition 3.1. *For a given constant state $(v_+, u_+) \in \mathbb{R}_+ \times \mathbb{R}$, there exist positive constants δ_0, ε_1 such that the following holds.*

Suppose that (v,u) *is the solution to* (1.1) *on* $[0,T]$ *for some* $T>0$, *and* $(\widetilde{v},\widetilde{u})$ *be the superposition wave of approximated rarefaction and viscous shock defined in* (3.4) *with shift* X *defined in* (3.9). *Assume that both the rarefaction and shock strengths satisfy* $\delta_R := |v_m - v_-|, \delta_S := |v_+ - v_m| < \delta_0$ *and that*

$$
\begin{aligned}
&v-\widetilde{v} \in C([0,T];H^1(\mathbb{R})),\\
&u-\widetilde{u} \in C([0,T];H^1(\mathbb{R})) \cap L^2(0,T;H^2(\mathbb{R})),
\end{aligned}
$$

and

$$
\|v-\widetilde{v}\|_{L^\infty(0,T;H^1(\mathbb{R}))} + \|u-\widetilde{u}\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq \varepsilon_1.
$$

Then, for all $t \leq T$,

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left(\|v-\widetilde{v}\|_{H^1(\mathbb{R})} + \|u-\widetilde{u}\|_{H^1(\mathbb{R})}\right) + \sqrt{\delta_S \int_0^t |\dot{X}|^2\,ds}\\
&\quad + \sqrt{\int_0^t (\mathcal{G}^S(U) + \mathcal{G}^R(U) + D(U) + D_1(U) + D_2(U))\,ds}\\
&\leq C_0 \left(\|v_0 - \widetilde{v}(0,\cdot)\|_{H^1(\mathbb{R})} + \|u_0 - \widetilde{u}(0,\cdot)\|_{H^1(\mathbb{R})}\right) + C_0 \delta_R^{1/6}
\end{aligned}
\tag{3.5}
$$

and

$$
|\dot{X}(t)| \leq C_0 \|(v-\widetilde{v})(t,\cdot)\|_{L^\infty(\mathbb{R})},
$$

where C_0 *is independent constant of* T *and*

$$
\begin{aligned}
&\mathcal{G}^S(U) := \int_{\mathbb{R}} |(v^S)^X_\xi| |v-\widetilde{v}|^2\,d\xi, \quad \mathcal{G}^R(U) := \int_{\mathbb{R}} |\widetilde{u}^R_\xi| |v-\widetilde{v}|^2\,d\xi,\\
&D(U) := \int_{\mathbb{R}} |\partial_\xi (p(v) - p(\widetilde{v}))|^2\,d\xi, \quad D_1(U) := \int_{\mathbb{R}} |(u-\widetilde{u})_\xi|^2\,d\xi,\\
&D_2(U) := \int_{\mathbb{R}} |(u-\widetilde{u})_{\xi\xi}|^2\,d\xi.
\end{aligned}
$$

The H^1-stability estimate (3.5) is the main estimate that will be used to derive the time-asymptotic behavior of the Navier-Stokes system, and in particular, the lower order estimate, i.e., the L^2-estimate is at the heart of the entire analysis. In order to obtain the L^2-estimate, we will basically use the method of a-contraction with shift. Below, we show several main steps for the L^2-estimate.

With an advantage in the calculation by introducing the variables $U = (v,h)$ with $h = u - (\ln v)_x$ associated with the BD entropy [1], it would be easier to get the L^2-estimate. So, we rewrite the system (1.3) into

$$
\begin{aligned}
&v_t - h_x = (\ln v)_{xx},\\
&h_t + p(v)_x = 0.
\end{aligned}
\tag{3.6}
$$

Notice that the above system has a parabolic regularization on the v-variable, contrary to the regularization on the u-variable for the original system (1.1). This would be better for our analysis, since the hyperbolic part of the system is linear in u (or h) but nonlinear

in v (via the pressure).
Let

$$\widetilde{U}(t,\xi) := \begin{pmatrix} \widetilde{v}(t,\xi) \\ \widetilde{h}(t,\xi) \end{pmatrix} = \begin{pmatrix} \widetilde{v}^R(t,\xi+\sigma t) + (\widetilde{v}^S)^X(\xi) - v_m \\ \widetilde{u}^R(t,\xi+\sigma t) + (\widetilde{h}^S)^X(\xi) - u_m \end{pmatrix}, \tag{3.7}$$

where the shift X is defined as below. Let $\delta_S := |v_+ - v_m| \sim |u_+ - u_m|$ and $\delta_R := |v_m - v_-| \sim |u_m - u_-|$. We construct the weight function $a(\xi) = a(x - \sigma t)$ as

$$a(\xi) = 1 + \frac{\lambda}{\delta_S}(p(v_m) - p(\widetilde{v}^S(\xi))), \tag{3.8}$$

where the constant λ is chosen such that $\delta_S \ll \lambda \leq C\sqrt{\delta_S}$. Moreover, we construct the shift $X(t)$ as a solution to the following ODE:

$$\dot{X}(t) = -\frac{M}{\delta_S}\left[\int_{\mathbb{R}} \frac{a^X}{\sigma}(\widetilde{h}^S)^X_\xi (p(v) - p(\widetilde{v}))\,d\xi - \int_{\mathbb{R}} a^X \partial_\xi p((\widetilde{v}^S)^X)(v - \widetilde{v})\,d\xi\right], \tag{3.9}$$

subject to the initial value $X(0) = 0$, where $\widetilde{h}^S := \widetilde{u}^S - (\ln \widetilde{v}^S)_\xi$ and M is chosen as $M := \frac{5(\gamma+1)\sigma_m^3}{8\gamma p(v_m)}$ with $\sigma_m := \sqrt{-p'(v_m)}$.

First, we compute the evolution of the weighted relative entropy with shift, and then maximize the worst hyperbolic terms related to the h variable over all h, from which the remaining hyperbolic terms are related to the v variable only, as in the following lemma.

Lemma 3.1. *Let U be a solution to* (3.6) *and $\widetilde{U}$ be the shifted wave defined in* (3.7). *Then,*

$$\frac{d}{dt}\int_{\mathbb{R}} a^X \eta(U(t,\xi)|\widetilde{U}(t,\xi))\,d\xi = \dot{X}(t)Y(U) + \mathcal{B}(U) - \mathcal{G}(U), \tag{3.10}$$

where

$$\begin{aligned}
Y(U) &:= -\int_{\mathbb{R}} a^X_\xi \eta(U|\widetilde{U})\,d\xi + \int_{\mathbb{R}} a^X \nabla^2\eta(\widetilde{U})(\widetilde{U}^S)^X_\xi (U - \widetilde{U})\,d\xi, \\
\mathcal{B}(U) &:= \frac{1}{2\sigma}\int_R a^X_\xi |p(v) - p(\widetilde{v})|^2\,d\xi + \sigma\int_R a^X (\widetilde{v}^S)^X_\xi p(v|\widetilde{v})\,d\xi \\
&\quad - \int_R a^X_\xi \frac{p(v) - p(\widetilde{v})}{\gamma p(v)}\partial_\xi(p(v) - p(\widetilde{v}))\,d\xi + \int_R a^X_\xi (p(v) - p(\widetilde{v}))^2 \frac{\partial_\xi p(\widetilde{v})}{\gamma p(v) p(\widetilde{v})}\,d\xi \\
&\quad - \int_R a^X \partial_\xi (p(v) - p(\widetilde{v}))\frac{p(\widetilde{v}) - p(v)}{\gamma p(v) p(\widetilde{v})}\partial_\xi p(\widetilde{v})\,d\xi + \int_R a^X (p(v) - p(\widetilde{v}))(\ln(\widetilde{v}^S)^X - \ln\widetilde{v})_{\xi\xi}\,d\xi \\
&\quad - \int_R a^X (h - \widetilde{h})(p(\widetilde{v}) - p(\widetilde{v}^R) - p((\widetilde{v}^S)^X))_\xi\,d\xi, \\
\mathcal{G}(U) &:= \frac{\sigma}{2}\int_{\mathbb{R}} a^X_\xi \left|h - \widetilde{h} - \frac{p(v) - p(\widetilde{v})}{\sigma}\right|^2 d\xi + \sigma\int_{\mathbb{R}} a^X_\xi Q(v|\widetilde{v})\,d\xi \\
&\quad + \int_{\mathbb{R}} a^X \widetilde{u}^R_\xi p(v|\widetilde{v})\,d\xi + \int_{\mathbb{R}} \frac{a^X}{\gamma p(v)}|\partial_\xi(p(v) - p(\widetilde{v}))|^2\,d\xi.
\end{aligned}$$

We may decompose each of Y, $\mathcal{B}$ and $\mathcal{G}$ as

$$\begin{aligned}
&Y(U) = \sum_{i=1}^{6} Y_i(U), \quad \mathcal{B}(U) := \sum_{i=1}^{5} B_i(U) + S_1(U) + S_2(U),\\
&\mathcal{G}(U) := G_1(U) + G_2(U) + G^R(U) + D(U),
\end{aligned}$$

where

$$\begin{aligned}
Y_1(U) &:= \int_{\mathbb{R}} \frac{a^X}{\sigma} (\widetilde{h}^S)^X_\xi (p(v) - p(\widetilde{v}))\, d\xi, \quad Y_2(U) := -\int_{\mathbb{R}} a^X p'((\widetilde{v}^S)^X)(\widetilde{v}^S)^X_\xi (v - \widetilde{v})\, d\xi,\\
Y_3(U) &:= \int_{\mathbb{R}} a^X (\widetilde{h}^S)^X_\xi \left(h - \widetilde{h} - \frac{p(v) - p(\widetilde{v})}{\sigma} \right) d\xi,\\
Y_4(U) &:= -\int_{\mathbb{R}} a^X (p'(\widetilde{v}) - p'((\widetilde{v}^S)^X)(\widetilde{v}^S)^X_\xi (v - \widetilde{v})\, d\xi,\\
Y_5(U) &:= -\frac{1}{2} \int_{\mathbb{R}} a^X_\xi \left(h - \widetilde{h} - \frac{p(v) - p(\widetilde{v})}{\sigma} \right) \left(h - \widetilde{h} + \frac{p(v) - p(\widetilde{v})}{\sigma} \right) d\xi,\\
Y_6(U) &:= -\int_{\mathbb{R}} a^X_\xi Q(v|\widetilde{v})\, d\xi - \int_{\mathbb{R}} \frac{a^X_\xi}{2\sigma^2} (p(v) - p(\widetilde{v}))^2\, d\xi,
\end{aligned}$$

$$\begin{aligned}
B_1(U) &:= \frac{1}{2\sigma} \int_{\mathbb{R}} a^X_\xi |p(v) - p(\widetilde{v})|^2\, d\xi, \quad B_2(U) := \sigma \int_{\mathbb{R}} a (\widetilde{v}^S)^X_\xi p(v|\widetilde{v})\, d\xi,\\
B_3(U) &:= -\int_{\mathbb{R}} a^X_\xi \frac{p(v) - p(\widetilde{v})}{\sigma} \partial_\xi (p(v) - p(\widetilde{v}))\, d\xi,\\
B_4(U) &:= \int_{\mathbb{R}} a^X_\xi (p(v) - p(\widetilde{v}))^2 \frac{\partial_\xi p(\widetilde{v})}{\gamma p(v) p(\widetilde{v})}\, d\xi,\\
B_5(U) &:= -\int_{\mathbb{R}} a^X \partial_\xi (p(v) - p(\widetilde{v})) \frac{p(\widetilde{v}) - p(v)}{\gamma p(v) p(\widetilde{v})} \partial_\xi p(\widetilde{v})\, d\xi,\\
S_1(U) &:= \int_{\mathbb{R}} a^X (p(v) - p(\widetilde{v}))(\ln (\widetilde{v}^S)^X - \ln \widetilde{v})_{\xi\xi}\, d\xi,\\
S_2(U) &:= -\int_{\mathbb{R}} a^X (h - \widetilde{h})(p(\widetilde{v}) - p(\widetilde{v}^R) - p((\widetilde{v}^S)^X))_\xi\, d\xi,
\end{aligned}$$

and

$$\begin{aligned}
G_1(U) &:= \frac{\sigma}{2} \int_{\mathbb{R}} a^X_\xi \left| h - \widetilde{h} - \frac{p(v) - p(\widetilde{v})}{\sigma} \right|^2 d\xi,\\
G_2(U) &:= \sigma \int_{\mathbb{R}} a^X_\xi Q(v|\widetilde{v})\, d\xi, \quad G^R(U) := \int_{\mathbb{R}} a^X (\widetilde{u}^R_\xi) p(v|\widetilde{v})\, d\xi,\\
D(U) &:= \int_{\mathbb{R}} \frac{a^X}{\gamma p(v)} |\partial_\xi (p(v) - p(\widetilde{v}))|^2\, d\xi.
\end{aligned}$$

However, it follows from the construction of the shift X that

$$\dot{X}(t) = -\frac{M}{\delta_S}(Y_1 + Y_2).$$

Using this, we estimate the time derivative of the weighted relative entropy as

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} a^X \eta(U(t,\xi)|\widetilde{U}(t,\xi))\,d\xi\\
&= \underbrace{-\frac{\delta_S}{2M}|\dot{X}|^2 + B_1 + B_2 - G_2 - \frac{3}{4}D}_{\text{main part}}\\
&\quad -\frac{\delta_S}{2M}|\dot{X}|^2 + \dot{X}\sum_{i=3}^{6} Y_i + \sum_{i=3}^{5} B_i + S_1 + S_2 - G_1 - G^R - \frac{1}{4}D,
\end{aligned}$$

where B_1 and B_2 are the main bad terms, as the hyperbolic that should be sharply controlled, and the remaining parabolic terms can be handled relatively easily. We use the sharp Poincaré-type inequality in Lemma 2.1 to control the main part. To this end, we rewrite the main parts in terms of the new variables y and w:

$$w := p(v) - p(\widetilde{v}) \quad \text{and} \quad y := \frac{p(v_m) - p(\widetilde{v}^S(\xi - X(t)))}{\delta_S}, \tag{3.11}$$

where for any fixed $t \in [0,T]$, the change of variable: $\xi - X(t) \in \mathbb{R} \mapsto y \in (0,1)$ makes sense, since $X(t)$ is bounded on $[0,T]$ by the a priori assumption.
Applying the Poincaré-type inequality to the main part, we obtain

$$\begin{aligned}
(\text{main part}) \le& -C\underbrace{\int_{\mathbb{R}} |(\widetilde{v}^S)^X_\xi||p(v)-p(\widetilde{v})|^2\,d\xi}_{=:G^S} + C\int_{\mathbb{R}} |a^X_\xi||p(v)-p(\widetilde{v})|^3\,d\xi\\
&+ C\int_{\mathbb{R}} |a^X_\xi||\widetilde{v}^R - v_m||p(v)-p(\widetilde{v})|^2\,d\xi.
\end{aligned}$$

Combining the estimates for the remaining parts, one can deduce the final estimate for the weighted relative entropy with the shift as

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} a^X\eta(U|\widetilde{U})\,d\xi \le& -\frac{\delta_S}{4M}|\dot{X}|^2 - \frac{1}{2}G_1 - \frac{C_1}{2}G^S - \frac{1}{8}D\\
&+ C\varepsilon_1\delta_S^{4/3}\delta_R^{4/3}e^{-C\delta_S t} + C\varepsilon_1^{2/3}\|(\widetilde{v}^R)_{\xi\xi}\|_{L^1}^{3/4} + C\varepsilon_1\|(\widetilde{v}^R)_\xi\|_{L^4}^2\\
&+ C(\varepsilon_1+\delta_R)\Big\||(\widetilde{v}^S)^X_\xi||\widetilde{v}^R - v_m| + |(\widetilde{v}^R)_\xi||(\widetilde{v}^S)^X - v_m| + |(\widetilde{v}^R)_\xi||(\widetilde{v}^S)^X_\xi|\Big\|_{L^2}.
\end{aligned}$$

We use Grönwall inequality, together with the estimates on the shock $\widetilde{v}^S$ and rarefaction $\widetilde{v}^R$, to derive the following control on the relative entropy:

$$\begin{aligned}
&\sup_{t\in[0,T]}\int_{\mathbb{R}}\eta(U|\widetilde{U})\,d\xi + \delta_S\int_0^t|\dot{X}|^2\,ds + \int_0^t (G_1 + G^S + D)\,ds\\
&\quad\le C\int_{\mathbb{R}}\eta(U_0|\widetilde{U}(0,\xi))\,d\xi + C\delta_R^{1/3}.
\end{aligned}\tag{3.12}$$

The estimate on the relative entropy (3.12) is the main step to prove the key proposition, Proposition 3.1. This estimate, together with the H^1-estimate on $u - \widetilde{u}$ yields the results in Proposition 3.1, in particular, the estimate (3.5). Finally, for the desired asymptotic behavior, we use the estimate (3.5) to show that the function:

$$g(t) := \|(v - \widetilde{v})_\xi\|^2_{L^2(\mathbb{R})} + \|(u - \widetilde{u})_\xi\|^2_{L^2(\mathbb{R})}, \tag{3.13}$$

satisfies

$$\int_0^\infty (|g(t)| + |g'(t)|)\, dt < \infty,$$

which implies

$$\lim_{t\to\infty} g(t) = 0.$$

Again, together with the estimate (3.5) and the interpolation inequality implies

$$\lim_{t\to\infty} \left(\|v - \widetilde{v}\|_{L^\infty(\mathbb{R})} + \|u - \widetilde{u}\|_{L^\infty(\mathbb{R})}\right) = 0.$$

Finally, we combine the above estimate to (3.3) to derive the desired convergence. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

We now establish the proof for Theorem 1.2. Since the outline of the proof is parallel to the proof of Theorem 1.1, we focus on discussing the difference and additional difficulty, when it comes to the composition of two shocks. To be specific, we consider the end states $(v_\pm, u_\pm)$ such that there exists a unique intermediate state (v_m, u_m) which is connected with (v_-, u_-) by 1-shock curve and with (v_+, u_+) by 2-shock curve. That is, there exists a unique (v_m, u_m) such that the following Rankine-Hugoniot condition and Lax entropy condition hold:

$$\begin{aligned}
&\begin{cases} -\sigma_1(v_m - v_-) - (u_m - u_-) = 0, \\ -\sigma_1(u_m - u_-) + (p(v_m) - p(v_-)) = 0, \end{cases} \qquad \sigma_1 := -\sqrt{-\frac{p(v_m) - p(v_-)}{v_m - v_-}}, \quad v_- > v_m, \quad u_- > u_m; \\
&\begin{cases} -\sigma_2(v_+ - v_m) - (u_+ - u_m) = 0, \\ -\sigma_2(u_+ - u_m) + (p(v_+) - p(v_m)) = 0, \end{cases} \qquad \sigma_2 := \sqrt{-\frac{p(v_+) - p(v_m)}{v_+ - v_m}}, \quad v_m < v_+, \quad u_m > u_+.
\end{aligned} \tag{4.1}$$

Then, the Euler equations (1.3) with (1.4)–(4.1) admit a unique self-similar solution, the so-called Riemann solution $(\bar{v}, \bar{u})$, represented by the composition $(\bar{v}, \bar{u}) = (v_1^s, u_1^s) + (v_2^s, u_2^s) - (v_m, u_m)$ of 1-shock wave (v_1^s, u_1^s) and 2-shock wave (v_2^s, u_2^s) defined as (see e.g. [21])

$$(v_1^s, u_1^s)(t, x) = \begin{cases} (v_-, u_-), & x < \sigma_1 t, \\ (v_m, u_m), & x > \sigma_1 t, \end{cases} \quad , \quad (v_2^s, u_2^s)(t, x) = \begin{cases} (v_m, u_m), & x < \sigma_2 t, \\ (v_+, u_+), & x > \sigma_2 t. \end{cases}$$

One of the main differences from Theorem 1.1 is that we need two shift functions X_1 and X_2, one for each shock wave. As a consequence, we will consider the superposition of two viscous shock waves as

$$(\widetilde{v}, \widetilde{u})(t,x) := \left(\widetilde{v}_1^{X_1}(x-\sigma_1 t) + \widetilde{v}_2^{X_2}(x-\sigma_2 t) - v_m, \widetilde{u}_1^{X_1}(x-\sigma_1 t) + \widetilde{u}_2^{X_2}(x-\sigma_2 t) - u_m\right), \tag{4.2}$$

which is composed of 1-viscous shock $(\widetilde{v}_1, \widetilde{u}_1)(x-\sigma_1 t)$ and 2-viscous shock $(\widetilde{v}_2, \widetilde{u}_2)(x-\sigma_2 t)$ satisfying (1.11).

Then, the parallel proposition to Proposition 3.1 reads as follows.

Proposition 4.1. *For a given constant state $(v_+, u_+) \in \mathbb{R}_+ \times \mathbb{R}$, there exist positive constants δ_0, ε_1 such that the following holds.*

Suppose that (v,u) is the solution to (1.1) on $[0,T]$ for some $T>0$, and $(\widetilde{v}, \widetilde{u})$ be the superposition wave of two viscous shock waves defined in (4.2). Assume that both shock strengths satisfy $\delta_1 := |v_m - v_-|, \delta_2 := |v_+ - v_-| < \delta_0$ and that

$$v - \widetilde{v} \in C([0,T]; H^1(\mathbb{R})),$$
$$u - \widetilde{u} \in C([0,T]; H^1(\mathbb{R})) \cap L^2(0,T; H^2(\mathbb{R})),$$

and

$$\|v - \widetilde{v}\|_{L^\infty(0,T;H^1(\mathbb{R}))} + \|u - \widetilde{u}\|_{L^\infty(0,T;H^1(\mathbb{R}))} \le \varepsilon_1.$$

Then, for all $t \le T$,

$$\begin{aligned}
&\sup_{t\in[0,T]} \left(\|v-\widetilde{v}\|_{H^1(\mathbb{R})} + \|u-\widetilde{u}\|_{H^1(\mathbb{R})}\right) + \sqrt{\int_0^t \sum_{i=1}^2 \delta_i |\dot{X}_i|^2 \, ds} \\
&\quad + \sqrt{\int_0^t (\mathcal{G}^S(U) + D(U) + D_1(U) + D_2(U)) \, ds} \\
&\le C_0 \left(\|v_0 - \widetilde{v}(0,\cdot)\|_{H^1(\mathbb{R})} + \|u_0 - \widetilde{u}(0,\cdot)\|_{H^1(\mathbb{R})}\right) + C_0 \delta_0^{1/4}
\end{aligned} \tag{4.3}$$

and

$$|\dot{X}_1(t)| + |\dot{X}_2(t)| + C_0 \le \|(v-\widetilde{v})(t,\cdot)\|_{L^\infty(\mathbb{R})},$$

where C_0 is the constant independent of T and

$$\mathcal{G}^S(U) := \sum_{i=1}^2 \int_{\mathbb{R}} |(\widetilde{v}_i)_x^{X_i}| |\phi_i (v-\widetilde{v})|^2 \, dx, \quad D(U) := \int_{\mathbb{R}} |\partial_x (p(v) - p(\widetilde{v}))|^2 \, dx,$$
$$D_1(U) := \int_{\mathbb{R}} |(u-\widetilde{u})x|^2 \, dx, \quad D_2(U) := \int_{\mathbb{R}} |(u-\widetilde{u})_{xx}|^2 \, dx,$$

and ϕ_i are cutoff functions defined by

$$\phi_1(t,x) := \begin{cases} 1 & \text{if} \quad x < \frac{X_1(t)+\sigma_1 t}{2}, \\ 0 & \text{if} \quad x > \frac{X_2(t)+\sigma_2 t}{2}, \\ \text{linearly decreasing from 1 to 0} & \text{otherwise}, \end{cases} \qquad \phi_2(t,x) := 1 - \phi_1(t,x).$$

As mentioned in the previous section, the H^1-stability estimate (4.3) is the main estimate to derive the time-asymptotic behavior of the Navier-Stokes system. However, in the case of the composition of two shock waves, some problematic issues arise, compared to the estimates in the previous section.

First, to use the method of a-contraction with shift applied to the case of a single shock wave, we should construct two shift functions X_1 and X_2, with two weight functions a_1 and a_2 that control perturbation near each shock wave respectively.

Second, we need to construct the cutoff functions ϕ_i as in Proposition 4.1, to apply the Poincaré-type inequality to each wave. Indeed, since the derivative of each weight or shock localizes all the bad terms, but not the diffusion term, we construct ϕ_i to localize the diffusion near i-wave. More specifically, ϕ_1 (resp. ϕ_2) localizes the perturbation near the 1-wave (resp. 2-wave) shifted by X_1 (resp. X_2) satisfying

$$X_1(t) + \sigma_1 t \le \frac{\sigma_1}{2} t < 0 < \frac{\sigma_2}{2} t \le X_2(t) + \sigma_2 t, \qquad t > 0,$$

and so the functions ϕ_1 and ϕ_2 are well-separated as time goes on.

Again, for convenience in the calculation, we use BD-variable $U = (v, h)$ instead of (v, u) and define $\widetilde{U}$ as

$$\widetilde{U}(t,x) := \begin{pmatrix} \widetilde{v}(t,x) \\ \widetilde{h}(t,x) \end{pmatrix} = \begin{pmatrix} \widetilde{v}_1^{X_1}(x - \sigma_1 t) + (\widetilde{v}_2)^{X_2}(x - \sigma_2 t) - v_m \\ \widetilde{h}_1^{X_1}(x - \sigma_1 t) + (\widetilde{h}_2)^{X_2}(x - \sigma_2 t) - u_m \end{pmatrix}, \tag{4.4}$$

where $\widetilde{h}_i := \widetilde{u}_i - (\ln \widetilde{v}_i)_x$, $i = 1, 2$. And let us discuss how to construct the weight function and corresponding shifts X_i. Since we have two shocks, we need to construct two weight functions a_i with $i = 1, 2$ as

$$a_i(x - \sigma_i t) = 1 + \frac{\lambda(p(v_m) - p(\widetilde{v}_i(x - \sigma_i t)))}{\delta_i},$$

and then construct the composition of the shifted weights as

$$a(t,x) := a_1^{X_1}(x - \sigma_1 t) + a_2^{X_2}(x - \sigma_2 t) - 1.$$

Then, similar to the construction of the shift as in the previous case, we define two shifts as the solution to the following ODEs:

$$\begin{aligned} \dot{X}_1 &= -\frac{M}{\delta_1} \left(\int_{\mathbb{R}} \frac{a}{\sigma_1} (\widetilde{h}_1)_x^{X_1} (p(v) - p(\widetilde{v}))\, dx - \int_{\mathbb{R}} a p(\widetilde{v}_1)_x^{X_1} (v - \widetilde{v})\, dx \right), \\ \dot{X}_2 &= -\frac{M}{\delta_2} \left(\int_{\mathbb{R}} \frac{a}{\sigma_2} (\widetilde{h}_2)_x^{X_2} (p(v) - p(\widetilde{v}))\, dx - \int_{\mathbb{R}} a p(\widetilde{v}_2)_x^{X_2} (v - \widetilde{v})\, dx \right), \\ X_1(0) &= X_2(0) = 0, \end{aligned}$$

Again, the main step of the proof of Proposition 4.1 is to estimate the weighted relative entropy with shifts:

$$\int_{\mathbb{R}} a(t,x) \eta(U(t,x) | \widetilde{U}(t,x))\, dx.$$

Following a similar process as in the previous section, one can obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} a\eta(U|\widetilde{U})\,dx \\
&= \underbrace{= -\sum_{i=1}^{2}\frac{\delta_i}{2M}|\dot{X}_i|^2 + \mathcal{B}_1 + \mathcal{B}_2 - \mathcal{G}_2 - \frac{3}{4}\mathcal{D}}_{=:\mathcal{R}_1} \\
&\quad \underbrace{-\sum_{i=1}^{2}\frac{\delta_i}{2M}|\dot{X}_i|^2 + \sum_{i=1}^{2}\left(\dot{X}_i\sum_{j=3}^{6}Y_{ij}\right) + \sum_{i=3}^{5}\mathcal{B}_i + \mathcal{S}_1 + \mathcal{S}_2 - \mathcal{G}_1 - \frac{1}{4}\mathcal{D}}_{=:\mathcal{R}_2}.
\end{aligned}
\tag{4.5}
$$

We refer to Section 4.5 in [4] for the exact definition of each term in (4.5). Again, the term $\mathcal{R}_1$ is the main part, which should be sharply controlled by using the Poincaré-type inequality in Lemma 2.1 and the remaining term $\mathcal{R}_2$ can be controlled in a rather rough way.

However, two things should be noted when using the Poincaré-type inequality. First, since there are two shift functions X_1 and X_2 for each of the viscous shock waves, we use the change of variable more carefully. Second, we need to localize the perturbation near each shock wave by using the cutoff functions ϕ_1 and ϕ_2. Since this procedure is extremely complicated and technical, we omit the details.

After we estimate $\mathcal{R}_1$ and $\mathcal{R}_2$, one can finally obtain the following control on the weight relative entropy with the shift:

$$
\begin{aligned}
&\int_{\mathbb{R}} a(t,x)\eta(U(t,x)|\widetilde{U}(t,x))\,dx + \int_0^t \left(\sum_{i=1}^{2}\delta_i|\dot{X}_i|^2 + G_1 + G^S + D\right) ds \\
&\quad \le C\int_R a(0,x)\eta(U_0(x)|\widetilde{U}(0,x))\,dx + C\delta_0,
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(U) &:= \sum_{i=1}^{2}|(a_i)_x^{X_i}|\left|h - \widetilde{h} - \frac{p(v)-p(\widetilde{v})}{\sigma_i}\right|^2 dx, \\
G^S(U) &:= \sum_{i=1}^{2}\int_{\mathbb{R}}|(\widetilde{v}_i)_x^{X_i}||\phi_i(p(v)-p(\widetilde{v}))|^2\,dx, \\
D(U) &:= \int_{\mathbb{R}}|\partial_x(p(v)-p(\widetilde{v}))|^2\,dx.
\end{aligned}
$$

Then, the remaining part is the same as in the previous section. We first obtain the H^1-estimate on $u-\widetilde{u}$, and then consider the same quantity $g(t)$ defined in (3.13) to obtain the desired time-asymptotic behavior. This completes the proof of Theorem 1.2.

5 Concluding remarks

The results of [4] and [11], stated in the main Theorems 1.1 and 1.2, are for the barotropic Navier-Stokes system in 1D as (1.1). The key idea to prove them is to use the method of a-contraction with shifts. As future works, it would be natural to extend the method of a-contraction with shifts to more complicated cases: (i) barotropic Navier-Stokes system in multi-D; (ii) Navier-Stokes-Fourier system in 1D. Recently, in [25], the method was extended to tackling the case (i) for a single (planar) shock wave. So, for the case (i), it would be interesting to handle the composition wave of either shock and rarefaction or shocks. On the other hand, it would be very challenging to extend the method to the case (ii), and so prove the open problem: the large-time behavior of solutions perturbed from the generic composite wave of shock, contact discontinuity, and rarefaction.
As further applications of the method, it would be interesting to study the long-time behavior or stability for the inflow (or outflow) problem of the compressible Navier-Stokes system in half space, and for the compressible Navier-Stokes system coupled with other physical phenomena.

References

[1] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models. J. Math. Pures Appl., 86(9):362–368, 2006.

[2] K. Choi, M.-J. Kang, Y. Kwon, and A. Vasseur. Contraction for large perturbations of traveling waves in a hyperbolic-parabolic system arising from a chemotaxis model. Math. Models Methods Appl. Sci., 30(2):387–437, 2020.

[3] J. Goodman. Nonlinear asymptotic stability of viscous shock profiles for conservation laws. Arch. Rational Mech. Anal., 95(4):325344, 1986.

[4] S. Han, M.-J. Kang, and J. Kim. Large-time behavior of composite waves of viscous shocks for the barotropic Navier-Stokes equations. arXiv.2207.05989.

[5] F. Huang and A. Matsumura. Stability of a composite wave of two viscous shock waves for the full compressible Navier-Stokes equation. Comm. Math. Phys., 289(3):841–861, 2009.

[6] M.-J. Kang. L^2-type contraction for shocks of scalar viscous conservation laws with strictly convex flux. J. Math. Pures Appl. 145, 1–43, 2021.

[7] M.-J. Kang and A. Vasseur. Criteria on contractions for entropic discontinuities of systems of conservation laws. Arch. Ration. Mech. Anal., 222(1):343–391, 2016.

[8] M.-J. Kang and A. Vasseur. Well-posedness of the Riemann problem with two shocks for the isentropic Euler system in a class of vanishing physical viscosity limits. J. Differential Equations, 338, 128–226, 2022.

[9] M.-J. Kang and A. Vasseur. Uniqueness and stability of entropy shocks to the isentropic Euler system in a class of inviscid limits from a large family of Navier-Stokes systems. Invent. Math., 224(1):55–146, 2021.

[10] M.-J. Kang and A. Vasseur. Contraction property for large perturbations of shocks of the barotropic Navier-Stokes system. J. Eur. Math. Soc. (JEMS), 23(2):585–638, 2021.

[11] M.-J. Kang, A. Vasseur, and Y. Wang. Time-asymptotic stability of composite waves of viscous shock and rarefaction for barotropic Navier-Stokes equations. arXiv:2104.06590, 2021.

[12] T.-P. Liu. Nonlinear stability of shock waves for viscous conservation laws. Mem. Amer. Math. Soc., 56(328):v+108, 1985.

[13] T.-P. Liu and Z. P. Xin. Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations. Comm. Math. Phys., 118(3):451–465, 1988.

[14] T.-P. Liu and Y. Zeng. Shock waves in conservation laws with physical viscosity. Mem. Amer. Math. Soc., 234(1105):vi+168, 2015.

[15] A. Matsumura. Waves in compressible fluids: viscous shock, rarefaction, and contact waves. In *Handbook of mathematical analysis in mechanics of viscous fluids*, pp. 2495–2548. Springer, Cham, 2018

[16] A. Matsumura and K. Nishihara. On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas. Japan J. Appl. Math., 2:17–25, 1985.

[17] A. Matsumura and K. Nishihara. Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas. Japan J. Appl. Math., 3:1–13, 1986.

[18] A. Matsumura and K. Nishihara. Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas. Comm. Math. Phys., 144:325–335, 1992.

[19] A. Matsumura and Y. Wang. Asymptotic stability of viscous shock wave for a one-dimensional isentropic model of viscous gas with density dependent viscosity. Methods Appl. Anal., 17(3):279–290, 2010.

[20] K. Nishihara, T. Yang, and H. Zhao. Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations. SIAM J. Math. Anal., 35(6):1561–1597, 2004.

[21] D. Serre. Systems of conservation laws I, II. Cambridge University Press, Cambridge, 1999.

[22] A. Szepessy and Z. P. Xin. Nonlinear stability of viscous shock waves. Arch. Rational Mech. Anal., 122(1):53–103, 1993.

[23] A. Vasseur. Recent results on hydrodynamic limits. In Handbook of differential equations: evolutionary equations. Vol. IV, Handb. Differ. Equ., pages 323–376. Elsevier/North-Holland, Amsterdam, 2008.

[24] A. Vasseur. Relative entropy and contraction for extremal shocks of conservation laws up to shift. In *Recent advances in partial differential equations and applications*, Vol. 666, *Contemp. Math.* pp 385–404. Amer. Math. Soc., Providence, RI, 2016.

[25] T. Wang and Y. Wang. Nonlinear stability of planer viscous shock wave to three-dimensional compressible Navier-Stokes equations. arXiv:2204.09428.

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