Continuous-time optimal execution under a transient market impact model in a Markovian environment*

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Abstract

This paper examines a continuous-time optimal trade execution problem under a transient market impact model. We also analyze the effect of an exogenous random factor that affects the market price on the optimal trade execution strategy. Our execution problem is formulated as a continuous-time stochastic control problem over a finite horizon of maximizing the expected utility from the final wealth of a risk-averse large trader. By examining the Hamilton-Jacobi-Bellman (HJB) equation, we characterize the optimal trade execution strategy and its associated optimal value function. The trade execution strategy becomes a time-dependent affine function of state variables. Further, the time-dependent coefficients could be derived from a solution of a system of ordinary differential equations (ODEs) with terminal conditions, which is numerically tractable. In addition, we conduct simulation-based numerical experiments and confirm that the optimal execution strategy captures various features observed in financial markets.

1 Introduction

Optimal execution problems, originated by the seminal papers Bertsimas and Lo [5] and Almgren and Chriss [2], have attracted widespread interest over the past two decades. In financial markets, an institutional trader called *large trader* (e.g., pension funds, life insurance companies) *executes* (trades) a large amount of orders, moving financial asset prices in an unfavorable direction. The effect caused by the large traders on the prices is called *market impact* (or price impact). The readers may refer to, e.g., Cartea et al. [12], Guéant [30], Laruelle and Lehalle [37], and Shimoshimizu [57] for a wide range of different configurations of optimal execution problems.

The following three market impacts are among the most discussed types: temporary, permanent, and transient market impacts. A temporary market impact refers to a part of the market impact that fades away immediately after the execution, while a permanent market impact is a part of the market impact that remains in the rest of the trading window. On the contrary, transient market impacts, recognized as a striking feature of market impacts, is referred to as the effect of market impact that decays over the trading window and is empirically confirmed (Bouchaud et al. [8]; Gatheral [25]). The transiency of the market impact indicates that a shortage of liquidity caused by a large trader's execution results in an imbalance between

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buy/sell orders in a financial market. There is considerable literature on execution problems in the presence of transient market impact (e.g., Gatheral et al. [27]; Kuno and Ohnishi [35]; Kuno et al. [36]; Schied and Zhang [55]; Tsoukalas et al. [60]; Luo and Schied [39]; Ohnishi and Shimoshimizu [47, 48, 49, 50]; Fukasawa et al. [23, 24]; Neuman and Voß [43, 44]; Cordoni and Lillo [16, 17] to mention only a few).

In addition to the market impacts, other exogenous factors often have a detrimental effect on the market price of a financial asset. For example, existing empirical and theoretical studies demonstrate that the order flow of small traders incurs a price impact (Potters and Bouchaud [52]; Cartea and Jaimungal [11, 10]) and influences an (optimal) execution strategy (Cartea and Jaimungal [11, 10]; Fukasawa et al. [23, 24]; Ohnishi and Shimoshimizu [48, 49, 50]). Another example of such an exogenous factor is the *order book imbalance* (OBI), which refers to the difference between the volume at the best buy price and the one at the best sell price divided by its sum and has a mean-reversion property (Lehalle and Neuman [38]). Existing studies show that OBI causes price fluctuation (e.g., Cont et al. [15]; Stoikov [58]) and thus influences an (optimal) execution strategy Lehalle and Neuman [38]. These examples indicate that an exogenous stochastic factor can affect the price of a financial asset, which in turn affects the optimal execution strategy or trade performance of a large trader.

Based on the background described so far, this paper examines an optimal execution strategy for a large trader under the existence of market impacts and an exogenous stochastic factor, both of which affect the price of a financial asset. In our analysis, we assume that temporary, permanent, and transient market impacts exist and that the exogenous stochastic factor is represented by a generalized Ornstein–Uhlenbeck (OU) process. The main objective of this paper is to analyze how a large trader, taking environmental uncertainties into account, optimally executes her orders. We particularly examine an optimal execution problem under the existence of an exogenous stochastic factor that affects the market price. We obtain an explicit optimal execution strategy and theoretically show that the exogenous stochastic factor directly influences the optimal execution strategy. Our theoretical analysis also enables us to predict the future inventory through only the information about the current state. Our numerical experiments reveal several features observed in financial markets: The higher environmental uncertainty (and the terminal cost per unit at the end of the trading horizon, respectively), the faster the optimal execution speed to avoid future price fluctuation driven by the uncertainty (to avoid the liquidation with high cost at the end).

The contribution of this paper is summarized as follows. We first derive an explicit optimal execution strategy that involves the effect of an exogenous stochastic factor on the price of a financial asset. Our model also provides an explicit formula for predicting the future inventory given the information about the current state variable for the model in the absence of transient market impacts. By the formula, practitioners as large traders in financial markets can manage their inventory, which may be of great importance from the viewpoint of execution control. Furthermore, our numerical examples quantify how environmental uncertainty influences the optimal execution strategy. In particular, we find that if a risk-averse large trader has an initial inventory below some fixed level, there exists a round trip trading through which the large trader can increase her expected utility.

This paper proceeds as follows. In Section 2, we construct a market model which characterizes the generalized market impact model. We solve the maximization problem of the expected utility of a risk-averse large trader in Section 3. Section 4 is devoted to conducting numerical experiments to see how problem parameters influence the optimal execution strategy. Finally, Section 5 concludes. The proofs are shown in the appendices.

2 Market Model

We assume that in a financial market, a risk-averse large trader is obligated to purchase $\mathfrak{Q}$ $(\in \mathbb{R})$ volume of a risky asset in a time window $[0, T]$. Let Q_t $(\in \mathbb{R})$ be the cumulative purchase up to time $t \in [0, T]$ of the large trader. Then, the number of shares that remained to purchase at time $t \in [0, T]$ is described as

$$\overline{Q}_t = \mathfrak{Q} - Q_t, \tag{2.1}$$

with the initial and terminal conditions: $\overline{Q}_0 = \mathfrak{Q}$ and $\overline{Q}_T = 0$. We consider a continuous trading strategy:

$$\mathrm{d}Q_t = \dot{Q}_t \mathrm{d}t. \tag{2.2}$$

It is assumed that Q_t is continuously differentiable in time $t \in [0, T]$. We denote by the positive and negative $\dot{Q}_t$ the acquisition and liquidation of the risky asset, respectively. This leads to a similar setup for a selling problem. The execution price of an asset $\widehat{P}$ is assumed to follow a linear market impact model (e.g., Bertsimas and Lo [5]; Almgren and Chriss [2]; Gatheral et al. [27]; Obizhaeva and Wang [46]; Kuno and Ohnishi [35]; Cartea and Jaimungal [10, 11]; Kuno et al. [36]; Lehalle and Neuman [38]; Ohnishi and Shimoshimizu [48, 49, 50]; Fukasawa et al. [23, 24]):

$$\widehat{P}_t = P_t + \lambda_t \dot{Q}_t, \tag{2.3}$$

where P_t $(\in \mathbb{R})$ represents the market price of the asset at time $t \in [0, T]$ and λ_t $(\in \mathbb{R}_{++})$ is a market impact coefficient at time $t \in [0, T]$.

In the sequel of this paper, we assume that the buy- and sell-trade of the large trader induce the same (instantaneous) market impact, although it would be different in the real market. We can, however, justify this assumption from the statistical analysis of market data shown by, for instance, Cartea and Jaimungal [10, 11]. The large trader's wealth process at time $t \in [0, T]$, denoted by W_t, evolves as

$$\mathrm{d}W_t = -\widehat{P}_t \mathrm{d}Q_t = -\widehat{P}_t \dot{Q}_t \mathrm{d}t = -\left(P_t + \lambda_t \dot{Q}_t\right) \dot{Q}_t \mathrm{d}t. \tag{2.4}$$

Besides the above (instantaneous) market impact, we consider the permanent and transient parts of the market impact. The residual effect of past market impacts is defined, with the deterministic linear temporary market impact coefficient $\alpha_t \in \mathbb{R}_+$, as

$$R_t = \mathrm{e}^{-\rho t} R_0 + \int_0^t \mathrm{e}^{-\rho(t-s)} \alpha_s \lambda_s \dot{Q}_s \mathrm{d}t, \tag{2.5}$$

or

$$\mathrm{d}R_t = -\rho R_t \mathrm{d}t + \alpha_t \lambda_t \dot{Q}_t \mathrm{d}t. \tag{2.6}$$

We define this transient market impact by employing the exponential decay kernel $G\colon \mathbb{R}_+ := [0, T] \to \mathbb{R}_+$:

$$G(t) := \mathrm{e}^{-\rho t}, \tag{2.7}$$

where $\rho \in \mathbb{R}_+$ stands for the deterministic resilience speed.¶ This residual effect indicates that the market impact decays gradually over the trading window $[0, T]$. R_0 is assumed to be zero

¶Much of theoretical analysis, such as in Obizhaeva and Wang [46], Tsoukalas et al. [60], deal with a (deterministic and) time-independent resilience speed. Many empirical kinds of research, however, demonstrate that liquidity is variable over time, suggesting that the resilience speed is time-dependent. Our analysis allows the time-dependence for the resilience speed, i.e., ρ_t for all $t \in [0, T]$, as considered in Fruth et al. [22]. Notwithstanding a meaningful extension from the viewpoint of real market analysis, we henceforth formulate the model without a time-dependent parameter (i.e., with ρ) since the dependence will not offer additional intriguing results in the following analysis.

in the following analysis. The assumption is quite plausible from the fact that the trader has no market impact on any risky asset before liquidating/acquiring the risky asset, and thereby there exist no residual effects caused by the trader on the price before the execution. Note that $\alpha_t \lambda_t \dot{Q}_t$ represents the temporary market impact caused by the large trader.

The market price is assumed to consist of the sum of the following two components:

$$P_t = P_t^f + R_t, \tag{2.8}$$

where P_t^f for $t \in [0, T]$ stands for what we call the *fundamental price* of the risky asset. This assumption stems from the definition of the transient market impact. The transient market impact is the discounted sum of the past temporary market impact. Thus, the transient market impact can be deemed to not influence the fundamental part of the market price. We define the dynamics of the fundamental price as follows:

$$\mathrm{d}P_t^f = \left(\beta_t \lambda_t \dot{Q}_t + \mathcal{I}_t\right) \mathrm{d}t + \mathrm{d}Z_t. \tag{2.9}$$

Here $\beta_t \lambda_t \dot{Q}_t$ represents the permanent market impact caused by the large trader. Contrary to the transient market impact, the permanent market impact influences the market price by definition, which in turn suggests that the permanent market impact directly affects the fundamental part of the market price. Z_t represents the effect of some public news/information about the economic situation which may affect the market price (or quoted price). Adding to these two factors, we assume that what we define as *Markovian environment* affects the fundamental price of the risky asset.

Remark 2.1 (Interpretation of Markovian environment)**.** The interpretation of the Markovian environment can be various and needs to be considered carefully. $\{\mathcal{I}_t\}_{0 \le t \le T}$ represents some "state" of the financial asset market or trading system of the exchange at time t that affects in a linear functional form the drift term of the price process of the financial asset that is the subject of our trade execution. We assume that it follows an Ornstein-Uhlenbeck type, Gaussian, and Markovian process.

- A classic and important example would be the continuous-time dynamic model version of the single factor model of asset price returns motivated by the Capital Asset Pricing Model (CAPM), a static equilibrium model of a financial asset market. Originating from the discrete-time CAPM (by, e.g., Sharpe [56]), the continuous-time CAPM relation is investigated by, for instance, Merton [40, 41], and implicitly by Cox et al. [18]. In our model, the rate of return on the market portfolio or its proxy, the asset market index, represents the state of the asset market.

- Many existing studies consider an effect on the risky asset price by a mean-reverting process expressing aggregate orders submitted by *small traders*. It is empirically shown that the market impacts caused by small traders are larger than those caused by large traders (Potters and Bouchaud [52]). Cartea and Jaimungal [10, 11] examine an expected cost minimization problem under the existence of small traders. Also, Fukasawa et al. [23] and Ohnishi and Shimoshimizu [48, 49] investigate an expected utility maximization problem incorporating the effect caused by small traders. These studies show that the aggregate past orders posed by small traders do influence the optimal execution strategy of a large trader. In line with these studies, we can consider $(\beta_t \lambda_t \dot{Q}_t + \mathcal{I}_t)$ as the permanent market impact caused by market traders.

- Some studies also study other features that influence the stock price. Lehalle and Neuman [38], for instance, finds that the *order book imbalance* (OBI in abbreviation), defined by Imb, with the volumes V^b of the best bid and V^a of the best ask defined as

$$\mathrm{Imb}_t := \frac{V_t^b - V_t^a}{V_t^b + V_t^a}, \tag{2.10}$$

has a predictability of future price movements. According to the empirical analysis, one can interpret $\mathcal{I}_t$ as the effect of the order book imbalance on the fundamental price. We should note that the OBI does *not* directly affect the fundamental price as the form (2.9). The OBI is assumed to, for example, influence the fundamental price as a linear form. That is, $\mathcal{I}_t$ is assumed to be a linear function of OBI as

$$\mathcal{I}_t = \kappa_t \mathrm{Imb}_t, \tag{2.11}$$

for some deterministic coefficient function κ_t. This relationship is empirically confirmed by Lehalle and Neuman [38].‖ Others, for example, Stoikov and Waeber [59], also show the predictability of the imbalance between the top of book bid and ask sizes, the so-called *order flow imbalance* (*OFI*). Bechler and Ludkovski [4] examine an optimal execution strategy taking into account the OFI.

Remark 2.2 (Related literature and the differences)**.** There are some papers closely related to our analysis. For example, Lehalle and Neuman [38] consider the effect of market microstructure signal on the risky asset price in the presence of transient market impacts. Another paper Cartea and Jaimungal [10] investigates an optimal execution problem in the presence of temporary and permanent market impacts. The market model is similar to those papers, in particular, the one of Lehalle and Neuman [38]. We should here note that some differences exist between their models and our model. First of all, as explicitly stated in the next section, our model considers an expected (CARA) utility maximization problem, although Lehalle and Neuman [38] and Cartea and Jaimungal [10] study an expected cost minimization problem. Examining the execution problem from an expected utility maximization enables us to consider the risk-averse property more accurately than other models focusing on the expected cost minimization problem. Lehalle and Neuman [38], in addition, do not consider the permanent market impact term as (2.9). (They assume that the market impact affects only the market price dynamics.) Also, Cartea and Jaimungal [10] does not include the transient part of the market impact, although the transient market impact plays a key role in our analysis. Our analysis sheds light on an optimal execution problem including all the fundamental aspects of market impact and the large

‖Existing studies illuminate the effect of microstructural factors on the market price of a financial asset. In particular, how price dynamics of a financial asset are determined through the order book has attracted widespread interest among academic researchers and practitioners. We may recognize the mid-price as the price of a financial asset without any price impact caused by (large) order submissions. Another feature that may be of interest to practitioners is the weighted mid-price defined by the *order book imbalance* the total volume at the best bid Q^b and the total volume at the best ask Q^a as $w := Q^b/(Q^b + Q^a)$.

Both types of "mid-price" have the advantage of being easily obtained from market data, although existing studies have shown their shortcomings (e.g., Gatheral and Oomen [26]; Robert and Rosenbaum [53]). Stoikov [58] then defines the notion of *micro-price*. The micro-price P^{micro} incorporates the effect of mid-price M, order book imbalance I, and the bid-ask spread $S := P^b - P^a$ into the components of the underlying price: in mathematical form, we can write the dependence as

$$P^{micro} := M + g(I, S), \tag{2.12}$$

using a function g, which can be empirically estimated.

Theoretical studies have analyzed a trade execution strategy that exploits predictable (microstructural) factors (e.g., Gârleanu and Pedersen [28, 29]; Chan and Sircar [14]; Cartea et al. [9]; Forde et al. [20]; Fouque et al. [21]; Neuman and Voß [43, 44]; Drissi [19]). Among them, Cartea and Jaimungal [10] and Lehalle and Neuman [38] are closely related to our analysis. They investigate the influence of Markovian microstructure signal on a single large trader's optimal execution strategy.

trader's property in one model. In addition, Bergault et al. [7] examine an optimal execution problem when the price dynamics of multiple risky assets are described by a multidimensional OU process. Drissi [19] studies an optimal market making problem in a similar price dynamics setting.** Our model differs from these two studies because we assume that the drift of the price dynamics is characterized by an OU process. We additionally account for transient market impacts in the price dynamics.

The more mathematically formal setting of the above model is as follows. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ be a filtered probability space. The processes of the Markovian environment $\mathcal{I}_t$ and the effect Z_t caused by public news or information on the (quoted) price are defined on the space as follows:

$$\mathrm{d}\mathcal{I}_t = \left(a_t^{\mathcal{I}} - b_t^{\mathcal{I}}\mathcal{I}_t\right)\mathrm{d}t + \sigma_t^{\mathcal{I}}\mathrm{d}B_t^{\mathcal{I}}; \tag{2.13}$$

$$\mathrm{d}Z_t = \mu_t^{Z}\mathrm{d}t + \sigma_t^{Z}\mathrm{d}B_t^{Z}, \tag{2.14}$$

where $B_t^{\mathcal{I}}$ and B_t^{Z} stand for standard Brownian motions with $B_0^{\mathcal{I}} = B_0^{Z} = 0$ a.s., $\mu \in \mathbb{R}$, and $\mathcal{I}_0 = 0$ except the numerical experiment in Section 4. Eq. (2.13) is a generalized Ornstein–Uhlenbeck (OU) type process. The explicit solution is given by

$$\mathcal{I}_t = \mathcal{I}_0\mathrm{B}^{\mathcal{I}}(0,t) + \int_0^t a_s^{\mathcal{I}}\mathrm{B}^{\mathcal{I}}(s,t)\mathrm{d}s + \int_0^t \sigma_s^{\mathcal{I}}\mathrm{B}^{\mathcal{I}}(s,t)\mathrm{d}B_s^{\mathcal{I}}, \tag{2.15}$$

where for $x, y \in \mathbb{R}$,

$$\mathrm{B}^{\mathcal{I}}(x,y) \coloneqq \exp\left\{-\int_x^y b_u^{\mathcal{I}}\mathrm{d}u\right\}. \tag{2.16}$$

We also assume that the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration generated by $(B_t^{\mathcal{I}}, B_t^{Z})$, that is,

$$\mathcal{F}_t \coloneqq \sigma\left\{(B_s^{\mathcal{I}}, B_s^{Z}), 0 \le s \le t\right\}, \tag{2.17}$$

and satisfies the usual conditions. The quadratic co-variation of $B_t^{\mathcal{I}}$ and B_t^{Z} takes the following form:

$$\mathrm{d}\langle B^{\mathcal{I}}, B^{Z}\rangle_t = \rho^{\mathcal{I},Z}\mathrm{d}t. \tag{2.18}$$

This equation implies that these two processes are allowed to be correlated with each other. Note that $a_t^{\mathcal{I}}, b_t^{\mathcal{I}}, \sigma_t^{\mathcal{I}}, \mu_t^{Z}, \sigma_t^{Z}$ in the dynamics of $\mathcal{I}_t$ and Z_t are all deterministic in time t.

Remark 2.3. When $a_t^{\mathcal{I}} = 0$ for all $t \in [0, T]$, Eq. (2.13) is called a *generalized Langevin equation*. Also, in the context of the (affine) term structure modeling, a general dynamics of short rate model described by Eq. (2.13) is flexible for real data by just allowing the time-dependency for $a_t^{\mathcal{I}}$:

$$\mathrm{d}\mathcal{I}_t = \left(a_t^{\mathcal{I}} - b^{\mathcal{I}}\mathcal{I}_t\right)\mathrm{d}t + \sigma^{\mathcal{I}}\mathrm{d}B_t^{\mathcal{I}}\left(= b^{\mathcal{I}}\left(\frac{a_t^{\mathcal{I}}}{b^{\mathcal{I}}} - \mathcal{I}_t\right)\mathrm{d}t + \sigma^{\mathcal{I}}\mathrm{d}B_t^{\mathcal{I}}\right). \tag{2.19}$$

(See, e.g., Hull and White [31] and Hull [32].) Therefore, we can assume that dynamics (2.13) or (2.19) describe any factors that affect a financial asset price and have a mean-reverting property.

**The author also analyzes an optimal execution problem with Bayesian learning of the drift in the asset price.

If we assume that the information flow accessible for the large trader is carried by the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, then the executed volume Q_t of the large trader by time $t\in[0,T]$ is an $\mathcal{F}_t$-measurable (real-valued) random variable. Thus, the set of admissible execution strategies is defined as follows:

$$\mathcal{A} := \Big\{ \{Q_t\}_{0\le t\le T} \Big| \{\mathcal{F}_t\}_{0\le t\le T}\text{–adapted process with a continuously differentiable path,} \quad Q_0 = 0, Q_T = \mathfrak{Q} \Big\}. \quad (2.20)$$

According to the dynamics of the market model, it turns out that the wealth process, price dynamics, remaining execution volume, and residual effect depend on the process of the cumulative purchase (or liquidation) denoted by $Q = \{Q_s\}_{0\le s\le t}$:

$$\begin{aligned}
\mathrm{d}W_t^Q &= -\widehat{P}_t^Q \mathrm{d}Q_t = -\widehat{P}_t^Q \dot{Q}_t \mathrm{d}t = -\left(P_t^Q + \lambda_t \dot{Q}_t\right) \dot{Q}_t \mathrm{d}t;\\
\mathrm{d}P_t^Q &= \beta_t \lambda_t \dot{Q}_t \mathrm{d}t + \mathcal{I}_t \mathrm{d}t + \mathrm{d}Z_t + \mathrm{d}R_t^Q;\\
\mathrm{d}\overline{Q}_t^Q &= -\mathrm{d}Q_t = -\dot{Q}_t \mathrm{d}t;\\
\mathrm{d}R_t^Q &= -\rho R_t^Q \mathrm{d}t + \alpha_t \lambda_t \dot{Q}_t \mathrm{d}t.
\end{aligned}$$

However, to simplify the notations, we suppress the superscript Q in the above expressions representing the dependence on Q to each state variable, and simply use the ones $(W_t, P_t, \overline{Q}_t, R_t)$ defined in the previous description except the cases when we should emphasize the dependence explicitly.

3 Performance Criteria and HJB Equation

In this section, we formulate and solve an HJB equation, from which we obtain an optimal execution strategy and a value function.

3.1 Performance Criteria of the Large Trader: A Hard Constraint

We first define the state of the process at time $t\in[0,T]$. The state, denoted by $\boldsymbol{s}_t$, is a 5-tuple and is defined as

$$\boldsymbol{s}_t := (W_t, P_t, \overline{Q}_t, R_t, \mathcal{I}_t)^\top \in \mathbb{R}^5 \; (= \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}) =: \mathcal{S}. \quad (3.21)$$

As we have mentioned above, each component of the state is dependent on the process of the cumulative purchase/liquidation: $\{Q_s\}_{0\le s\le t}$.

The utility function of the large trader is assumed to take the form of a Constant Absolute Risk Aversion (CARA) von Neumann–Morgenstern (vN-M) utility function. The utility payoff (or reward) arises only from the terminal wealth at maturity:

$$g_T(\boldsymbol{s}_T) := \begin{cases} -\exp\{-\gamma W_T\}, & \text{if } \overline{Q}_T = 0;\\ -\infty, & \text{if } \overline{Q}_T \neq 0, \end{cases} \quad (3.22)$$

where $\gamma\in\mathbb{R}_{++}$ denotes the risk-aversion parameter. We define the (conditional) expected utility of the large trader at time $t\in[0,T]$ on an execution strategy $Q = \{Q_t\}_{0\le t\le T}$ as

$$V_t^Q := \mathbb{E}\left[g_T(\boldsymbol{s}_T^Q)\Big|\mathcal{F}_t\right] = \mathbb{E}\Big[-\exp\Big\{-\gamma W_T^Q\Big\}\cdot \mathbb{1}_{\{\overline{Q}_T^Q=0\}} + (-\infty)\cdot \mathbb{1}_{\{\overline{Q}_T^Q\neq 0\}}\Big|\mathcal{F}_t\Big], \quad t\in[0,T], \quad (3.23)$$

where $\mathbb{1}_A$ represents the indicator function for a measurable set (or an event) $A \in \mathcal{F}$.

Let the optimal (expected utility) value from time $t \in [0,T]$ by

$$V_t := \operatorname*{ess\,sup}_{Q \in \mathcal{A}} V_t^Q, \quad t \in [0,T]. \tag{3.24}$$

From the Markov property of the state dynamics, V_t depends on the history or information $\mathcal{F}_t$ only through the (controlled) state $\boldsymbol{s}_t \in \mathcal{S}$ and we denote this functional dependence by the optimal value function as

$$V\left[t, W_t, P_t, \overline{Q}_t, R_t, \mathcal{I}_t\right] := V_t, \quad t \in [0,T]. \tag{3.25}$$

3.2 In the Case with Target Close Order: A Soft Constraint

We consider a model with a closing price. The time framework $t \in [0,T]$ is the same in the model described above. However, we add an assumption that a large trader can execute her remaining execution volume at the terminal, $\overline{Q}_T$, with closing price P_T. We further assume that the trading at time T imposes the large trader to pay the additive cost χ_T per unit of the remaining volume.

According to the above settings, the value function at maturity becomes

$$V\left[T, \boldsymbol{s}_T\right] = -\exp\Big\{ -\gamma\big[W_T - (P_T + \chi_T\overline{Q}_T)\overline{Q}_T\big]\Big\}, \tag{3.26}$$

and the conditional expectational utility at time $t \in [0,T]$ is defined by

$$V_t^Q := \mathbb{E}\left[-\exp\Big\{ -\gamma\big[W_T - (P_T + \chi_T\overline{Q}_T)\overline{Q}_T\big]\Big\}\Big|\mathcal{F}_t\right]. \tag{3.27}$$

This formulation is similar to Cartea and Jaimungal [10, 11], which incorporate the (quadratic) cost of trading the remaining execution at maturity into the performance criteria. The term $\chi_T(\overline{Q}_T)^2$ corresponds to a quadratic transaction cost emerging at the terminal. Existing studies formulate a model with quadratic transaction costs and give it some economic meaning. For example, Gârleanu and Pedersen [28, 29] solve a portfolio selection problem with quadratic transaction costs, meaning that the buying or selling a unit of risky asset incurs an additional cost proportional to the number of buying/selling assets (which we can regard as market impact). A plausible interpretation for the formulation of Eq. (3.26) would seem to be that the large trader can execute the orders remaining at the terminal T in a dark pool. Consider the following case that a brokerage has to buy $\mathfrak{Q}$ $(\in \mathbb{R})$ volume which a client asked her to manage in a daily lit market. These insights infer that analyzing the optimal execution strategies in the soft constraint case lays the foundation of how a large trader should allocate the execution volumes (or speeds) in the lit and dark pool.

Formulating the terminal condition as Eq. (3.26) also plays an indispensable role from a mathematical viewpoint. Changing the terminal condition from Eqs. (3.22) to (3.26) makes a system of (Riccati type) ordinary differential equations (ODE) (accompanied by the derivation of optimal execution strategies) analytically tractable. The relationship between the two formulations explained above concludes this subsection.

Remark 3.1 (Relationship between Hard and Soft Constraints)**.** We can obtain the optimal execution strategy for the primary problem (the optimal execution problem with a hard constraint) by $\chi_T \to \infty$ in the following problem (the optimal execution problem with a soft constraint). If the additional trading cost χ_T goes to infinity, then the large trader will not use the dark pool and execute only in the lit pool, as shown in [Kuno et al. [36], Proposition]. This fact indicates that the large trader does not rely on a dark pool with too high trading costs (commitment fee), even if the dark pool protects against the leakage of information about her trading in the venue.

3.3 HJB Equation

From the dynamic programming principle, the optimal value function, denoted by

$$V\left[t, \boldsymbol{s}_t\right] \coloneqq V\left[t, W_t, P_t, \overline{Q}_t, R_t, \mathcal{I}_t\right] \tag{3.28}$$

with the terminal condition:

$$V\left[T, W_T, P_T, \overline{Q}_T, R_T, \mathcal{I}_T\right] = -\exp\Big\{ -\gamma\big[W_T - (P_T + \chi_T\overline{Q}_T)\overline{Q}_T\big]\Big\}, \tag{3.29}$$

satisfies the following Hamilton–Jacobi–Bellman (HJB) equation (or dynamic programming equation) for the optimal execution speed $\dot{Q}$:

$$\begin{aligned}\sup_{\dot{Q}_t\in\mathbb{R}} \Big[&\partial_t V - (P_t + \lambda_t\dot{Q}_t)\dot{Q}_t\partial_W V + \big\{ -\rho R_t + (\alpha_t+\beta_t)\lambda_t\dot{Q}_t + \mathcal{I}_t + \mu_t^Z\big\}\partial_P V - \dot{Q}_t\partial_{\overline{Q}}V \\ &+ \big(-\rho R_t + \alpha_t\lambda_t\dot{Q}_t\big)\partial_R V + (a_t^{\mathcal{I}} - b_t^{\mathcal{I}}\mathcal{I}_t)\partial_{\mathcal{I}}V \\ &+ \frac{1}{2}\left\{(\sigma_t^Z)^2\partial_{PP}V + 2\sigma_t^Z\sigma_t^{\mathcal{I}}\rho^{Z,\mathcal{I}}\partial_{P\mathcal{I}}V + (\sigma_t^{\mathcal{I}})^2\partial_{\mathcal{I}\mathcal{I}}V\right\}\Big] = 0, \quad 0 \le t < T \end{aligned} \tag{3.30}$$

if we assume that the function $V\colon [0,T]\times\mathcal{S}\to\mathbb{R}$ is in $\mathcal{C}^{1,2}$: V is continuously differentiable with respect to time and continuously twice differentiable with respect to each state variable. Rewriting this results in

$$\begin{aligned}\sup_{\dot{Q}_t\in\mathbb{R}} &\left[-\left(P_t + \lambda_t\dot{Q}_t\right)\dot{Q}_t\partial_W V + (\alpha_t+\beta_t)\lambda_t\dot{Q}_t\partial_P V - \dot{Q}_t\partial_{\overline{Q}}V + \alpha_t\lambda_t\dot{Q}_t\partial_R V\right] \\ &+ \partial_t V + \left(-\rho R_t + \mathcal{I}_t + \mu_t^Z\right)\partial_P V + (-\rho R_t)\partial_R V + (a_t^{\mathcal{I}} - b_t^{\mathcal{I}}\mathcal{I}_t)\partial_{\mathcal{I}}V \\ &+ \frac{1}{2}\left\{(\sigma_t^Z)^2\partial_{PP}V + 2\sigma_t^{\mathcal{I}}\sigma_t^Z\rho^{\mathcal{I},Z}\partial_{P\mathcal{I}}V + (\sigma_t^{\mathcal{I}})^2\partial_{\mathcal{I}\mathcal{I}}V\right\} = 0, \quad 0\le t<T. \end{aligned} \tag{3.31}$$

We can derive the optimal execution strategy and its associated value function of Eq. (3.25) explicitly by appropriately guessing the form of the optimal value function and verifying the obtained solution.

3.4 Optimal Value Function and Optimal Execution Strategy

Theorem 3.1 (Optimal Execution Strategy and Optimal Value Function)**.**
Under a set of regularity conditions:

1. The optimal execution speed at time $t\in[0,T]$, denoted as $\dot{Q}_t^*$, becomes an affine function of the remaining execution volume $\overline{Q}_t$ and the cumulative residual effect R_t and the Markovian environment $\mathcal{I}_t$ at time t:

$$\dot{Q}_t^* = f_t(\boldsymbol{s}_t) = a_t + b_t\overline{Q}_t + c_tR_t + d_t\mathcal{I}_t, \quad 0\le t\le T. \tag{3.32}$$

2. The optimal value function $V[t,\boldsymbol{s}_t] \coloneqq V[t,W_t,P_t,\overline{Q}_t,R_t,\mathcal{I}_t]$ at time $t\in[0,T]$ is represented as follows:

$$\begin{aligned}V[t,W_t,P_t,\overline{Q}_t,R_t,\mathcal{I}_t] = -\exp\Big\{-\gamma\Big[&W_t - P_t\overline{Q}_t + G_t(\overline{Q}_t)^2 + H_t\overline{Q}_t + I_t\overline{Q}_tR_t + J_t(R_t)^2 + L_tR_t \\ &+ M_t\overline{Q}_t\mathcal{I}_t + N_tR_t\mathcal{I}_t + X_t(\mathcal{I}_t)^2 + Y_t\mathcal{I}_t + K_t\Big]\Big\},\end{aligned} \tag{3.33}$$

where a_t, b_t, c_t, d_t; $G_t, H_t, I_t, J_t, L_t, M_t, N_t, X_t, Y_t, K_t$ for $t \in [0, T]$ are deterministic functions of time t which are dependent on the model parameters, and these are assumed to exist as a unique solution of a simultaneous system of ordinary differential equations derived in the proof.

Proof. See Appendix A. □

From Theorem 3.1, the optimal execution speed $\dot{Q}_t^*$ depends on the state $\boldsymbol{s}_t = (W_t, P_t, \overline{Q}_t, R_t, \mathcal{I}_t)$ of the controlled process only through the remaining execution volume $\overline{Q}_t$, the cumulative residual effect R_t, and the Markovian environment $\mathcal{I}_t$, not through the wealth W_t or market price P_t. In addition, by definition of Markovian environment, the optimal execution volume $\dot{Q}_t$ includes a nondeterministic term (random variable), the optimal execution strategy being a stochastic one. Then, the following corollary immediately holds.

Corollary 3.1. If the Markovian environment $\mathcal{I}_t$ for all $t \in [0, T]$ are deterministic, the optimal execution speed $\dot{Q}_t^*$ at time $t \in [0, T]$ also becomes a deterministic function of time in a class of the static and non-randomized execution strategy.

The optimal execution expressed by Eq. (3.32) implies that the optimal trading strategy is adaptive to different values of the Markovian environment. Lehalle and Neuman [38] show that when considering the transient market impact, the optimal trading strategy is time-consistent so that no different values of $\mathcal{I}_t$ do not change the optimal strategy. Contrary to that, the model considered in this paper allows us to derive the optimal execution strategy for a transient market impact model with an effect of a Markovian environment on price dynamics.

Academic literature that focus on the execution problems of a single large trader derive an optimal execution strategy in a deterministic class. For example, Schied et al. [54] consider an execution problem of a large trader in a continuous-time setting and derive an optimal execution strategy in a deterministic class. However, we can confirm from our analysis that the optimal execution strategy does not always become deterministic if a stochastic exogenous factor affects the financial asset price.

The optimal execution speed satisfies the following functional equation (for each $\omega \in \Omega$):

$$\begin{aligned}\dot{Q}_t^* &= a_t + b_t\overline{Q}_t + c_tR_t + d_t\mathcal{I}_t \\ &= a_t + b_t\int_0^t \dot{Q}_u^*\mathrm{d}u + c_t\left(\mathrm{e}^{-\rho t}R_0 + \int_0^t \alpha_u\lambda_u\mathrm{e}^{-\rho(t-u)}\dot{Q}_u^*\mathrm{d}u\right) + d_t\mathcal{I}_t \\ &=: g_t + \int_0^t K(t,u)\dot{Q}_u^*\mathrm{d}u, \end{aligned} \tag{3.34}$$

where

$$g_t \coloneqq a_t + c_t\mathrm{e}^{-\rho t}R_0 + d_t\mathcal{I}_t, \tag{3.35}$$

$$K(t,u) \coloneqq b_t + \alpha_u\lambda_u\mathrm{e}^{-\rho(t-u)}c_t. \tag{3.36}$$

Eq. (3.34) is a *Volterra integral equation of the second kind.* Since the function g and the integral kernel K are continuous, the equation has a unique solution (Yosida [61]).

3.5 Special case: No transient market impact

As a special case, we examine an optimal execution strategy for the model described so far without transient market impacts. If α_t is zero, the residual effect of past market impacts becomes zero since we assume that $R_0 = 0$. Moreover, in the case that the residual speed goes to infinity, the residual effect of past market impacts becomes zero for all $t \in [0, T]$ since from Eq. (2.5),

$$\lim_{\rho \to \infty} R_t = \lim_{\rho \to \infty} \left(\mathrm{e}^{-\rho t} R_0 + \int_0^t \mathrm{e}^{-\rho(t-s)} \alpha_s \lambda_s \dot{Q}_s \mathrm{d}t \right) = 0, \tag{3.37}$$

holds for all $t \in [0, T]$. In both cases, the market price dynamics become

$$\mathrm{d}P_t = \beta_t \lambda_t \dot{Q}_t \mathrm{d}t + \mathcal{I}_t \mathrm{d}t + \mathrm{d}Z_t, \tag{3.38}$$

that is, we have a permanent impact model with Markovian environment. In this case, our model partially includes the so-called "target zone models," in which the price of an asset traded by a large trader has one or more barriers and is reflected at the barriers (e.g., Krugman [34]; Neuman and Schied [42]; Belak et al. [6]). To be precise, if the large trader executes no orders, the price is capped by the mean-reverting process $\mathcal{I}_t$. This model is motivated by the fact that monetary authorities (e.g., central banks) keep a currency exchange rate above some threshold.

In the case without transient market impacts, by setting the state variables as $\boldsymbol{s}_t := (W_t, P_t, \overline{Q}_t, \mathcal{I}_t)$ we have the following results:

Corollary 3.2. Under the financial market considered in Section 2 without transient market impacts, the optimal execution speed becomes an *affine* function of the current remained execution volume and Markovian environment:

$$\dot{Q}_t^* = \tilde{a}_t + \tilde{b}_t \overline{Q}_t + \tilde{d}_t \mathcal{I}_t, \tag{3.39}$$

where $\tilde{a}_t, \tilde{b}_t$, and $\tilde{d}_t$ are all deterministic functions of time. Then, the inventory at time t for $0 \leq t \leq T$ has an explicit form as follows:

$$\overline{Q}_t = \mathrm{B}^b(0,t)\mathfrak{Q} + \int_0^t \mathrm{B}^b(u,t)\tilde{a}_u \mathrm{d}u + \int_0^t \mathrm{B}^b(u,t)\tilde{d}_u \mathcal{I}_u \mathrm{d}u, \tag{3.40}$$

where for $x, y \in \mathbb{R}$,

$$\mathrm{B}^b(x,y) := -\exp\left\{ -\int_x^y \tilde{b}_s \mathrm{d}s \right\}. \tag{3.41}$$

In this case, the ansatz of the optimal value function is given by

$$V[t, \boldsymbol{s}_t] = -\exp\Big\{ -\gamma\Big[W_t - P_t\overline{Q}_t + \tilde{G}_t(\overline{Q}_t)^2 + \tilde{H}_t\overline{Q}_t + \tilde{M}_t\overline{Q}_t\mathcal{I}_t + \tilde{X}_t(\mathcal{I}_t)^2 + \tilde{Y}_t\mathcal{I}_t + \tilde{K}_t \Big]\Big\}. \tag{3.42}$$

Proof. See Appendix B. □

By stochastic Fubini theorem,

$$\begin{aligned}
\int_0^t \mathrm{B}^b(u,t)\tilde{d}_u\mathcal{I}_u\mathrm{d}u &= \int_0^t \mathrm{B}^b(u,t)\tilde{d}_u \left(\mathcal{I}_0\mathrm{B}^{\mathcal{I}}(0,u) + \int_0^u a_l^{\mathcal{I}}\mathrm{B}^{\mathcal{I}}(l,u)\mathrm{d}l + \int_0^u \sigma_l^{\mathcal{I}}\mathrm{B}^{\mathcal{I}}(l,u)\mathrm{d}B_l^{\mathcal{I}} \right) \mathrm{d}u \\
&= \mathcal{I}_0 \int_0^t \mathrm{B}^{\mathcal{I}}(0,u)\mathrm{B}^b(u,t)\tilde{d}_u\mathrm{d}u + \int_0^t a_l^{\mathcal{I}} \left(\int_l^t \mathrm{B}^{\mathcal{I}}(l,u)\mathrm{B}^b(u,t)\tilde{d}_u\mathrm{d}u \right) \mathrm{d}l \\
&\quad + \int_0^t \sigma_l^{\mathcal{I}} \left(\int_l^t \mathrm{B}^{\mathcal{I}}(l,u)\mathrm{B}^b(u,t)\tilde{d}_u\mathrm{d}u \right) \mathrm{d}B_l^{\mathcal{I}},
\end{aligned} \tag{3.43}$$

holds. Thus, we can rewrite Eq. (3.40) as

$$\overline{Q}_t = \mathrm{B}(0,t)\mathfrak{Q} + \int_0^t \mathfrak{X}_l \mathrm{d}l + \int_0^t \sigma_l^{\mathcal{I}} \left(\int_l^t \mathrm{B}^{\mathcal{I}}(l,u)\mathrm{B}^b(l,t)\tilde{d}_u \mathrm{d}u \right) \mathrm{d}B_l^{\mathcal{I}}, \tag{3.44}$$

where

$$\mathfrak{X}_l := \mathrm{B}^b(l,t)\tilde{a}_l + \mathcal{I}_0 \mathrm{B}^{\mathcal{I}}(0,l)\mathrm{B}^b(l,t)\tilde{d}_l + a_l^{\mathcal{I}} \left(\int_l^t \mathrm{B}^{\mathcal{I}}(l,u)\mathrm{B}^b(l,t)\tilde{d}_u \mathrm{d}u \right). \tag{3.45}$$

Eq. (3.40) implies that the process of the remaining execution volume becomes a Gaussian (Ito) process under the optimal execution strategy.

From these results, the following proposition holds:

Proposition 3.1. The mean (or average) inventory at time t conditioned on $\mathcal{F}_s$ for $0 \le s \le t \le T$, captured by $\mathbb{E}\left[\overline{Q}_t \middle| \mathcal{F}_s\right]$, becomes

$$\mathbb{E}\left[\overline{Q}_t \middle| \mathcal{F}_s\right] = \mathrm{e}^{-\int_s^t b_l \mathrm{d}l} \overline{Q}_s + \mathfrak{y}_{s,t}\mathcal{I}_s + \mathfrak{z}_{s,t}, \tag{3.46}$$

where

$$\mathfrak{y}_{s,t} := \int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l} \mathrm{e}^{-\int_s^u b_m^{\mathcal{I}} \mathrm{d}m} \tilde{d}_u \mathrm{d}u; \tag{3.47}$$

$$\mathfrak{z}_{s,t} := \int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l} \tilde{a}_u \mathrm{d}l + \int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l} \left(\int_s^t a_m^{\mathcal{I}} \mathrm{e}^{-\int_m^t b_k^{\mathcal{I}} \mathrm{d}k} \mathrm{d}m \right) \tilde{d}_u \mathrm{d}u. \tag{3.48}$$

Also, the conditional covariance of the inventory at time t conditioned on $\mathcal{F}_s$ for $0 \le s \le t \le v \le T$, captured by $\mathrm{Cov}[\overline{Q}_t, \overline{Q}_v | \mathcal{F}_s]$, is given by

$$\mathrm{Cov}[\overline{Q}_t, \overline{Q}_v | \mathcal{F}_s] = \int_s^t \mathfrak{A}(l,t)\mathfrak{A}(l,v)\mathrm{d}l, \tag{3.49}$$

where

$$\mathfrak{A}(l,t) := \sigma_l^{\mathcal{I}} \int_l^t \mathrm{B}^{\mathcal{I}}(l,u)\mathrm{B}^b(u,t)\tilde{d}_u \mathrm{d}u. \tag{3.50}$$

Proof. See Appendix C. □

Eq. (3.46) enables us to predict the future inventory given the current state. In particular, we have

$$\mathbb{E}[\overline{Q}_T | \mathcal{F}_s] = \mathrm{e}^{-\int_s^T b_l \mathrm{d}l} \overline{Q}_s + \mathfrak{y}_{s,T}\mathcal{I}_s + \mathfrak{z}_{s,T}, \tag{3.51}$$

for $0 \le s \le T$, indicating that we can predict the inventory at the end of the trading horizon through the information about the current state at time s ($\overline{Q}_s$ and $\mathcal{I}_s$). Also, the conditional variance for the future inventory is quantified as

$$\mathbb{V}[\overline{Q}_T | \mathcal{F}_s] = \int_s^T \mathfrak{A}(l,t)^2 \mathrm{d}l = \int_s^T (\sigma_l^{\mathcal{I}})^2 \left(\int_l^T \mathrm{B}^{\mathcal{I}}(l,u)\mathrm{B}^b(u,t)\tilde{d}_u \mathrm{d}u \right)^2 \mathrm{d}l. \tag{3.52}$$

Remark 3.2. If $b^{\mathcal{I}} = 0$, by a similar calculation conducted in Appendix C we obtain

$$\mathbb{E}\left[\overline{Q}_t \middle| \mathcal{F}_s\right] = \mathrm{e}^{-\int_s^t b_l \mathrm{d}l} \overline{Q}_s + \tilde{\mathfrak{y}}_{s,t}\mathcal{I}_s + \tilde{\mathfrak{z}}_{s,t}, \tag{3.53}$$

where

$$\tilde{\mathfrak{y}}_{s,t} := \int_s^t \mathrm{e}^{-\int_u^t b_l \mathrm{d}l} d_u \mathrm{d}u; \tag{3.54}$$

$$\tilde{\mathfrak{z}}_{s,t} := \int_s^t \mathrm{e}^{-\int_u^t b_l \mathrm{d}l} a_u \mathrm{d}u + a^{\mathcal{I}} \int_s^t \mathrm{e}^{-\int_u^t b_l \mathrm{d}l} d_u (u-s) \mathrm{d}u. \tag{3.55}$$

Table 1: Benchmark values for parameters.

Parameters	μ_t^Z	σ_t^Z	$\sigma_t^{\mathcal{I}}$	$\rho^{\mathcal{I},Z}$	α_t	β_t	λ_t	$a_t^{\mathcal{I}}$	$b_t^{\mathcal{I}}$	ρ	γ	χ_T	T
Benchmark values	0	0.1	0.1	0	0.5	0.5	0.001	0.5	0.5	0.1	0.001	0.01	1

4 Numerical examples

Theorem 3.1 characterizes how Markovian environment affects the large trader's optimal execution strategy. We can identify the effect of Markovian environment on the optimal execution strategy as environmental uncertainty. In this section, we examine how a large trader facing environmental uncertainty executes her orders in a financial market by comparative statics and simulation-based analyses.

We assume the time homogeneity of the time-dependent parameters. The benchmark values for parameters in this section are shown in Table 1. It should be noted that the initial value of Markovian environment is set as zero in the rest of this paper except Subsubsection 4.3.2.

Remark 4.1 (Implications of parameters)**.** We set $\alpha_t = \beta_t = 0.5$ as the benchmark values. These parameter values imply that the instantaneous market impact caused by the large traders at time $t \in [0, T]$ is just half decomposed into temporary and permanent market impacts. In addition, under the benchmark parameters setting, the dynamics of Markovian environment are given by

$$\mathrm{d}\mathcal{I}_t = (0.5 - 0.5\mathcal{I}_t)\mathrm{d}t + 0.1\mathrm{d}B_t^{\mathcal{I}} = 0.5(1 - \mathcal{I}_t)\mathrm{d}t + 0.1\mathrm{d}B_t^{\mathcal{I}}, \tag{4.56}$$

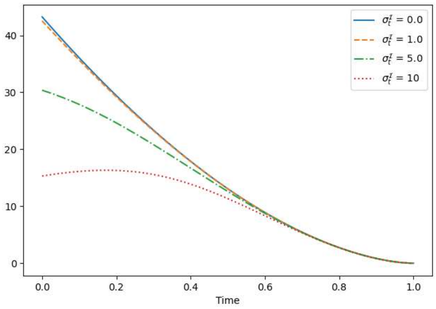
which implies that Markovian environment tends to push up the market price. Also, the parameter $\sigma_t^{\mathcal{I}}$ denotes how high the environmental uncertainty is.

4.1 Comparative statics for coefficients

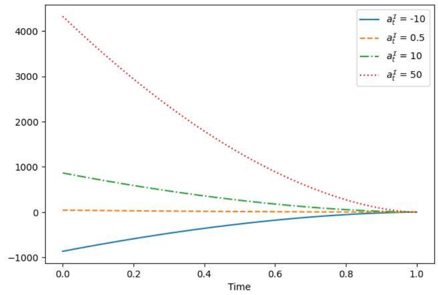
We first conduct comparative statics for the coefficients of state variables in the optimal execution speed (i.e., $a_t, b_t, c_t,\ d_t$). In particular, we focus on how these coefficients are influenced by the parameters defining Markovian environment ($a^{\mathcal{I}}, b^{\mathcal{I}}$, and $\sigma^{\mathcal{I}}$) and execution cost at the terminal (χ_T). This analysis allows us to see how a large trader incorporates Markovian environment into the optimal execution strategy. Figures 1 to 4 illustrate the comparative statics for the coefficients with different values of the parameters.

From Figs. 1 to 4, we numerically find that the coefficient of residual effect (c_t) seems to be negative. We can interpret this result from the following insights: The residual effect is the discounted sum of the past market impacts caused by the large trader's order submissions. Thus, the negativity of c_t implies that a large trader reduces the execution speed in accordance with the degree to which she has incurred the market impacts. This result reflects the following viewpoint: The more the large trader has caused market impacts, the more unpreferable the future price movement is. Thus, the large trader slows down the execution if she poses large orders before some trading time. There are many other insights observed from Figs. 1 to 4 as follows:

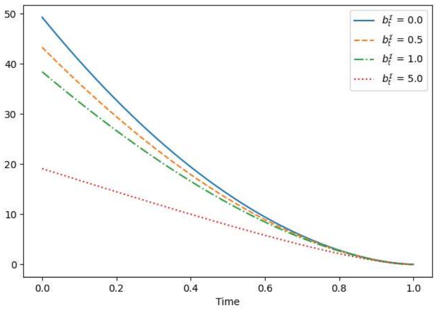
- The part corresponding to the constant term (a_t) decrease as environmental uncertainty increases. This result infers that the large trader tends to avoid the loss from price fluctuation driven by environmental uncertainty. In addition, as $a^{\mathcal{I}}$ (and thus mean-reversion level) increases, the coefficient rises more, and the feature is remarkable at the beginning of the trading window. When $a^{\mathcal{I}}$ is large, Markovian environment pushes up the market price, which is unpreferable for the large trader with positive initial inventory, and vice



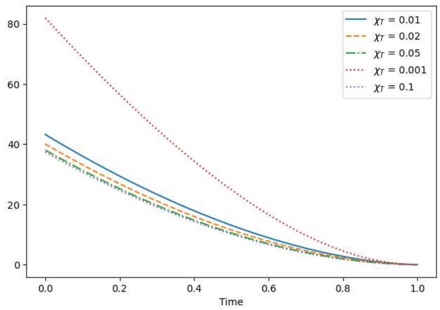
(a) Effect of $\sigma^{\mathcal{I}}$



(b) Effect of $a^{\mathcal{I}}$

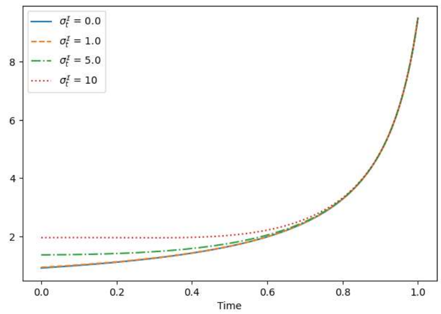


(c) Effect of $b^{\mathcal{I}}$

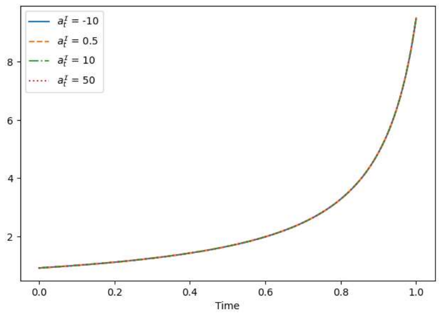


(d) Effect of χ_T

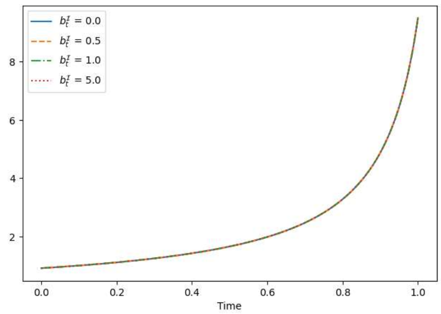
Figure 1: Comparative Statics for a_t



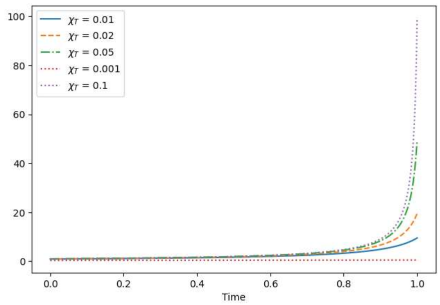
(a) Effect of $\sigma^{\mathcal{I}}$



(b) Effect of $a^{\mathcal{I}}$

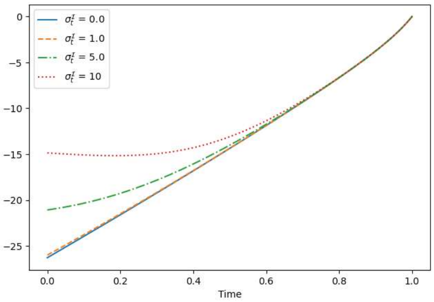


(c) Effect of $b^{\mathcal{I}}$

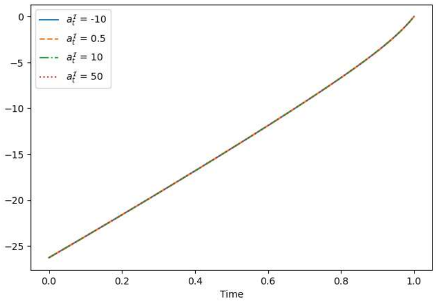


(d) Effect of χ_T

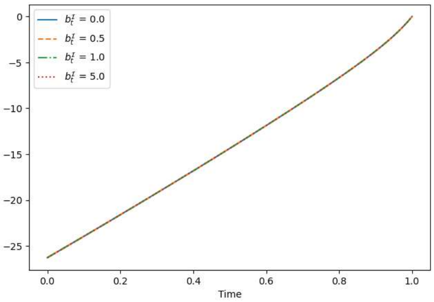
Figure 2: Comparative Statics for b_t



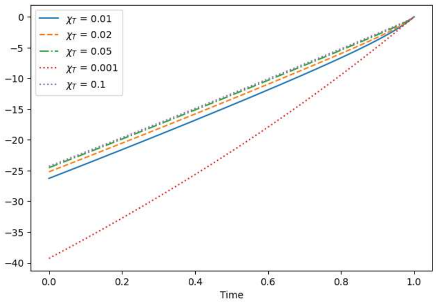
(a) Effect of $\sigma^{\mathcal{I}}$



(b) Effect of $a^{\mathcal{I}}$

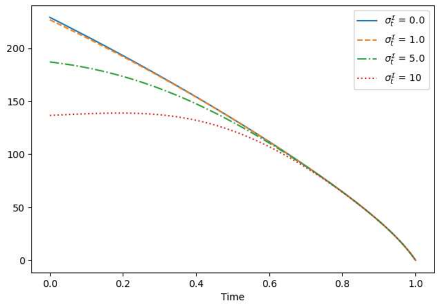


(c) Effect of $b^{\mathcal{I}}$

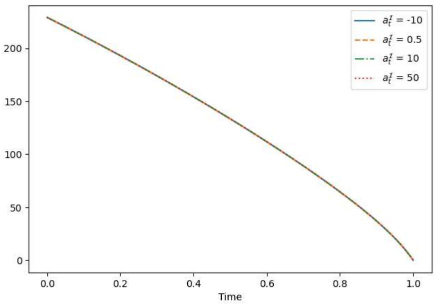


(d) Effect of χ_T

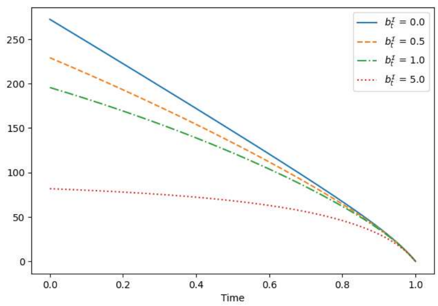
Figure 3: Comparative Statics for c_t



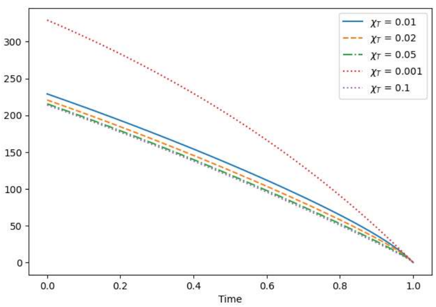
(a) Effect of $\sigma^{\mathcal{I}}$



(b) Effect of $a^{\mathcal{I}}$



(c) Effect of $b^{\mathcal{I}}$



(d) Effect of χ_T

Figure 4: Comparative Statics for d_t

versa. Therefore, the large trader with positive initial inventory should liquidate her orders at the beginning before the market (execution) price becomes high.

- The coefficient of the remaining execution volume (b_t) increases in the first half of the trading window as environmental uncertainty increases. The large trader is risk-averse and avoids future fluctuation caused by environmental uncertainty. Thus, this result implies that the large trader executes fast when environmental uncertainty increases. On the contrary, the coefficient decreases at the end of the trading window as the terminal cost (per unit) increases. The large trader executes faster at the end of the trading window as the terminal cost increases more to reduce the terminal payment of additional costs.

- The coefficient of the residual effect of past market impacts (c_t) increases in the first half of the trading window as environmental uncertainty increases. Combining this result with the negativity of the coefficient, the large trader reduces the degree of incorporating the residual effect of past market impacts into optimal execution strategy and facilitates her execution to avoid environmental uncertainty. In addition, high terminal cost leads to the increase of the coefficient. With high terminal costs, the large trader is keen to execute orders fast not to remain some volumes that she must unwind at maturity.

- The coefficient of Markovian environment (d_t) increases in the first half of the trading window as environmental uncertainty increases. In addition, as the terminal cost is higher, the coefficient decreases more. These results imply that a risk-averse large trader avoids incorporating Markovian environment into the optimal execution strategy under a market with high environmental uncertainty or high terminal cost. Also, the faster the mean-reversion speed, the less the coefficient becomes.

Remark 4.2 (Effect of mean-reversion level of markovian environment)**.** The parameter $a^{\mathcal{I}}$ and thus the mean-reversion level of Markovian environment does not influence the coefficients b_t, c_t, d_t. We can mathematically confirm this result from the system of ODEs (Eqs. (A.72) to (A.81)).

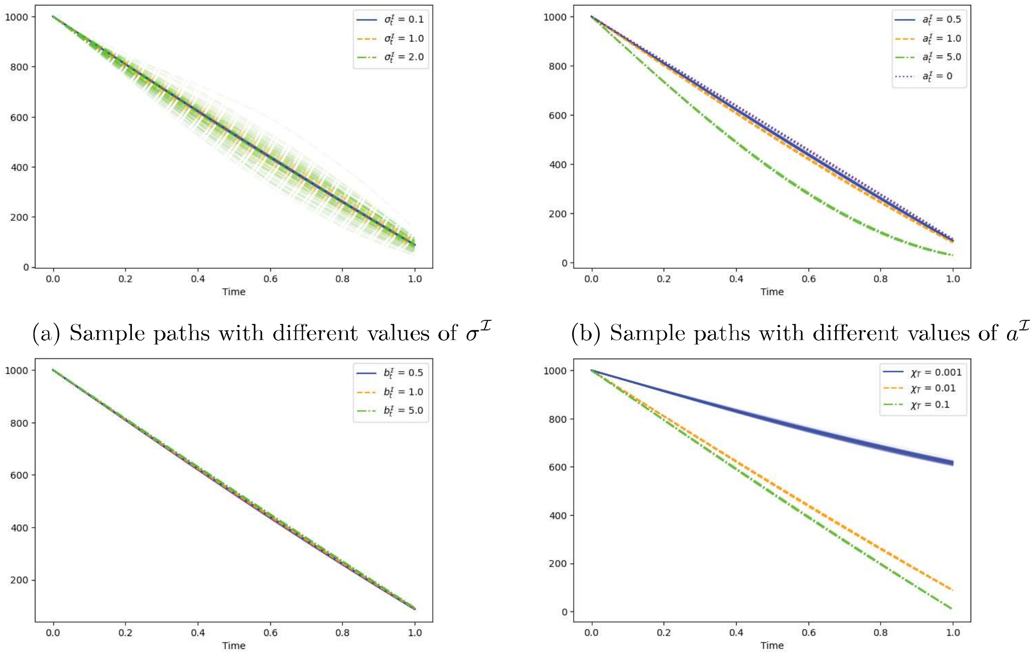
Remark 4.3 (Effect of mean-reversion speed of markovian environment)**.** The coefficient d_t may *not* become zero even when parameter $b^{\mathcal{I}}$ is zero. This result implies that our continuous-time model is *not* the limit version of the discrete-time analog since the coefficient d_t becomes zero if $b^{\mathcal{I}}$ is zero.

4.2 Simulation of optimal execution

We next examine how optimal execution strategy differs depending on the market situation as well as the large trader's initial inventory. As stated in Subsection 3.4, the optimal execution speed lies in a class of stochastic strategies. Therefore, we run the simulation for $N = 100$ sample pathes to obtain 100 realized optimal execution strategy captured by $\overline{Q}_t(k)$ for $t \in [0, T]$ and $k \in \{1, \ldots, N\}$. For each $k \in \{1, \ldots, N\}$, $\overline{Q}_t(k)$ represents the kth sample path of the large trader's remaining execution volume for $t \in [0, T]$. The reason that we only show 100 sample paths is that more sample paths make the realization of optimal execution strategies unidentifiable.

4.2.1 Sample paths of $\overline{Q}_t$

Our first objective in this subsection is to identify how variations in parameters affect the optimal execution strategy and whether a kind of statistical arbitrage exists or not. We first show the



(a) Sample paths with different values of $\sigma^{\mathcal{I}}$

(b) Sample paths with different values of $a^{\mathcal{I}}$

(c) Sample paths with different values of $b^{\mathcal{I}}$

(d) Sample paths with different values of χ_T

Figure 5: Comparative Statics for 100 sample paths of $\overline{Q}_t$ with $\mathfrak{Q} = 1000$

100 paths of optimal execution strategy for large initial inventory $\mathfrak{Q} = 1,000$ with different values of parameters: $\sigma^{\mathcal{I}}, a^{\mathcal{I}}, b^{\mathcal{I}}$, and $\chi_T^{\mathcal{I}}$.

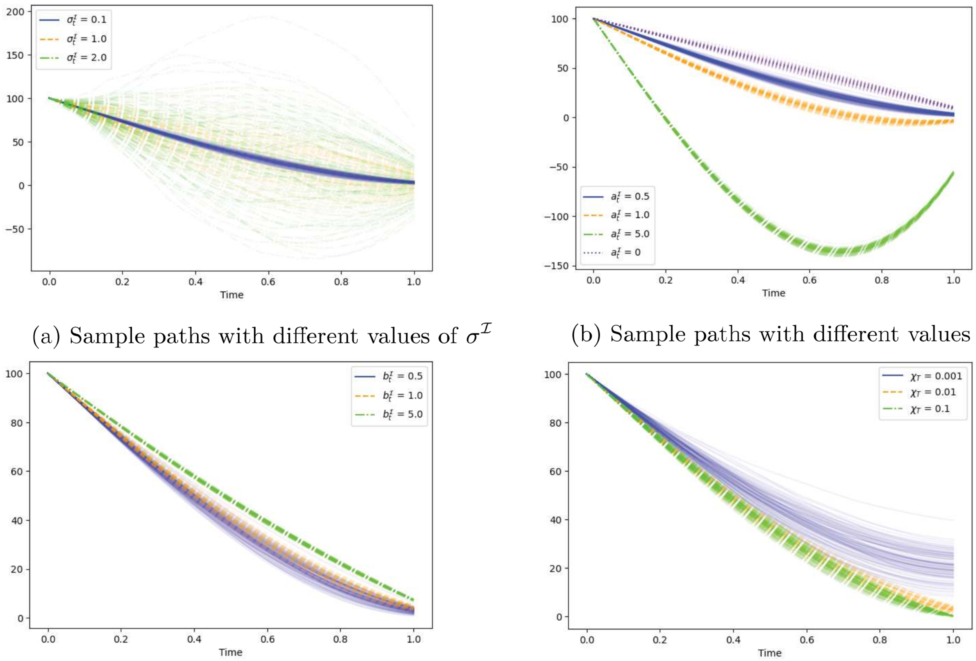
Fig. 5 illustrates the 100 sample paths of optimal execution strategy with $\mathfrak{Q} = 1,000$. Panel (a) in Fig. 5 shows that high uncertainty driven by Markovian environment induces the large trader to change the strategy depending on the realization of Markovian environment. This result demonstrates that a risk-averse large trader chooses an optimal execution strategy adapted to Markovian environment to take environmental uncertainty into account. The effect of $a^{\mathcal{I}}$ is shown in Panel (b), suggesting that as the value of $a^{\mathcal{I}}$ increases, the large trader executes faster. The mean-reverting level of the OU process describing Markovian environment is given by $a^{\mathcal{I}}/b^{\mathcal{I}}$ so that an increase of $a^{\mathcal{I}}$ infers that of future risky asset price (in the sense of expectation). Thus, a risk-averse large trader executes faster when $a^{\mathcal{I}}$ is large Moreover, we observe from anel (d) in Fig. 5 that the higher the terminal cost per execution, the faster the large trader executes his/her orders over the trading horizon (and the less at the end of the trading horizon). This result confirms that a risk-averse large trader tends to avoid the terminal liquidation if the terminal cost per execution is high.

4.2.2 Utility-based statistical arbitrage

Alfonsi et al. [1] and Palmari et al. [51] consider the following concept of transaction-triggerd price manipulation in a cost minimization framework.

Definition 4.1 (Transaction-triggerd price manipulation (Alfonsi et al. [1] and Palmari et al. [51])**.** A continuous trade execution model admits *transaction-triggerd price manipulation* if there exists an initial inventory $\mathfrak{Q} \in \mathbb{R}$ and a corresponding admissible strategy $\hat{Q} = \{\hat{Q}_t\}_{0 \le t \le T} \in \mathcal{A}$ such that

$$\mathfrak{C}(\hat{Q}) > \sup_{Q \in \mathcal{A}} \Big\{ \mathfrak{C}(Q) \,\Big|\, \mathrm{sign}(\dot{Q}_t) = \mathrm{sign}(\mathfrak{Q}), \forall t \in [0,T] \Big\}, \tag{4.57}$$



(a) Sample paths with different values of $\sigma^{\mathcal{I}}$

(b) Sample paths with different values of $a^{\mathcal{I}}$

(c) Sample paths with different values of $b^{\mathcal{I}}$

(d) Sample paths with different values of χ_T

Figure 6: Comparative Statics for 100 sample paths of $\overline{Q}_t$ with $\mathfrak{Q} = 100$

where $\mathcal{A}$ is the set of admissible strategies that satisfies $\mathbb{E}[\int_0^T \dot{Q}_t^2 \mathrm{d}t] < \infty$, and $\mathfrak{C}(Q)$ for $Q \in \mathcal{A}$ is an execution cost driven by the strategy $Q \in \mathcal{A}$ defined as

$$\mathfrak{C}(Q) := \mathbb{E}\left[W_T^Q - \mathfrak{Q} P_0 \middle| \mathcal{F}_0 \right]. \tag{4.58}$$

Motivated by this notion of price manipulation, we define a concept of utility-based statistical arbitrage.

Definition 4.2 (Utility-based statistical arbitrage)**.** A financial market with a stochastic process $\{\mathcal{I}_t\}_{0\le t\le T}$ expressing Markovian environment admits a *utility-based statistical arbitrage* for a large trader with CARA-type utility if there exists an initial inventory $\mathfrak{Q} \in \mathbb{R}$, and a trade execution strategy $\hat{Q} := \{Q_t\}_{0\le t\le T}$ such that

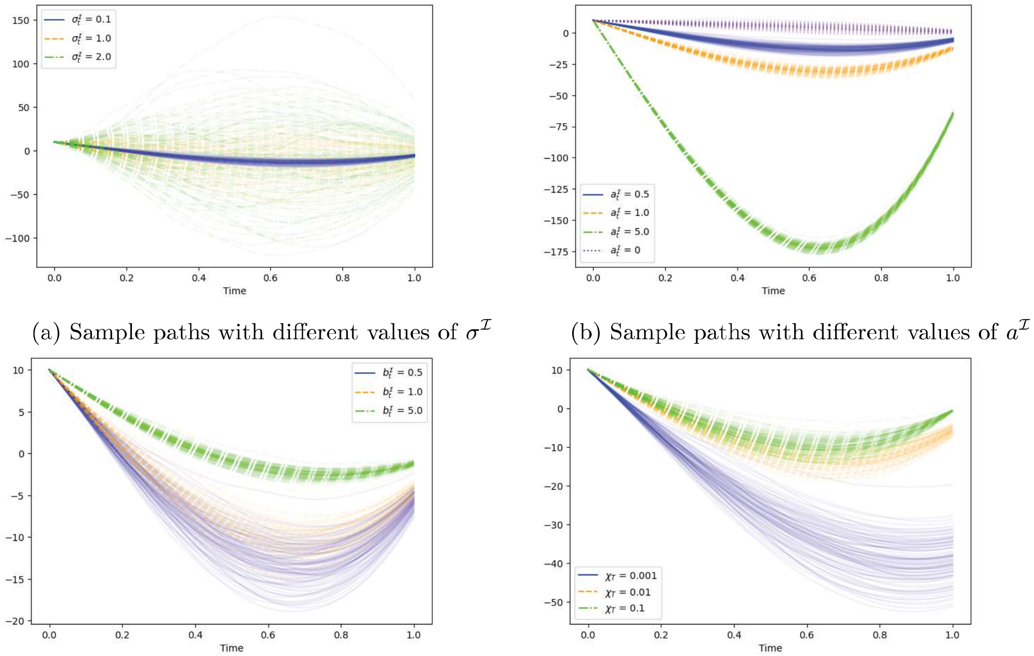
$$\mathbb{E}\left[-\exp\left\{ -\gamma W_T^{\hat{Q}} \right\} \middle| \mathcal{F}_0 \right] > \sup_{Q\in\tilde{\mathcal{A}}} \left\{ \mathbb{E}\left[-\exp\left\{ -\gamma W_T^{Q} \right\} \middle| \mathcal{F}_0 \right] \right\}, \tag{4.59}$$

where $\tilde{\mathcal{A}}$ ($\subset \mathcal{A}$) is the (sub)set of admissible strategies defined as

$$\tilde{\mathcal{A}} := \Big\{ \{Q_t\}_{0\le t\le T} : \Big| \{\mathcal{F}_t\}_{0\le t\le T}\text{–adapted process with a continuously differentiable path,} \\ Q_0 = 0, Q_T = \mathfrak{Q}, \mathrm{sign}(\dot{Q}_t) = \mathrm{sign}(\mathfrak{Q}), \forall t \in [0,T] \Big\}. \tag{4.60}$$

In this subsection, we examine whether there exists a utility-based statistical arbitrage in a financial market by simulation-based numerical experiments.

Figure 6 illustrates 100 sample paths of simulated remaining execution volume under the optimal execution strategy. Panels (a) and (b) in Figure 6 show that a financial market admits a utility-based statistical arbitrage for high environmental uncertainty and high mean-reversion



(a) Sample paths with different values of $\sigma^{\mathcal{I}}$

(b) Sample paths with different values of $a^{\mathcal{I}}$

(c) Sample paths with different values of $b^{\mathcal{I}}$

(d) Sample paths with different values of χ_T

Figure 7: Comparative Statics for 100 sample paths of $\overline{Q}_t$ with $\mathfrak{Q} = 10$

level of Markovian environment. On the contrary, from Figure (c) and (d) in Figure 6 we find that no statistical arbitrages seemingly arise even if the mean-reversion speed and terminal cost per unit are high.

We also observe from Figure 7 that the possibility of utility-based statistical arbitrage is high when the initial inventory is close to zero ($\mathfrak{Q} = 10$).

4.3 Optimal execution speed with zero initial inventory

We finally examine a sample path of optimal execution strategy with different values of parameters for $\mathfrak{Q} = 0$. In this case, we call the trading strategy as *round trip trading*.

4.3.1 Existence of round trip trading

Figure 8 illustrates that there exists a *round trip trading* that increases the expected utility (or Eq. (3.31)). Let $\mathbf{0}$ denote the zero-trade schedule in which $\dot{Q}_t = 0$ for all $t \in [0, T]$.[††] Then, if $V^{\mathbf{0}}[\boldsymbol{s}_0] < V^{Q}[\boldsymbol{s}_0]$ holds for some round trip trading strategy $Q^* := \{\dot{Q}^*_t\}_{0 \le t \le T}$ such that $\mathfrak{Q} = 0$ and

$$\int_0^T \dot{Q}^*_s \mathrm{d}s + \overline{Q}_T^{Q^*} = 0, \tag{4.61}$$

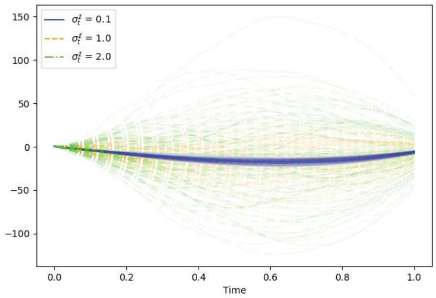
we have

$$-\exp\{-\gamma W_0\} = V^{\mathbf{0}}[\boldsymbol{s}_0] < V^{Q}[\boldsymbol{s}_0] = \mathbb{E}[-\exp\{-\gamma W_T\}|\mathcal{F}_0] \le -\exp\{-\gamma \mathbb{E}[W_T|\mathcal{F}_0]\}, \tag{4.62}$$

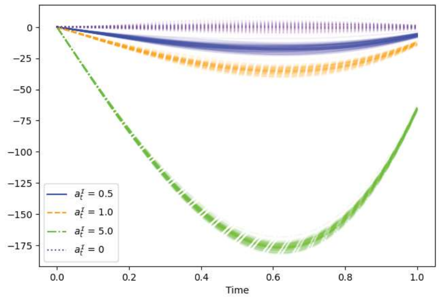
[††]Under this condition, we have

$$\int_0^T \dot{Q}_s \mathrm{d}s = 0,$$

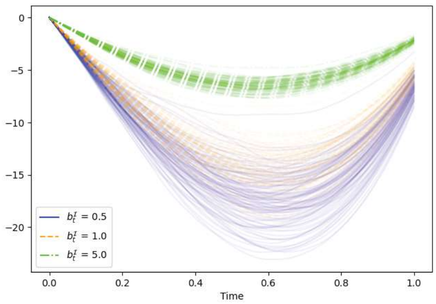
and thus $\overline{Q}_T = 0$.



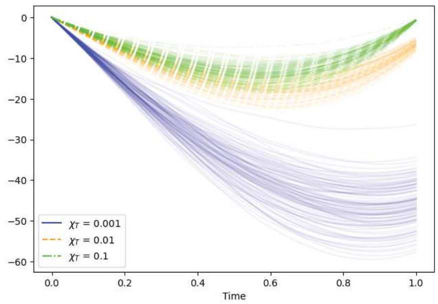
(a) Sample paths with different values of $\sigma^{\mathcal{I}}$



(b) Sample paths with different values of $a^{\mathcal{I}}$



(c) Sample paths with different values of $b^{\mathcal{I}}$



(d) Sample paths with different values of χ_T

Figure 8: Comparative Statics for 100 sample paths of $\overline{Q}_t$ with $\mathfrak{Q} = 0$

by conditional Jensen's inequality, which implies that $W_0 < \mathbb{E}[W_T|\mathcal{F}_0]$.

4.3.2 Market situation

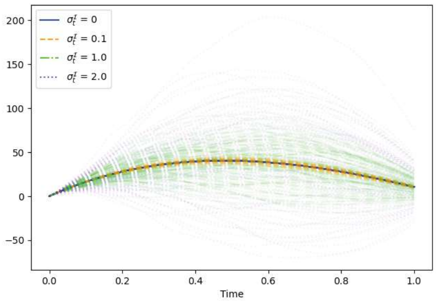
Large traders often observe that financial markets are unstable and financial asset prices fluctuate in a short time window. Typical examples are "Flash Clash" (Kirilenko et al. [33]) or "hot-potate game" (CFTC-SEC [13]; Schied and Zhang [55]). Markovian environment can express market instability by changing parameter values. To examine whether a large trader has a statistical arbitrage in such a (volatile) market, in this section we consider the situation that the initial value of Markovian environment is negative but the mean-reversion level is positive.

From Figs 8 and 9, we observe that the sign of optimal trade execution speed for $\mathcal{I}_0 = -1$ is different from the one for $\mathcal{I}_0 = 0$ (except Panel (a)). In addition, Panel (b) in Fig. 9 shows that the sign of the optimal trade execution speed differs depending on the mean-reversion level of Markovian environment.

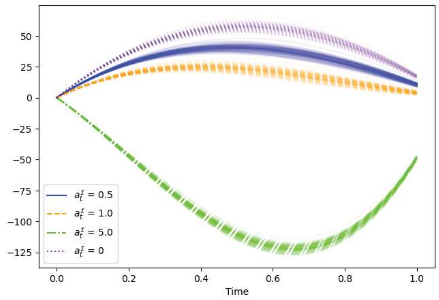
4.3.3 In the case of zero mean-reversion

The interesting case is when $a^{\mathcal{I}}$ is zero but $b^{\mathcal{I}}$ and $\sigma^{\mathcal{I}}$ are different from zero. In this case, Markovian environment is expressed by a Langevin equation. In particular, we are interested in whether there are manipulations in the following cases: (i) when environmental uncertainty is high (i.e., when the value of $\sigma^{\mathcal{I}}$ is large); (ii) when the mean-reversion speed of Markovian environment is fast (i.e., when the value of $b^{\mathcal{I}}$ is large).

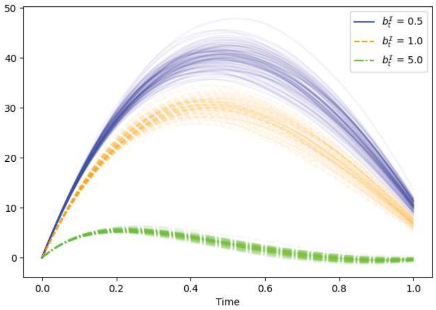
Figure 10 illustrates 100 sample paths for different values of $\sigma^{\mathcal{I}}$ with $\mathfrak{Q} = 0$, $a^{\mathcal{I}} = 0$ and $b^{\mathcal{I}} = 0.5$ fixed. As shown, the increase in environmental uncertainty raises the possibility that a round trip trade starting from zero initial inventory may increase the large trader's expected utility (and thus the expected wealth).



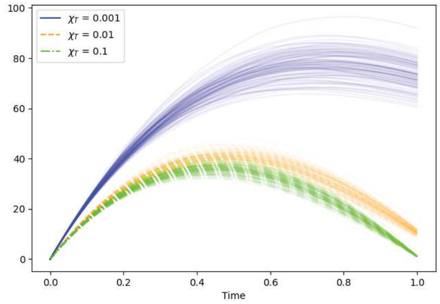
(a) Sample paths with different values of $\sigma^{\mathcal{I}}$



(b) Sample paths with different values of $a^{\mathcal{I}}$



(c) Sample paths with different values of $b^{\mathcal{I}}$



(d) Sample paths with different values of χ_T

Figure 9: Comparative Statics for 100 sample paths of $\overline{Q}_t$ with $\mathfrak{Q} = 0$

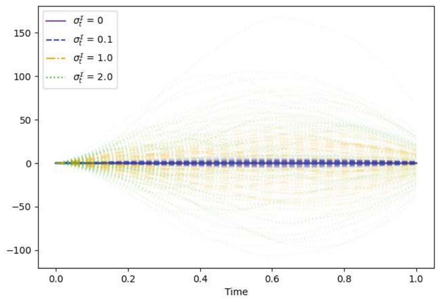


Figure 10: 100 sample paths of $\overline{Q}_t$ for different values of $\sigma^{\mathcal{I}}$ with $\mathfrak{Q} = 0$ and $a^{\mathcal{I}} = 0$

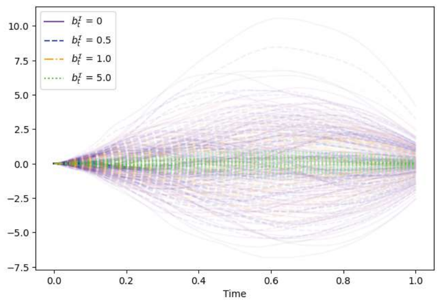


Figure 11: 100 sample paths of $\overline{Q}_t$ for different values of $b^{\mathcal{I}}$ with $\mathfrak{Q} = 0$ and $a^{\mathcal{I}} = 0$

Figure 11 shows 100 sample paths for different values of $\sigma^{\mathcal{I}}$ with $\mathfrak{Q} = 0$, $a^{\mathcal{I}} = 0$ and $\sigma^{\mathcal{I}} = 0.5$ fixed. We find that as the mean-reversion speed increases, the extent to which the large trader executes orders decreases. This result seems plausible from the following consideration: In the setting that $a^{\mathcal{I}} = 0$, we observe that the mean-reversion level is zero. Therefore, the fast mean-reversion of Markovian environment indicates that Markovian environment takes values near zero and thus possibly has little effect on the asset price over the trading window. Thus, there is little opportunity for the large trader to make use of Markovian environment to increase her expected utility from zero initial inventory when the mean-reversion speed is fast.

5 Conclusion

We have examined an optimal execution model for a single large trader in a finite continuous-time framework. The large trader maximizes the expected CARA utility arising from her wealth at the end of the trading epoch in a market. By formulating a generalized market impact model, the backward induction method of dynamic programming based on the dynamic programming principle permitted us to derive the optimal execution strategy. This kind of work concerned with an execution problem through the backward induction procedure of dynamic programming will be explored from a more in-depth and extensive perspective, which we can expect will also give us a more illuminating insight into all the other problems left in this field of research as follows.

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Appendix

A Proof of Theorem 3.1

From the discrete time analogue of Fukasawa et al. [23], we guess the objective (or value) function as follows:

$$\begin{aligned} V\left[t, W_t, P_t, \overline{Q}_t, R_t, \mathcal{I}_t\right] = -\exp\Big\{-\gamma\big[W_t - P\overline{Q}_t + G_t(\overline{Q}_t)^2 + H_t\overline{Q}_t + I_t\overline{Q}_t R_t + J_t(R_t)^2 + L_t R_t \\ + M_t\overline{Q}_t\mathcal{I}_t + N_t R_t\mathcal{I}_t + X_t(\mathcal{I}_t)^2 + Y_t\mathcal{I}_t + K_t\big]\Big\}, \quad \text{(A.63)} \end{aligned}$$

with the terminal condition:

$$V\left[T, W_T, P_T, \overline{Q}_T, R_T, \mathcal{I}_T\right] = -\exp\Big\{-\gamma\big[W_T - (P_T + \chi_T \overline{Q}_T)\overline{Q}_T\big]\Big\}. \tag{A.64}$$

The partial differentiation of $V\left[t, W_t, P_t, \overline{Q}_t, R_t, \mathcal{I}_t\right]$ with respect to time and each state variable is calculated as follows:

$$\begin{aligned}
&\partial_t V = -\gamma\left\{\dot{G}_t(\overline{Q}_t)^2 + \dot{H}_t\overline{Q}_t + \dot{I}_t\overline{Q}_t R_t + \dot{J}_t R_t^2 + \dot{L}_t R_t + \dot{M}_t\overline{Q}_t\mathcal{I}_t + \dot{N}_t R_t\mathcal{I}_t + \dot{X}_t\mathcal{I}_t^2 + \dot{Y}_t\mathcal{I}_t + \dot{K}_t\right\} V;\\
&\partial_W V = -\gamma V;\\
&\partial_P V = \gamma\overline{Q}_t V;\\
&\partial_{\overline{Q}} V = -\gamma\left(-P_t + 2G_t\overline{Q}_t + H_t + I_t R_t + M_t\mathcal{I}_t\right) V;\\
&\partial_R V = -\gamma\left(I_t\overline{Q}_t + 2J_t R_t + L_t + N_t\mathcal{I}_t\right) V;\\
&\partial_{\mathcal{I}} V = -\gamma(M_t\overline{Q}_t + N_t R_t + 2X_t\mathcal{I}_t + Y_t)V;\\
&\partial_{PP} V = \gamma^2\overline{Q}_t^2 V;\\
&\partial_{P\mathcal{I}} V = -\gamma^2\overline{Q}_t\left(M_t\overline{Q}_t + N_t R_t + 2X_t\mathcal{I}_t + Y_t\right) V;\\
&\partial_{\mathcal{I}\mathcal{I}} V = -2\gamma X_t V + \gamma^2(M_t\overline{Q}_t + N_t R_t + 2X_t\mathcal{I}_t + Y_t)^2 V.
\end{aligned}$$

Therefore, by substituting these into Eq. (3.31), we have

$$\begin{aligned}
\sup_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[&\lambda_t\dot{Q}_t^2 + \big[\big\{(\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t\big\}\overline{Q}_t + (I_t - 2\alpha_t\lambda_t J_t)\, R_t\\
&+ (M_t - \alpha_t\lambda_t N_t)\,\mathcal{I}_t + (H_t - \alpha_t\lambda_t L_t)\big]\dot{Q}_t\Big]V\\
&+ \gamma\Big\{-\dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} M_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2 M_t^2\Big\}\overline{Q}_t^2 V\\
&+ \gamma\Big\{-\dot{H}_t + \mu_t^Z - a_t^{\mathcal{I}} M_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} Y_t + \gamma(\sigma_t^{\mathcal{I}})^2 M_t Y_t\Big\}\overline{Q}_t V\\
&+ \gamma\Big\{-\dot{I}_t - \rho + \rho I_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} N_t + \gamma(\sigma_t^{\mathcal{I}})^2 M_t N_t\Big\}\overline{Q}_t R_t V\\
&+ \gamma\Big\{-\dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2 N_t^2\Big\} R_t^2 V\\
&+ \gamma\Big\{-\dot{L}_t + \rho L_t - a_t^{\mathcal{I}} N_t + \gamma(\sigma_t^{\mathcal{I}})^2 N_t Y_t\Big\} R_t V\\
&+ \gamma\Big\{-\dot{M}_t + 1 + b_t^{\mathcal{I}} M_t - 2\gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 M_t X_t\Big\}\overline{Q}_t\mathcal{I}_t V\\
&+ \gamma\Big\{-\dot{N}_t + \rho N_t + b_t^{\mathcal{I}} N_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 N_t X_t\Big\} R_t\mathcal{I}_t V\\
&+ \gamma\Big\{-\dot{X}_t + 2b_t^{\mathcal{I}} X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 X_t^2\Big\}\mathcal{I}_t^2 V\\
&+ \gamma\Big\{-\dot{Y}_t - 2a_t^{\mathcal{I}} X_t + b_t^{\mathcal{I}} Y_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 X_t Y_t\Big\}\mathcal{I}_t V\\
&+ \gamma\Big\{-\dot{K}_t - a_t^{\mathcal{I}} Y_t - (\sigma_t^{\mathcal{I}})^2 X_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2 Y_t^2\Big\} V\\
=\ & 0.
\end{aligned} \tag{A.65}$$

Since we assume the negative exponential utility function above,

$$
\begin{aligned}
&\sup_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[\lambda_t\dot{Q}_t^2 + \big[\{(\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t\}\overline{Q}_t + (I_t - 2\alpha_t\lambda_t J_t)\, R_t \\
&\quad + (M_t - \alpha_t\lambda_t N_t)\,\mathcal{I}_t + (H_t - \alpha_t\lambda_t L_t)\,\big]\dot{Q}_t\Big]V \\
&= V \inf_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[\lambda_t\dot{Q}_t^2 + \big[\{(\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t\}\overline{Q}_t + (I_t - 2\alpha_t\lambda_t J_t)\, R_t \\
&\quad + (M_t - \alpha_t\lambda_t N_t)\,\mathcal{I}_t + (H_t - \alpha_t\lambda_t L_t)\,\big]\dot{Q}_t\Big] \\
&= V \inf_{\dot{Q}_t} \gamma\Big[a_t\dot{Q}_t^2 + \big[b_t\overline{Q}_t + c_tR_t + d_t\mathcal{I}_t + e_t\big]\,\dot{Q}_t\Big], \qquad \text{(A.66)}
\end{aligned}
$$

where

$$
\begin{aligned}
a_t &\coloneqq \lambda_t; \\
b_t &\coloneqq (\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t; \\
c_t &\coloneqq I_t - 2\alpha_t\lambda_t J_t; \\
d_t &\coloneqq M_t - \alpha_t\lambda_t N_t; \\
e_t &\coloneqq H_t - \alpha_t\lambda_t L_t. \qquad \text{(A.67)}
\end{aligned}
$$

Therefore, Eq. (A.66) attains the infimum at the optimal execution speed:

$$
\dot{Q}_t^* = -\frac{b_t\overline{Q}_t + c_tR_t + d_t\mathcal{I}_t + e_t}{2a_t} =: a_t^* + b_t^*\overline{Q}_t + c_t^*R_t + d_t^*\mathcal{I}_t, \qquad \text{(A.68)}
$$

where

$$
a_t^* \coloneqq -\frac{e_t}{2a_t}; \quad b_t^* \coloneqq -\frac{b_t}{2a_t}; \quad c_t^* \coloneqq -\frac{c_t}{2a_t}; \quad \text{and} \quad d_t^* \coloneqq -\frac{d_t}{2a_t}, \qquad \text{(A.69)}
$$

and the value of Eq. (A.66) at the infimum becomes

$$
V \inf_{\dot{Q}_t} \gamma\Big[a_t\dot{Q}_t^2 + \big[b_t\overline{Q}_t + c_tR_t + d_t\mathcal{I}_t + e_t\big]\,\dot{Q}_t\Big] = -\frac{(b_t\overline{Q}_t + c_tR_t + d_t\mathcal{I}_t + e_t)^2}{4a_t}V\gamma. \qquad \text{(A.70)}
$$

Substituting this into Eq. (A.65) yields

$$
\begin{aligned}
&\sup_{\dot{Q}_t\in\mathbb{R}} \gamma\Big[\lambda_t\dot{Q}_t^2 + \big[\big\{(\alpha_t+\beta_t)\lambda_t + 2G_t - \alpha_t\lambda_t I_t\big\}\overline{Q}_t + (I_t - 2\alpha_t\lambda_t J_t)\,R_t \\
&\quad + (M_t - \alpha_t\lambda_t N_t)\,\mathcal{I}_t + (H_t - \alpha_t\lambda_t L_t)\big]\dot{Q}_t\Big]V \\
&\quad + \gamma\Big\{ - \dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}M_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2M_t^2\Big\}\overline{Q}_t^2V \\
&\quad + \gamma\Big\{ - \dot{H}_t + \mu_t^Z - a_t^{\mathcal{I}}M_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}Y_t + \gamma(\sigma_t^{\mathcal{I}})^2M_tY_t\Big\}\overline{Q}_tV \\
&\quad + \gamma\Big\{ - \dot{I}_t - \rho + \rho I_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}N_t + \gamma(\sigma_t^{\mathcal{I}})^2M_tN_t\Big\}\overline{Q}_tR_tV \\
&\quad + \gamma\Big\{ - \dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2N_t^2\Big\}R_t^2V \\
&\quad + \gamma\Big\{ - \dot{L}_t + \rho L_t - a_t^{\mathcal{I}}N_t + \gamma(\sigma_t^{\mathcal{I}})^2N_tY_t\Big\}R_tV \\
&\quad + \gamma\Big\{ - \dot{M}_t + 1 + b_t^{\mathcal{I}}M_t - 2\gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2M_tX_t\Big\}\overline{Q}_t\mathcal{I}_tV \\
&\quad + \gamma\Big\{ - \dot{N}_t + \rho N_t + b_t^{\mathcal{I}}N_t + 2\gamma(\sigma_t^{\mathcal{I}})^2N_tX_t\Big\}R_t\mathcal{I}_tV \\
&\quad + \gamma\Big\{ - \dot{X}_t + 2b_t^{\mathcal{I}}X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2X_t^2\Big\}\mathcal{I}_t^2V \\
&\quad + \gamma\Big\{ - \dot{Y}_t - 2a_t^{\mathcal{I}}X_t + b_t^{\mathcal{I}}Y_t + 2\gamma(\sigma_t^{\mathcal{I}})^2X_tY_t\Big\}\mathcal{I}_tV \\
&\quad + \gamma\Big\{ - \dot{K}_t - a_t^{\mathcal{I}}Y_t - (\sigma_t^{\mathcal{I}})^2X_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2Y_t^2\Big\}V \\
&= \gamma\left\{-\dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}M_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2M_t^2 - \frac{b_t^2}{4a_t}\right\}\overline{Q}_t^2V \\
&\quad + \gamma\left\{-\dot{H}_t + \mu_t^Z - a_t^{\mathcal{I}}M_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}Y_t + \gamma(\sigma_t^{\mathcal{I}})^2M_tY_t - \frac{b_te_t}{2a_t}\right\}\overline{Q}_tV \\
&\quad + \gamma\left\{-\dot{I}_t - \rho + \rho I_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}N_t + \gamma(\sigma_t^{\mathcal{I}})^2M_tN_t - \frac{b_tc_t}{2a_t}\right\}\overline{Q}_tR_tV \\
&\quad + \gamma\left\{-\dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2N_t^2 - \frac{c_t^2}{4a_t}\right\}R_t^2V \\
&\quad + \gamma\left\{-\dot{L}_t + \rho L_t - a_t^{\mathcal{I}}N_t + \gamma(\sigma_t^{\mathcal{I}})^2N_tY_t - \frac{c_te_t}{2a_t}\right\}R_tV \\
&\quad + \gamma\left\{-\dot{M}_t + 1 + b_t^{\mathcal{I}}M_t - 2\gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}}X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2M_tX_t - \frac{b_td_t}{2a_t}\right\}\overline{Q}_t\mathcal{I}_tV \\
&\quad + \gamma\left\{-\dot{N}_t + \rho N_t + b_t^{\mathcal{I}}N_t + 2\gamma(\sigma_t^{\mathcal{I}})^2N_tX_t - \frac{c_td_t}{2a_t}\right\}R_t\mathcal{I}_tV \\
&\quad + \gamma\left\{-\dot{X}_t + 2b_t^{\mathcal{I}}X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2X_t^2 - \frac{d_t^2}{4a_t}\right\}\mathcal{I}_t^2V \\
&\quad + \gamma\left\{-\dot{Y}_t - 2a_t^{\mathcal{I}}X_t + b_t^{\mathcal{I}}Y_t + 2\gamma(\sigma_t^{\mathcal{I}})^2X_tY_t - \frac{d_te_t}{2a_t}\right\}\mathcal{I}_tV \\
&\quad + \gamma\left\{-\dot{K}_t - a_t^{\mathcal{I}}Y_t - (\sigma_t^{\mathcal{I}})^2X_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2Y_t^2 - \frac{e_t^2}{4a_t}\right\}V \\
&= 0. \qquad\qquad \text{(A.71)}
\end{aligned}
$$

Since this equation holds for all states, the following conditions must hold:

$$-\dot{G}_t + \frac{1}{2}\gamma(\sigma_t^Z)^2 - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} M_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2 M_t^2 - \frac{b_t^2}{4a_t} = 0; \quad \text{(A.72)}$$

$$-\dot{H}_t + \mu_t^Z - a_t^{\mathcal{I}} M_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} Y_t + \gamma(\sigma_t^{\mathcal{I}})^2 M_t Y_t - \frac{b_t e_t}{2a_t} = 0; \quad \text{(A.73)}$$

$$-\dot{I}_t - \rho + \rho I_t - \gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} N_t + \gamma(\sigma_t^{\mathcal{I}})^2 M_t N_t - \frac{b_t c_t}{2a_t} = 0; \quad \text{(A.74)}$$

$$-\dot{J}_t + 2\rho J_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2 N_t^2 - \frac{c_t^2}{4a_t} = 0; \quad \text{(A.75)}$$

$$-\dot{L}_t + \rho L_t - a_t^{\mathcal{I}} N_t + \gamma(\sigma_t^{\mathcal{I}})^2 N_t Y_t - \frac{c_t e_t}{2a_t} = 0; \quad \text{(A.76)}$$

$$-\dot{M}_t + 1 + b_t^{\mathcal{I}} M_t - 2\gamma\rho^{Z,\mathcal{I}}\sigma_t^Z\sigma_t^{\mathcal{I}} X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 M_t X_t - \frac{b_t d_t}{2a_t} = 0; \quad \text{(A.77)}$$

$$-\dot{N}_t + \rho N_t + b_t^{\mathcal{I}} N_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 N_t X_t - \frac{c_t d_t}{2a_t} = 0; \quad \text{(A.78)}$$

$$-\dot{X}_t + 2b_t^{\mathcal{I}} X_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 X_t^2 - \frac{d_t^2}{4a_t} = 0; \quad \text{(A.79)}$$

$$-\dot{Y}_t - 2a_t^{\mathcal{I}} X_t + b_t^{\mathcal{I}} Y_t + 2\gamma(\sigma_t^{\mathcal{I}})^2 X_t Y_t - \frac{d_t e_t}{2a_t} = 0; \quad \text{(A.80)}$$

$$-\dot{K}_t - a_t^{\mathcal{I}} Y_t - (\sigma_t^{\mathcal{I}})^2 X_t + \frac{1}{2}\gamma(\sigma_t^{\mathcal{I}})^2 Y_t^2 - \frac{e_t^2}{4a_t} = 0. \quad \text{(A.81)}$$

with the terminal conditions:

$$G_T = -\chi_T; \quad H_T = I_T = J_T = L_T = M_T = N_T = X_T = Y_T = K_T = 0. \quad \text{(A.82)}$$

By substituting the dynamics of a_t, b_t, c_t, d_t, e_t into the condition derived above and rearranging, we obtain a system of ordinary differential equations consisting of $G_t, H_t, I_t, J_t, L_t, M_t, N_t, X_t, Y_t, K_t$. □

B Proof of Corollary 3.2

The derivation of Eq. (3.39) is similar to that of optimal execution speed (3.32) with the ansatz of the optimal value function (3.42). We show that Eq. (3.40) holds from Eq. (3.39). If the unique solution of the system of ODEs exists, $\tilde{a}_t, \tilde{b}_t$, and $\tilde{d}_t$ are all continuous functions defined on $[0, T]$, and thus are bounded on the domain. Also, $\{\mathcal{I}_t\}_{0\leq t\leq T}$ is a generalized OU process, and thus is also a centered and continuous Gaussian process. Therefore, $\mathbb{E}[\sup_{t\in[0,T]} \mathcal{I}_t]$ is finite (Nourdin [45]), implying that $\sup_{t\in[0,T]} \mathcal{I}_t$ is finite a.s. Since

$$\mathrm{d}\overline{Q}_t = -\dot{Q}_t \mathrm{d}t, \quad \text{(B.83)}$$

By stochastic integration by parts, Eq. (3.40) can be rewritten as

$$\begin{aligned}
\dot{Q}_t = \tilde{a}_t + \tilde{b}_t\overline{Q}_t + \tilde{d}_t\mathcal{I}_t &\iff (\dot{Q}_t - \tilde{b}_t\overline{Q}_t) = \tilde{a}_t + \tilde{d}_t\mathcal{I}_t \\
&\iff -\mathrm{e}^{\int_0^t \tilde{b}_s \mathrm{d}s}(-\dot{Q}_t + \tilde{b}_t\overline{Q}_t) = \mathrm{e}^{\int_0^t \tilde{b}_s \mathrm{d}s}(\tilde{a}_t + \tilde{d}_t\mathcal{I}_t) \\
&\iff \mathrm{e}^{\int_0^t \tilde{b}_s \mathrm{d}s}\overline{Q}_t = \mathfrak{Q} - \int_0^t \left\{\mathrm{e}^{\int_0^u \tilde{b}_l \mathrm{d}l}(\tilde{a}_u + \tilde{d}_u\mathcal{I}_u)\right\}\mathrm{d}u \\
&\iff \overline{Q}_t = \mathrm{e}^{-\int_0^t \tilde{b}_s \mathrm{d}s}\mathfrak{Q} + \int_0^t \left(-\mathrm{e}^{\int_u^t \tilde{b}_l \mathrm{d}l}\right)\tilde{a}_u \mathrm{d}u + \int_0^t \left(-\mathrm{e}^{\int_u^t \tilde{b}_l \mathrm{d}l}\right)\tilde{d}_u\mathcal{I}_u \mathrm{d}u.
\end{aligned} \quad \text{(B.84)}$$

□

C Proof of Proposition 3.1

A similar calculation for solving ODE (3.39) results in

$$\begin{aligned}
\mathbb{E}[\overline{Q}_t|\mathcal{F}_s] &= \mathrm{e}^{-\int_0^t \tilde{b}_l \mathrm{d}l}\mathfrak{Q} + \int_0^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l}\tilde{a}_u \mathrm{d}u + \int_0^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l}\tilde{d}_u \mathbb{E}[\mathcal{I}_u|\mathcal{F}_s]\mathrm{d}u \\
&= \mathrm{e}^{-\int_s^t \tilde{b}_l \mathrm{d}l}\left(\mathrm{e}^{-\int_0^s \tilde{b}_l \mathrm{d}l}\mathfrak{Q} + \int_0^s \mathrm{e}^{-\int_u^s \tilde{b}_l \mathrm{d}l}\tilde{a}_u \mathrm{d}u + \int_0^s \mathrm{e}^{-\int_u^s \tilde{b}_l \mathrm{d}l}\tilde{d}_u \mathbb{E}[\mathcal{I}_u|\mathcal{F}_s]\mathrm{d}u\right) \\
&\quad + \int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l}\tilde{a}_u \mathrm{d}l + \int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l}\tilde{d}_u \mathbb{E}[\mathcal{I}_u|\mathcal{F}_s]\mathrm{d}u \\
&= \mathrm{e}^{-\int_s^t \tilde{b}_l \mathrm{d}l}\overline{Q}_s + \left(\int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l}\mathrm{e}^{-\int_s^u b_m^{\mathcal{I}}\mathrm{d}m}\tilde{d}_u \mathrm{d}u\right)\mathcal{I}_s \\
&\quad + \int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l}\tilde{a}_u \mathrm{d}l + \int_s^t \mathrm{e}^{-\int_u^t \tilde{b}_l \mathrm{d}l}\left(\int_s^t a_m^{\mathcal{I}}\mathrm{e}^{-\int_m^t b_k^{\mathcal{I}}\mathrm{d}k}\mathrm{d}m\right)\tilde{d}_u \mathrm{d}u,
\end{aligned} \tag{C.85}$$

where we use Fubini theorem in the first equality and the fact that for all $0 \leq s \leq u$,

$$\mathbb{E}[\mathcal{I}_u|\mathcal{F}_s] = \mathrm{e}^{-\int_s^u b_m^{\mathcal{I}}\mathrm{d}m}\mathcal{I}_s + \int_s^u a_m^{\mathcal{I}}\mathrm{e}^{-\int_m^u b_k^{\mathcal{I}}\mathrm{d}k}\mathrm{d}m. \tag{C.86}$$

For the derivation of conditional covariance, define

$$\mathfrak{A}(l,t) := \sigma_l^{\mathcal{I}}\int_l^t \mathrm{B}^{\mathcal{I}}(l,u)\mathrm{B}^b(u,t)\tilde{d}_u \mathrm{d}u, \tag{C.87}$$

and assume that $0 \leq s \leq t \leq v \leq T$. Then we have

$$\begin{aligned}
\mathrm{Cov}[\overline{Q}_t, \overline{Q}_v|\mathcal{F}_s] &= \mathrm{Cov}\left[\int_0^t \mathfrak{A}(l,t)\mathrm{d}B_l^{\mathcal{I}}, \int_0^v \mathfrak{A}(m,v)\mathrm{d}B_m^{\mathcal{I}}\middle|\mathcal{F}_s\right] \\
&= \mathrm{Cov}\left[\int_s^t \mathfrak{A}(l,t)\mathrm{d}B_l^{\mathcal{I}}, \int_s^t \mathfrak{A}(m,v)\mathrm{d}B_m^{\mathcal{I}} + \int_t^v \mathfrak{A}(m,v)\mathrm{d}B_m^{\mathcal{I}}\middle|\mathcal{F}_s\right] \\
&= \mathrm{Cov}\left[\int_s^t \mathfrak{A}(l,t)\mathrm{d}B_l^{\mathcal{I}}, \int_s^t \mathfrak{A}(m,v)\mathrm{d}B_m^{\mathcal{I}}\middle|\mathcal{F}_s\right] + \mathrm{Cov}\left[\int_s^t \mathfrak{A}(l,t)\mathrm{d}B_l^{\mathcal{I}}, \int_t^v \mathfrak{A}(m,v)\mathrm{d}B_m^{\mathcal{I}}\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\int_s^t \mathfrak{A}(l,t)\mathrm{d}B_l^{\mathcal{I}}\int_s^t \mathfrak{A}(m,v)\mathrm{d}B_m^{\mathcal{I}}\middle|\mathcal{F}_s\right] + \mathrm{Cov}\left[\int_s^t \mathfrak{A}(l,t)\mathrm{d}B_l^{\mathcal{I}}, \int_t^v \mathfrak{A}(m,v)\mathrm{d}B_m^{\mathcal{I}}\middle|\mathcal{F}_s\right] \\
&= \int_s^t \mathfrak{A}(l,t)\mathfrak{A}(l,v)\mathrm{d}l.
\end{aligned} \tag{C.88}$$

The last equality stems from Ito isometry. □