Sheaf theoretical study of bicomplex hyperfunctions

By

Yutaka MATSUI*

Abstract

In this article, we introduce the results of the sheaf theoretical study of bicomplex hyperfunctions in [6].

§ 1. Introduction

The hyperfunctions was introduced by M. Sato in [10] as a generalization of functions. It is well-known that they are more natural and useful than distributions in studying linear partial differential equations with real analytic coefficients. The theory of hyperfunctions has vastly developed as algebraic analysis [11].

Bicomplex algebra was introduced by Segre inspired by the work of Hamilton and Clifford on quaternions. It is defined by

$$\mathbb{BC} = \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}, \tag{1.1}$$

where j is another imaginary unit commuting with the imaginary unit i of $\mathbb{C}$. Since $\mathbb{BC}$ is commutative and has zero divisors, it is more difficult to study bicomplex functions than complex functions. Nevertheless, we can define the notion of holomorphicity of bicomplex functions similarly to that of complex functions. The study of bicomplex holomorphic functions is called bicomplex analysis ([3], [7]). See [4] and [5] for the author's recent works.

Colombo et al. introduced the notion of bicomplex hyperfunctions in [1] as a natural generalization of classical hyperfunctions to bicomplex analysis. They proved a vanishing theorem of a relative cohomology group of the sheaf of bicomplex holomorphic functions and the flabbiness of the sheaf of bicomplex hyperfunctions. See also [9] and [12] for further developments.

2010 Mathematics Subject Classification(s): Primary 30G35; Secondary 32A45.
Key Words: Bicomplex analysis, Hyperfunctions
*Department of mathematics, Kindai University, 3-4-1, Kowakae, Higashi-Osaka, Osaka, 577-8502, Japan.

In this article, we introduce the result of [6] on the idempotent representation of bicomplex hyperfunctions, which is based on the idempotent representation of bicomplex holomorphic functions and the functorial techniques of sheaf theory. By our methods, we can reconstruct the theory of bicomplex hyperfunctions and develop it into the theory of bicomplex microfunctions.

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

§ 2. Preliminaries

§ 2.1. Bicomplex numbers

In this subsection, we review the definitions and fundamental properties of bicomplex numbers. See [3] and [7] for more details.

Let $\mathbb{C}$ be the field of complex numbers with the imaginary unit i. We define the set of bicomplex numbers by

$$\mathbb{BC} = \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}, \tag{2.1}$$

where j is another imaginary unit independent of and commuting with i:

$$i \neq j, \quad ij = ji, \quad i^2 = j^2 = -1. \tag{2.2}$$

By defining the addition and multiplication naturally, $\mathbb{BC}$ has a structure of a commutative ring. The set of zero divisors of $\mathbb{BC}$ with 0 is described as

$$\mathfrak{S}_0 = \{Z = z_1 + z_2 j \in \mathbb{BC} \mid z_1^2 + z_2^2 = 0\}. \tag{2.3}$$

Note that a zero divisor is a non-unit element of $\mathbb{BC}$. Setting

$$\mathbf{e} = \frac{1 + ij}{2}, \quad \mathbf{e}^\dagger = \frac{1 - ij}{2}, \tag{2.4}$$

$\mathbf{e}$ and $\mathbf{e}^\dagger$ are the non-complex idempotent elements satisfying with $\mathbf{e}\mathbf{e}^\dagger = 0$.

For $Z = z_1 + z_2 j \in \mathbb{BC}$, we define the surjective ring homomorphisms $\Phi_{\mathbf{e}} \colon \mathbb{BC} \longrightarrow \mathbb{C}$, $\Phi_{\mathbf{e}^\dagger} \colon \mathbb{BC} \longrightarrow \mathbb{C}$ by

$$\Phi_{\mathbf{e}}(Z) = z_1 - z_2 i, \quad \Phi_{\mathbf{e}^\dagger}(Z) = z_1 + z_2 i \tag{2.5}$$

respectively. Then any bicomplex number Z has the idempotent representation

$$Z = \Phi_{\mathbf{e}}(Z)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger. \tag{2.6}$$

By the idempotent representation, the set of zero divisors with 0 is represented by

$$\mathfrak{S}_0 = \{Z = \Phi_{\mathbf{e}}(Z)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger \in \mathbb{BC} \mid \Phi_{\mathbf{e}}(Z)\Phi_{\mathbf{e}^\dagger}(Z) = 0\} = \mathbb{C}\mathbf{e} \cup \mathbb{C}\mathbf{e}^\dagger. \tag{2.7}$$

In the case of several bicomplex variables, for $Z = (Z_1, \ldots, Z_n) \in \mathbb{BC}^n$, we also define the maps $\Phi_{\mathbf{e}} \colon \mathbb{BC}^n \longrightarrow \mathbb{C}^n$ and $\Phi_{\mathbf{e}^\dagger} : \mathbb{BC}^n \longrightarrow \mathbb{C}^n$ by

$$\Phi_{\mathbf{e}}(Z) = (\Phi_{\mathbf{e}}(Z_1), \ldots, \Phi_{\mathbf{e}}(Z_n)), \quad \Phi_{\mathbf{e}^\dagger}(Z) = (\Phi_{\mathbf{e}^\dagger}(Z_1), \ldots, \Phi_{\mathbf{e}^\dagger}(Z_n)) \tag{2.8}$$

respectively. Then we have the idempotent representation of $Z \in \mathbb{BC}^n$ as

$$Z = \Phi_{\mathbf{e}}(Z)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger. \tag{2.9}$$

In order to emphasize components, we may identify an image of a point by $\Phi_{\mathbf{e}}$ (resp. $\Phi_{\mathbf{e}^\dagger}$) with a point of the $\mathbf{e}$-axis $\mathbb{C}^n\mathbf{e}$ (resp. the $\mathbf{e}^\dagger$-axis $\mathbb{C}^n\mathbf{e}^\dagger$) in $\mathbb{BC}^n$. Namely, we may use the notation of $\Phi_{\mathbf{e}}(\mathbb{BC}^n) = \mathbb{C}^n\mathbf{e}$, $\Phi_{\mathbf{e}^\dagger}(\mathbb{BC}^n) = \mathbb{C}^n\mathbf{e}^\dagger$ and so on.

We can also define the notion of mutlicomplex numbers, similarly. We omit the details.

§ 2.2. Bicomplex holomorphic functions

In this subsection, we review the definitions and fundamental results in bicomplex analysis. See [3], [7] and [8] for more details.

For any bicomplex number $Z = z_1 + z_2 j \in \mathbb{BC}$, we define the norm $\|Z\|$ of Z by

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2}. \tag{2.10}$$

$\mathbb{BC}$ has a structure of a topological space induced by it, which is isomorphic to the Euclidian space $\mathbb{C}^2$. Moreover, the maps $\Phi_{\mathbf{e}}$ and $\Phi_{\mathbf{e}^\dagger}$ are continuous and open.

Let $\Omega \subset \mathbb{BC}$ be an open set, $F \colon \Omega \longrightarrow \mathbb{BC}$ a bicomplex function on Ω and $Z_0 \in \Omega$. We say that F is bicomplex differentiable at Z_0 if the limit

$$\lim_{\substack{Z \to Z_0 \\ Z - Z_0 \notin \mathfrak{S}_0}} \frac{F(Z) - F(Z_0)}{Z - Z_0} \tag{2.11}$$

exists, which is denoted by $F'(Z_0)$. We also say that F is bicomplex holomorphic on Ω if F is bicomplex differentiable at any point of Ω. We denote the set of bicomplex holomorphic functions on Ω by $\mathcal{O}_{\mathbb{BC}}(\Omega)$. Then $\mathcal{O}_{\mathbb{BC}}$ has a sheaf structure.

The idempotent representation of bicomplex holomorphic functions plays an important role in bicomplex analysis.

Theorem 2.1. *Let $Z_0 \in \mathbb{BC}$ and F a bicomplex function on a sufficiently small neighborhood Ω of Z_0. Then F is bicomplex holomorphic on Ω if and only if there exist complex holomorphic functions $F_{\mathbf{e}}$ and $F_{\mathbf{e}^\dagger}$ on $\Phi_{\mathbf{e}}(\Omega)$ and $\Phi_{\mathbf{e}^\dagger}(\Omega)$ respectively such that*

$$F(Z) = F_{\mathbf{e}}(Z_{\mathbf{e}})\mathbf{e} + F_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger \tag{2.12}$$

holds on Ω.

By Theorem 2.1, we can immediately generalize fundamental properties of complex holomorphic functions of one variable such as Taylor's theorem, theorem of identity and so on to those of bicomplex holomorphic functions.

We denote the sheaf of complex holomorphic functions of one variable by $\mathcal{O}_{\mathbb{C}}$. We can represent Theorem 2.1 in terms of sheaves as follows.

Corollary 2.2. *We have an isomorphism of sheaves*

$$\mathcal{O}_{\mathbb{BC}} \simeq \Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}\mathbf{e}}\mathbf{e} \oplus \Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}\mathbf{e}^\dagger}\mathbf{e}^\dagger. \tag{2.13}$$

In the case of several bicomplex variables, let $\Omega \subset \mathbb{BC}^n$ be an open set, $F\colon \Omega \longrightarrow \mathbb{BC}$ a bicomplex function on Ω. We say that F is bicomplex holomorphic on Ω if and only if F is partially holomorphic in each variable on Ω. We denote the sheaf of bicomplex homolorphic functions of several variables by $\mathcal{O}_{\mathbb{BC}^n}$. Then we also have an isomorphism

$$\mathcal{O}_{\mathbb{BC}^n} \simeq \Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}}\mathbf{e} \oplus \Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger}\mathbf{e}^\dagger, \tag{2.14}$$

where $\mathcal{O}_{\mathbb{C}^n}$ is the sheaf of complex holomorphic functions of several variables.

We can also define the notion of mutlicomplex holomorphicity, similarly. We omit the details.

§ 2.3. Bicomplex hyperfunctions

Let $V \subset \mathbb{R}^n$ be an open set and $\Omega \subset \mathbb{BC}^n$ a bicomplex neighborhood of V. Namely, $V = \mathbb{R}^n \cap \Omega$ holds. Colombo et al. in [1] proved that the cohomology group $H^p_V(\Omega; \mathcal{O}_{\mathbb{BC}^n})$ of $\mathcal{O}_{\mathbb{BC}^n}$ supported by V vanishes if $p \neq 3n$ and defined the notion of bicomplex hyperfunctions as

$$\mathcal{B}_{\mathbb{BC}^n}(V) = H^{3n}_V(\Omega; \mathcal{O}_{\mathbb{BC}^n}) \tag{2.15}$$

by using an abstract Dolbeault complex, which is a resolution of the sheaf $\mathcal{O}_{\mathbb{BC}^n}$ of bicomlex holomorphic functions. They also proved that $\mathcal{B}_{\mathbb{BC}^n}$ is a flabby sheaf on $\mathbb{R}^n$ and the duality theorem

$$H^{3n}_K(\mathbb{BC}^n; \mathcal{O}_{\mathbb{BC}^n}) = (\mathcal{O}_{\mathbb{BC}^n}(K))' \tag{2.16}$$

holds for any compact convex subset K of $\mathbb{BC}^n$.

Similarly, Vajiac-Vajiac in [12] defined the notion of multicomplex hyperfunctions of one variable.

§ 3. Sheaf theoretical study of bicomplex hyperfunctions

In this article, we study bicomplex hyperfunctions by sheaf theoretical way. We mainly treat objects in the derived categories $\mathbf{D}^b(\mathbb{R}^n)$ and $\mathbf{D}^b(\mathbb{BC}^n)$. Here, for a topological space X, we denote the derived category of bounded complexes of sheaves of $\mathbb{C}_X$-modules on X by $\mathbf{D}^b(X)$. See [2] for more details on the derived category.

In the derived category $\mathbf{D}^b(\mathbb{R}^n)$, the sheaf of classical (complex) hyperfunctions, denoted by $\mathcal{B}_{\mathbb{C}^n}$ in this article, is isomorphic to the complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})|_{\mathbb{R}^n}[n]$, which is concentrated in degree 0. Note that we omit the orientation sheaf since it is trivial.

Considering the previous research of bicomplex hyperfunctions, it is natural to study the complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ of sections the sheaf $\mathcal{O}_{\mathbb{BC}^n}$ supported by $\mathbb{R}^n$. In order to study it, let us consider the following diagonal embedding

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^n\mathbf{e} + \mathbb{R}^n\mathbf{e}^\dagger \hookrightarrow \mathbb{C}^n\mathbf{e} + \mathbb{C}^n\mathbf{e}^\dagger = \mathbb{BC}^n \tag{3.1}$$

of the real space $\mathbb{R}^n$ into the bicomplex space $\mathbb{BC}^n$. The idempotent representation (2.14) of bicomplex holomorphic functions induces the following isomorphism.

Theorem 3.1. *We have an isomorphism*

$$\begin{aligned} R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n} \\ \simeq \Phi_{\mathbf{e}}|_{\mathbb{R}^n}^{-1} R\Gamma_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})|_{\mathbb{R}^n\mathbf{e}}\mathbf{e}[-2n] \oplus \Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n}^{-1} R\Gamma_{\mathbb{R}^n\mathbf{e}^\dagger}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})|_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbf{e}^\dagger[-2n] \end{aligned} \tag{3.2}$$

in $\mathbf{D}^b(\mathbb{R}^n)$.

By the fundamental results of classical (complex) hyperfunctions $\mathcal{B}_{\mathbb{C}^n}$, we can reprove the vanishing theorem of the complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n}$ and the flabbiness of the sheaf $\mathcal{B}_{\mathbb{BC}^n}$ of bicomplex hyperfunctions as a corollary of Theorem 3.1.

Corollary 3.2. *(i) The complex* $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n}$ *is concentrated in degree* $3n$.
(ii) Redefining $\mathcal{B}_{\mathbb{BC}^n} = H^{3n}(R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n})$ *as a sheaf, we obtain an isomorphism*

$$\mathcal{B}_{\mathbb{BC}^n} \simeq \Phi_{\mathbf{e}}|_{\mathbb{R}^n}^{-1}\mathcal{B}_{\mathbb{C}^n\mathbf{e}}\mathbf{e} \oplus \Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n}^{-1}\mathcal{B}_{\mathbb{C}^n\mathbf{e}^\dagger}\mathbf{e}^\dagger. \tag{3.3}$$

(iii) The sheaf $\mathcal{B}_{\mathbb{BC}^n}$ *is flabby on* $\mathbb{R}^n$.

Similarly, we can define the notion of multicomplex hyperfunction of several variables and obtain its idempotent representation. We omit the details in this article. See [6] for the details.

§ 4. Bicomplex microfunctions

Let us consider microlocally the study in the previous section.

In the derived category $\mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{C}^n)$, the sheaf of classical (complex) microfunctions, denoted by $\mathcal{C}_{\mathbb{C}^n}$ in this article, is isomorphic to the complex $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})[n]$, which is concentrated in degree 0. Here $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})$ is the microlocalization of $\mathcal{O}_{\mathbb{C}^n}$ along $\mathbb{R}^n$, which is an object of $\mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{C}^n)$. See [2] for more details on the microlocalization functor μ.

Considering our study of bicomplex hyperfunctions in Section 3, it is natural to study the microlocalization $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ of $\mathcal{O}_{\mathbb{BC}^n}$ along $\mathbb{R}^n$. In order to study it, let us also consider the diagonal embedding

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^n\mathbf{e} + \mathbb{R}^n\mathbf{e}^\dagger \hookrightarrow \mathbb{C}^n\mathbf{e} + \mathbb{C}^n\mathbf{e}^\dagger = \mathbb{BC}^n \tag{4.1}$$

of the real space $\mathbb{R}^n$ into the bicomplex space $\mathbb{BC}^n$ and the following morphisms

$$T^*_{\mathbb{R}^n}\mathbb{BC}^n \xleftarrow{{}^t\Phi'_{\mathbf{e}}} \mathbb{R}^n \underset{\mathbb{R}^n\mathbf{e}}{\times} T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} \xrightarrow{\Phi_{\mathbf{e}\pi}} T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}, \tag{4.2}$$

$$T^*_{\mathbb{R}^n}\mathbb{BC}^n \xleftarrow{{}^t\Phi'_{\mathbf{e}^\dagger}} \mathbb{R}^n \underset{\mathbb{R}^n\mathbf{e}^\dagger}{\times} T^*_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger \xrightarrow{\Phi_{\mathbf{e}^\dagger\pi}} T^*_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger \tag{4.3}$$

induced by the maps $T\Phi_{\mathbf{e}}\colon T\mathbb{BC}^n \longrightarrow T\mathbb{C}^n\mathbf{e}$ and $T\Phi_{\mathbf{e}^\dagger}\colon T\mathbb{BC}^n \longrightarrow T\mathbb{C}^n\mathbf{e}^\dagger$. Then the idempotent representation (2.14) of bicomplex holomorphic functions induces the following isomorphism.

Theorem 4.1. *We have an isomorphism*

$$\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n}) \simeq {}^t\Phi'_{\mathbf{e}*}\Phi^{-1}_{\mathbf{e}\pi}\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-2n]\mathbf{e} \oplus {}^t\Phi'_{\mathbf{e}^\dagger*}\Phi^{-1}_{\mathbf{e}^\dagger\pi}\mu_{\mathbb{R}^n\mathbf{e}^\dagger}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})[-2n]\,\mathbf{e}^\dagger \tag{4.4}$$

in $\mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{BC}^n)$.

By the vanishing theorem of the complex $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})$, we can prove the vanishing theorem of the complex $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ and define the notion of bicomplex microfunctions.

Theorem 4.2. *The complex* $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ *is concentrated in degree* $3n$.

Definition 4.3. We define the sheaf of bicomplex microfunctions by

$$\mathcal{C}_{\mathbb{BC}^n} = H^{3n}(\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})). \tag{4.5}$$

By the fundamental results of classical (complex) microfunctions $\mathcal{C}_{\mathbb{C}^n}$, we obtain the fundamental properties of bicomplex microfunctions as a corollary of Theorem 4.1.

Theorem 4.4. *(i) We have the idempotent representation of* $\mathcal{C}_{\mathbb{BC}^n}$

$$\mathcal{C}_{\mathbb{BC}^n} \simeq {}^t\Phi'_{\mathbf{e}*}\Phi^{-1}_{\mathbf{e}\pi}\mathcal{C}_{\mathbb{C}^n\mathbf{e}}\,\mathbf{e} \oplus {}^t\Phi'_{\mathbf{e}^\dagger*}\Phi^{-1}_{\mathbf{e}^\dagger\pi}\mathcal{C}_{\mathbb{C}^n\mathbf{e}^\dagger}\,\mathbf{e}^\dagger. \tag{4.6}$$

(ii) The sheaf $\mathcal{C}_{\mathbb{BC}^n}|_{\dot{T}^*_{\mathbb{R}^n}\mathbb{BC}^n}$ *is conically flabby on* $\dot{T}^*_{\mathbb{R}^n}\mathbb{BC}^n$.

(iii) There is a natural exact sequence on $\mathbb{R}^n$

$$0 \longrightarrow \mathcal{O}_{\mathbb{BC}^n}|_{\mathbb{R}^n} \longrightarrow \mathcal{B}_{\mathbb{BC}^n} \longrightarrow \dot{\pi}_*\mathcal{C}_{\mathbb{BC}^n} \longrightarrow 0, \tag{4.7}$$

where $\dot{\pi}\colon \dot{T}^*_{\mathbb{R}^n}\mathbb{BC}^n \longrightarrow \mathbb{R}^n$ *is the natural projection.*

(iv) There exists the spectrum isomorphism of sheaves on $\mathbb{R}^n$

$$\mathrm{sp}\colon \mathcal{B}_{\mathbb{BC}^n} \xrightarrow{\sim} \pi_*\mathcal{C}_{\mathbb{BC}^n}, \tag{4.8}$$

where $\pi\colon T^*_{\mathbb{R}^n}\mathbb{BC}^n \longrightarrow \mathbb{R}^n$ *is the natural projection.*

Similarly, we define the notion of multicomplex microfunctions of several variables and obtain its idempotent representation. We omit the details in this article. See [6] for the details.

References

[1] Colombo, E., Sabadini, I., Struppa, D. C., Vajiac, A., Vajiac, M., Bicomplex hyperfunctions, *Annali di Matematica* **49** (2011), 247–261.

[2] Kashiwara, M., Schapira, P., *Sheaves on manifolds*, Grundlehren der Math. Wiss. 292, Springer-Verlag, Berlin-Heidelberg-New York, 1990.

[3] Luna-Elizarrarás, M. E., Shapiro, M., Struppa, D. C., Vajiac, A., *Bicomplex holomorphic functions: The algebra, geometry and analysis of bicomplex numbers*, Frontiers in Mathematics, Birkhäuser, Basel, 2015.

[4] Matsui, Y., Sato, Y., Characterization theorems of Riley type for bicomplex holomorphic functions, *Commun. Korean Math. Soc.* **35** (2020), no.3, 825–841.

[5] Matsui, Y., Sato, Y., Generalized self-inversive bicomplex polynomials with respect to the j-conjugation, *Bull. Korean Math. Soc.* **58** (2021), no.4, 885–895.

[6] Matsui, Y., Bicomplex hyperfunctions and bicomplex microfunctions, in preparation.

[7] Price, G. B., *An introduction to multicomplex spaces and function*, Monograph and textbooks in pure and applied mathematics 140, Marcel Dekker Inc., New York, 1991.

[8] Riley, J. D., Contributions to the theory of functions of a bicomplex variable, *Tohoku Math. J.* **5** (1953), 132-165.

[9] Sabadini, I., Sommen, F., Struppa, D. C., Sato's hyperfunctions and boundary values of monogenic functions, *Adv. Appl. Clifford Algebra* **24** (2014), 1131–1143.

[10] Sato, M., Theory of hyperfunctions I, II, *J. Fac. Sci. Univ. Tokyo*, **8** (1959), 139–193.

[11] Sato, M., Kawai, T., Kashiwara, M., Microfunctions and pseudo-differential equations, *Hyperfunctions and Pseudo-Differential Equations*, Lecture Notes in Math. **287** (1973), 265–529.

[12] Vajiac, A., Vajiac, M., Multicomplex hyperfunctions, *Complex Var. Elliptic Equ.* **57** (2012), 751–762.