An extension of contiguity relations of the confluent hypergeometric system of Kummer type

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Abstract

The generalized confluent hypergeometric theory by Kimura, Haraoka, and Takano includes systems of differential equations, functions as solutions of them, and the contiguity relations both for the systems and for the functions. We report our recent study on an extension of that theory without the proofs. The details will be published in a forthcoming paper.

1 Introduction

Classical hypergeometric and confluent hypergeometric functions can be expressed in an integral representation whose integrand is a multiplication of exponentials and power functions with an exponent of each. These exponents are special ones for their base, so one has to treat each function as a distinguish stuff. For example, a classical Kummer function has an integral representation

$$y(a, c; x) = \int_0^1 e^{xu} u^{a-1} (1-u)^{c-a-1} du, \tag{1.1}$$

which has two paremeters a and c to give the integrand consisted of the three factors with the exponents xu, $a-1$, and $c-a-1$.

Kimura, Haraoka, and Takano[3] gave a solution to this difficulty. They, based on the idea of Aomoto[1] and Gelfand[2], introduced a family of functions, the generalized confluent hypergeometric functions, which can be considered to be an extension from classical hypergeometric and confluent hypergeometric functions. If one use a generalized confluent hypergeometric function, the expression (1.1) is written as

$$\Phi_{(2,1,1)}(z;\alpha;\gamma)=\int_{\gamma}(tz_1)^{\alpha_1}e^{\alpha_2\frac{tz_2}{tz_1}}(tz_3)^{\alpha_3}(tz_4)^{\alpha_4}(t_0dt_1-t_1dt_0), \tag{1.2}$$

where $z=(z_1\ z_2\ z_3\ z_4)$ is a 2×4 matrix in the set $Z_{(2,1,1)}$ defined by some way. The notation $\alpha=(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ is a parameter vector of size 4, which has a restriction of $\alpha_1+\alpha_3+\alpha_4=-2$. By lifting up $y(a,c;x)$ to $\Phi_{(2,1,1)}(z;\alpha;\gamma)$, we can see the parameters from an equal perspective. The parameters a and c in (1.1) has a different role in the integrand, while α_3 and α_4 in (1.2) can be seen as the two which have the same duty. This symmetry is due to how to construct $\Phi_{(2,1,1)}(z;\alpha;\gamma)$ from a Lie group $H_{(2,1,1)}$.

Kimura, Haraoka, and Takano[4] gave a system of differential equations $M_{r,\lambda}(\alpha)$, where the notation λ is a partition of a natural number $N\in\mathbb{Z}_{\geq r}$, $r\in\mathbb{Z}_{\geq 2}$ and α is a parameter vector of size N. This system has a generalized confluent hypergeometric function $\Phi_{r,\lambda}(z;\alpha;\gamma)$ as its solution, where the variable z is an $r\times N$ matrix.

The form $r\times N$ of the matrix variable z in $\Phi_{r,\lambda}(z;\alpha;\gamma)$ comes from the corresponding integrand of a classical function which has the $r-1$ variables $(u_1,u_2,\cdots,u_{r-1})\in\mathbb{C}^{r-1}$. The integration $\Phi_{r,\lambda}(z;\alpha;\gamma)$ of a function $\chi_{r,\lambda}(tz;\alpha)$ is a Radon transform of $\chi_{r,\lambda}(h;\alpha)$. The integrand $\chi_{r,\lambda}(tz;\alpha)$ is defined on the projective space $\mathbb{P}^{r-1}$ with the homogeneous variable $t=(t_0,t_1,t_2,\cdots,t_{r-1})\in\mathbb{C}^r$, $(t_0,t_1,t_2,\cdots,t_{r-1})\neq(0,0,0.,\cdots,0)$, where r corresponds to the number of row vectors of the matrix z. The special case $\Phi_{(2,1,1)}(z;\alpha;\gamma)$ in (1.2) has a 2×4 variable z since a Kummer function (1.1) is the corresponding one whose integrand has the only one variable $u\in\mathbb{C}$.

They also gave contiguity relations for the solution space $\mathcal{S}_{r,\lambda}(\alpha)$ with respect to a differential operator

$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}:\mathcal{S}_{r,\lambda}(\alpha)\to\mathcal{S}_{r,\lambda}(\alpha+\varepsilon^{(k)}-\varepsilon^{(l)}). \tag{1.3}$$

Since $\Phi_{r,\lambda}(z;\alpha;\gamma)$ is an element of $\mathcal{S}_{r,\lambda}(\alpha)$, there is a contiguity relation also for this function which has the same shift of the parameter as the relation (1.3):

$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}\Phi_{r,\lambda}(z;\alpha;\gamma)=\alpha^{(l)}_{n_l-1}\Phi_{r,\lambda}(z;\alpha+\varepsilon^{(k)}-\varepsilon^{(l)};\gamma). \tag{1.4}$$

If (1.3) and (1.4) are reduced to the case of Kummer type, both relations become

$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}: \mathcal{S}_{(2,1,1)}(\alpha) \to \mathcal{S}_{(2,1,1)}(\alpha + \varepsilon^{(k)} - \varepsilon^{(l)}), \tag{1.5}$$

$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \Phi_{(2,1,1)}(z;\alpha;\gamma) = \alpha^{(l)}_{n_l-1} \Phi_{(2,1,1)}(z;\alpha + \varepsilon^{(k)} - \varepsilon^{(l)};\gamma). \tag{1.6}$$

A root subspace $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}} \subset \mathfrak{gl}(N)$ is determined with respect to a Lie subalgebra $\mathfrak{h}_\lambda$ of $\mathfrak{gl}(N)$. They took an eigenvector $E_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ from $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ to give a linear differential operator $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ and got the results of (1.3) and (1.4).

In this article, we are going to see how the relations (1.5) and (1.6) are extended. While Kumira, Haraoka, and Takano[4] used only an eigenvector $E_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ to make an operator, we will use all the linearly independent vectors in the generalized eigenspace $\hat{\mathfrak{g}}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ corresponding to $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$. By this method, a square matrix of size 2, $\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ will be given as an extension of $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$. We will give an extended system $\hat{M}^{[2]}_{(2,1,1)}(\alpha,\beta)$, its solution space $\hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta)$, and a function $\hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha,\beta;\gamma)$, which correspond to $M_{(2,1,1)}(\alpha)$, $\mathcal{S}_{(2,1,1)}(\alpha)$, and $\Phi_{(2,1,1)}(z;\alpha;\gamma)$, respectively. Here we take two parameter vectors α and β in the extension.

2 Generalized confluent hypergeometric theory

2.1 Root space decomposition

Let us consider the linear map ad_X defined by

$$\mathrm{ad}_X : \mathfrak{gl}(4) \ni Y \mapsto [X,Y] \in \mathfrak{gl}(4),$$

where, as an element in the linear subspace of $\mathfrak{gl}(4)$, X is a square matrix of size 4 whose entries are complex numbers:

$$X \in \mathfrak{h}_{(2,1,1)} = \left\{ \left(\begin{array}{cc|c|c} X_1 & X_2 & & \\ & X_1 & & \\ \hline & & X_3 & \\ \hline & & & X_4 \end{array} \right) \right\}.$$

Since the vector space $\mathfrak{h}_{(2,1,1)}$ is an abelian Lie subalgebra of $\mathfrak{gl}(4)$, the family of the linear maps $\{\mathrm{ad}_X\}_{X\in\mathfrak{h}_{(2,1,1)}}$ is commutative. Hence, a subspace of $\mathfrak{gl}(4)$ is decomposed into its simultaneous eigenspaces with respect to $\{\mathrm{ad}_X\}_{X\in\mathfrak{h}_{(2,1,1)}}$:

$$\mathfrak{g}_0 + \sum_{1\le k\ne l\le 3} \mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}} \subset \mathfrak{gl}(4), \tag{2.1}$$

where the space $\mathfrak{g}_0 = \mathfrak{h}_{(2,1,1)}$ corresponds to the eigenvalue 0, and the space $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ the eigenvalue $(\varepsilon^{(k)} - \varepsilon^{(l)})(X)$, $X \in \mathfrak{h}_{(2,1,1)}$, $1 \le k \ne l \le 3$. The notation $\varepsilon^{(k)}$, $1 \le k \le 3$ expresses the linear map which sends the value as the common elements X_j, $j = 1,3,4$ in the k-th diagonal block of $\mathfrak{h}_{(2,1,1)}$ into the space of the complex numbers $\mathbb{C}$:

$$\varepsilon^{(1)}(X) = X_1, \varepsilon^{(2)}(X) = X_3, \varepsilon^{(3)}(X) = X_4,$$

$$X = \left(\begin{array}{cc|c|c} X_1 & X_2 & & \\ & X_1 & & \\ \hline & & X_3 & \\ \hline & & & X_4 \end{array}\right) = \left(\begin{array}{cc|c|c} X_0^{(1)} & X_1^{(1)} & & \\ & X_0^{(1)} & & \\ \hline & & X_0^{(2)} & \\ \hline & & & X_0^{(3)} \end{array}\right).$$

The linear map $\varepsilon^{(k)} - \varepsilon^{(l)}$ expressing the nonzero eigenvalue is called a root, so the space $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ is said to be a root subspace of $\mathfrak{gl}(4)$. Thus, we have obtained the root subspace decomposition (2.1).

On one hand, the space $\mathfrak{g}_0$ is spanned by the four vectors

$$\left(\begin{array}{cc|c|c} 1 & & & \\ & 1 & & \\ \hline & & & \\ \hline & & & \end{array}\right), \left(\begin{array}{cc|c|c} & 1 & & \\ & & & \\ \hline & & & \\ \hline & & & \end{array}\right), \left(\begin{array}{cc|c|c} & & & \\ & & & \\ \hline & & 1 & \\ \hline & & & \end{array}\right), \left(\begin{array}{cc|c|c} & & & \\ & & & \\ \hline & & & \\ \hline & & & 1 \end{array}\right),$$

so let us use these vectors as its base. On the other hand, we are interested in what form the subspace $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ has. This space has only one nonzero entry at the position of the first row and the last column in the (k,l) block. For example, the space $\mathfrak{g}_{\varepsilon^{(1)}-\varepsilon^{(2)}}$ is

$$\mathfrak{g}_{\varepsilon^{(1)}-\varepsilon^{(2)}} = \left\{\left(\begin{array}{cc|c|c} & & * & \\ & & 0 & \\ \hline & & & \\ \hline & & & \end{array}\right)\right\}.$$

The eigenvector $E_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ in $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ can be taken as the square matrix of size 4 with the only one nonzero entry 1 at the same position of the nonzero entry of the space:

$$E_{\varepsilon^{(1)}-\varepsilon^{(2)}} = \left(\begin{array}{cc|c|c} & & 1 & \\ & & 0 & \\ \hline & & & \\ \hline & & & \end{array}\right).$$

Therefore, the space in the left hand side in (2.1) can be expressed as

$$\mathfrak{g}_0 + \sum_{1\le k\ne l\le 3} \mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}} = \left\{ \left(\begin{array}{cc|c|c} X_1 & X_2 & * & * \\ & X_1 & & \\ \hline \multicolumn{2}{c|}{*} & X_3 & * \\ \hline \multicolumn{2}{c|}{*} & * & X_4 \end{array} \right) \right\}.$$

2.2 Confluent hypergeometric system of Kummer type

2.2.1 The system $M_{(2,1,1)}(\alpha)$

In Section 2.1, we got the space $\mathfrak{g}_0$ and the spaces $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$. From the former, the differential operator $L_X - \alpha(X)$ in the confluent hypergeometric system of Kummer type $M_{(2,1,1)}(\alpha)$ is made, and from the latter the differential operators $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ giving a contiguity relation for the solution space $\mathcal{S}_{(2,1,1)}(\alpha)$ of the system.

The notation α, which is a row vector of size 4, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha_0^{(1)}, \alpha_1^{(1)}, \alpha_0^{(2)}, \alpha_0^{(3)}) \in \mathbb{C}^4$, means the parameter of the system $M_{(2,1,1)}(\alpha)$. However, by the same notation, let us define the linear map

$$\begin{aligned} &\alpha : \mathfrak{h}_{(2,1,1)} \ni X \\ &\mapsto \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 = \alpha_0^{(1)} X_0^{(1)} + \alpha_1^{(1)} X_1^{(1)} + \alpha_0^{(2)} X_0^{(2)} + \alpha_0^{(3)} X_0^{(3)} \in \mathbb{C}, \end{aligned}$$

$$X = \left(\begin{array}{cc|c|c} X_1 & X_2 & & \\ & X_1 & & \\ \hline & & X_3 & \\ \hline & & & X_4 \end{array} \right) = \left(\begin{array}{cc|c|c} X_0^{(1)} & X_1^{(1)} & & \\ & X_0^{(1)} & & \\ \hline & & X_0^{(2)} & \\ \hline & & & X_0^{(3)} \end{array} \right).$$

If we consider α as this map, we use this notation as an element of the dual space: $\alpha \in \mathfrak{h}^*_{(2,1,1)}$.

By the matrices $X \in \mathfrak{h}_{(2,1,1)}$ and $Y \in \mathfrak{gl}(2)$, let us define the differential operators which operate on a function F whose variable is a 2×4 matrix $z \in \mathrm{Mat}(2,4)$:

$$L_X F(z) = \frac{d}{ds} F(z e^{sX}) \bigg|_{s=0}, \quad M_X F(z) = \frac{d}{ds} F(e^{sY} z) \bigg|_{s=0}.$$

Moreover, we define the differential operator

$$\Box_{p,q} = \frac{\partial}{\partial z_{0,p}} \frac{\partial}{\partial z_{1,q}} - \frac{\partial}{\partial z_{0,q}} \frac{\partial}{\partial z_{1,p}} \quad (1 \le p, q \le 4).$$

By these operators and a linear map $\alpha \in \mathfrak{h}^*_{(2,1,1)}$, and by the notation $\mathrm{Tr}(\cdot)$ such that

$$\mathrm{Tr}(Y) = a + d, \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the confluent hypergeometric system of Kummer type $M_{(2,1,1)}(\alpha)$ is defined by the next definition in Kimura, Haraoka, and Takano[3].

Definition 2.1. Let $\alpha \in \mathfrak{h}^*_{(2,1,1)}$ be a linear map and let $F(z), z \in \mathrm{Mat}(2,4)$ be a function. The confluent hypergeometric system $M_{(2,1,1)}(\alpha)$ is defined by

$$\begin{array}{lcll} \{L_X - \alpha(X)\} & F & = & 0 \qquad (X \in \mathfrak{h}_{(2,1,1)}), \\ \{M_Y + \mathrm{Tr}(Y)\} & F & = & 0 \qquad (Y \in \mathfrak{gl}(2)), \\ \Box_{p,q} & F & = & 0 \qquad (1 \le p, q \le 4). \end{array}$$

Let us denote by $\mathcal{S}_{(2,1,1)}(\alpha)$ the set of the solutions in a neighborhood of a fixed point z°.

They also gave the system $M_{r,\lambda}(\alpha)$ for an arbitrary natural number $r \in \mathbb{Z}_{\ge 2}$ and for an arbitrary partition λ of a natural number $N \in \mathbb{Z}_{\ge r}$.

Definition 2.2. Let r and N be natural numbers satisfying $2 \le r \le N$, and $\lambda = (n_1, n_2, \cdots, n_\ell)$ a partition of N, i.e., $n_1 \ge n_2 \ge \cdots \ge n_\ell$ and $n_1 + n_2 + \cdots + n_\ell = N$. We denote an $r \times N$ matrix z by

$$(2.2) \quad z = \begin{pmatrix} z_{0,1} & z_{0,2} & \cdots & z_{0,N} \\ z_{1,1} & z_{1,2} & \cdots & z_{1,N} \\ \vdots & \vdots & & \vdots \\ z_{r-1,1} & z_{r-1,2} & \cdots & z_{r-1,N} \end{pmatrix}$$
$$= (z_1 \ z_2 \ \cdots \ z_N) = (z^{(1)} \ z^{(2)} \ \cdots \ z^{(\ell)}) \in \mathrm{Mat}(r, N),$$

$$z^{(k)} = \begin{pmatrix} z^{(k)}_{0,0} & z^{(k)}_{0,1} & \cdots & z^{(k)}_{0,n_k-1} \\ z^{(k)}_{1,0} & z^{(k)}_{1,1} & \cdots & z^{(k)}_{1,n_k-1} \\ \vdots & \vdots & & \vdots \\ z^{(k)}_{r-1,0} & z^{(k)}_{r-1,1} & \cdots & z^{(k)}_{r-1,n_k-1} \end{pmatrix} \in \mathrm{Mat}(r, n_k), \quad 1 \le k \le \ell,$$

a Lie subalgebra $\mathfrak{h}_\lambda$ of $\mathfrak{gl}(N)$ by

$$(2.3) \qquad \mathfrak{h}_\lambda = \left\{ \left(\begin{array}{c|c|c} \mathrm{j}(n_1) & & \\ \hline & \ddots & \\ \hline & & \mathrm{j}(n_\ell) \end{array} \right) \right\} \in \mathfrak{gl}(N),$$

$$\mathfrak{j}(n_k) = \begin{pmatrix} X_0^{(k)} & X_1^{(k)} & \cdots & X_{n_k-1}^{(k)} \\ & X_0^{(k)} & \ddots & \vdots \\ & & \ddots & X_1^{(k)} \\ & & & X_0^{(k)} \end{pmatrix} \in \mathfrak{gl}(n_k), \quad 1 \le k \le \ell,$$

and a parameter vector α by

$$\alpha = (\alpha_1\ \alpha_2\ \cdots \alpha_N) = (\alpha^{(1)}\ \alpha^{(2)}\ \cdots\ \alpha^{(\ell)}) \in \mathrm{Mat}(1, N). \tag{2.4}$$

Besides, we define the differential operators $\Box_{p,q}^{s,t}$ by

$$\Box_{p,q}^{s,t} = \frac{\partial}{\partial z_{s,p}}\frac{\partial}{\partial z_{t,q}} - \frac{\partial}{\partial z_{s,q}}\frac{\partial}{\partial z_{t,p}}, \quad 0 \le s, t \le r-1,\ 1 \le p, q \le N. \tag{2.5}$$

The confluent hypergeometric system $M_{r,\lambda}(\alpha)$ for unknown $F(z)$ is defined by

$$\begin{array}{rcll} \{L_X - \alpha(X)\} & F & = & 0 \qquad (X \in \mathfrak{h}_\lambda), \\ \{M_Y + \mathrm{Tr}(Y)\} & F & = & 0 \qquad (Y \in \mathfrak{gl}(r)\ ), \\ \Box_{p,q}^{s,t} & F & = & 0 \quad (0 \le s, t \le r-1, 1 \le p, q \le N). \end{array}$$

Let us denote by $\mathcal{S}_{r,\lambda}(\alpha)$ the set of the solutions in a neighborhood of a fixed point z°.

2.2.2 The contiguity relations

In Section 2.2.1, the operator $L_X - \alpha(X)$ in the system $M_{(2,1,1)}(\alpha)$ was defined by an element $X \in \mathfrak{h}_{(2,1,1)}$. In this section, we define the operator $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ by a simultaneous eigenvector $E_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ in the space $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$. This operator gives a contiguity relation for the solution space $\mathcal{S}_{(2,1,1)}(\alpha)$.

By a square matrix of size 4, $E_{\varepsilon^{(k)}-\varepsilon^{(l)}} \in \mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$, we get $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ defined by

$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} F(z) = \frac{d}{ds} F(z e^{sE_{\varepsilon^{(k)}-\varepsilon^{(l)}}})\Big|_{s=0}$$

which operates on a function F with a 2×4 matrix variable $z \in \mathrm{Mat}(2, 4)$. Let us consider the product $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}\{L_X - \alpha(X)\}$. This can be transformed into

$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}\{L_X - \alpha(X)\} = \{L_X - (\alpha + \varepsilon^{(k)} - \varepsilon^{(l)})(X)\}L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \tag{2.6}$$

by a calculation of a bracket product of matrices. This equality is a key for a contiguity relation for the solution space $\mathcal{S}_{(2,1,1)}(\alpha)$: Take a function $F \in \mathcal{S}_{(2,1,1)}(\alpha)$ and operate the both hand sides of the equality (2.6) on this function, and we get the value 0 as

(2.7)
$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}\{L_X - \alpha(X)\}F = \{L_X - (\alpha + \varepsilon^{(k)} - \varepsilon^{(l)})(X)\}L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}F = 0.$$

From the equalities (2.7), we know that the function $G = L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}F$ is one such that

$$\{L_X - (\alpha + \varepsilon^{(k)} - \varepsilon^{(l)})(X)\}G = 0.$$

Moreover, we can confirm that

$$\begin{array}{cccl} \{M_Y + \mathrm{Tr}(Y)\} & G & = & 0 \quad (\, Y \in \mathfrak{gl}(2) \,), \\ \square_{p,q} & G & = & 0 \quad (1 \le p, q \le 4). \end{array}$$

Hence, the function G is an element of the solution space $G \in \mathcal{S}_{(2,1,1)}(\alpha + \varepsilon^{(k)} - \varepsilon^{(l)})$. We summarize this story as a theorem in the case of $(r, N) = (2, 4)$ with the partition $\lambda = (2, 1, 1)$.

Theorem 2.3 (Kimura, Haraoka, and Takano[4]; Kimura[5], p.177)**.** *The differential operator $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ gives the contiguity relation for the solution space*

$$L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} : \mathcal{S}_{(2,1,1)}(\alpha) \to \mathcal{S}_{(2,1,1)}(\alpha + \varepsilon^{(k)} - \varepsilon^{(l)}).$$

Moreover, if the parameters satisfy that

$$(\alpha^{(k)}_{n_k-1} + \delta_{1,n_k})\alpha^{(l)}_{n_l-1} \neq 0, \tag{2.8}$$

this linear map has its inverse

$$\frac{1}{(\alpha^{(k)}_{n_k-1} + \delta_{1,n_k})\alpha^{(l)}_{n_l-1}} L_{E_{\varepsilon^{(l)}-\varepsilon^{(k)}}} : \mathcal{S}_{(2,1,1)}(\alpha) \to \mathcal{S}_{(2,1,1)}(\alpha + \varepsilon^{(l)} - \varepsilon^{(k)}),$$

and so is isomorphic. Here if we use the correspondence of the entries in the parameter α

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha^{(1)}_0, \alpha^{(1)}_1, \alpha^{(2)}_0, \alpha^{(3)}_0),$$

the condition (2.8) *means that, for the pair*

$$(k,l) = \begin{array}{lll} (1,2), & (1,3), & (2,3), \\ (2,1), & (3,1), & (3,2), \end{array}$$

the inequalities

$$\begin{array}{rrr} \alpha_2\alpha_3 \neq 0, & \alpha_2\alpha_4 \neq 0, & (\alpha_3+1)\alpha_4 \neq 0, \\ (\alpha_3+1)\alpha_2 \neq 0, & (\alpha_4+1)\alpha_2 \neq 0, & (\alpha_4+1)\alpha_3 \neq 0 \end{array}$$

hold, respectively.

2.3 The generalized confluent hypergeometric function of Kummer type

In Section 2.2.2, we saw the contiguity relation for the solution space $\mathcal{S}_{(2,1,1)}(\alpha)$ of the system $M_{(2,1,1)}(\alpha)$. We can construct a function satisfying the system, which is called the generalized confluent hypergeometric function of Kummer type.

Consider the set

$$H_{(2,1,1)} = \left\{ \left(\begin{array}{cc|c|c} h_1 & \frac{h_2}{h_1} & & \\ & h_1 & & \\ \hline & & h_3 & \\ \hline & & & h_4 \end{array} \right) : h_1, h_3, h_4 \neq 0 \right\},$$

which is one of the Cartan subgroups of GL(4). We write the universal cover of this group as $\tilde{H}_{(2,1,1)}$. Then, as the character of the group $\tilde{H}_{(2,1,1)}$, we get the function

$$\begin{array}{lcll} \chi_{(2,1,1)}(h;\alpha) & = & h_1^{\alpha_1} e^{\alpha_2 \frac{h_2}{h_1}} h_3^{\alpha_3} h_4^{\alpha_4}, & \\ h & = & (h_1,h_2,h_3,h_4) & \in \tilde{\mathbb{C}}^\times \times \mathbb{C} \times \tilde{\mathbb{C}}^\times \times \tilde{\mathbb{C}}^\times \\ \alpha & = & (\alpha_1,\alpha_2,\alpha_3,\alpha_4) & \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}, \end{array}$$

where we assign the condition $\alpha_1+\alpha_3+\alpha_4 = -2$ to the parameter α: the map $\chi_{(2,1,1)}(h;\alpha) : \tilde{H}_{(2,1,1)} \to \mathbb{C}^\times$ is a group homomorphism. From $\chi_{(2,1,1)}(h;\alpha)$, the function on the projective line

$$\chi_{(2,1,1)}(tz;\alpha) = (tz_1)^{\alpha_1} e^{\alpha_2 \frac{tz_2}{tz_1}} (tz_3)^{\alpha_3} (tz_4)^{\alpha_4}$$

is obtained, where $t = (t_0, t_1) \in \mathbb{P}^1$ is a homogeneous variable of this function, and the coefficients of t_0 and t_1 consist of the entries in the 2×4 matrix

$$z \in Z_{(2,1,1)} = \left\{ (z_1\ z_2\ z_3\ z_4) : \det(z_i, z_j) \neq 0, \ \begin{matrix} (i,j) = (1,2), \\ (1,3), (1,4), (3,4) \end{matrix} \right\},$$

$$(z_1\ z_2\ z_3\ z_4) = \begin{pmatrix} z_{0,1} & z_{0,2} & z_{0,3} & z_{0,4} \\ z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} \end{pmatrix}.$$

By integrating $\chi_{(2,1,1)}(tz; \alpha)$ along an adequate path γ which has zero points of the function $\chi_{(2,1,1)}(tz; \alpha)$, $[t_0 : t_1] = [-z_{1,1} : z_{0,1}]$, $[-z_{1,3} : z_{0,3}]$, $[-z_{1,4} : z_{0,4}]$, as its two end points, we get the generalized confluent hypergeometric function of Kummer type

$$\Phi_{(2,1,1)}(z; \alpha; \gamma) = \int_\gamma \chi_{(2,1,1)}(tz; \alpha)(t_0 dt_1 - t_1 dt_0).$$

This function was defined by [3] in a general form.

Remark 2.4. The conditions $\det(z_i, z_j) \neq 0$, $(i,j) = (1,2), (1,3), (1,4), (3,4)$ for the set $Z_{(2,1,1)}$ are assumed in order not to degenerate the function $\chi_{(2,1,1)}(tz; \alpha)$. For example, if $\det(z_1, z_2) = 0$, then there exists a constant $s_0 \in \mathbb{C}$ such that $tz_2 = s_0 tz_1$ and we have

$$\chi_{(2,1,1)}(tz; \alpha) = (tz_1)^{\alpha_1} e^{\alpha_2 s_0} (tz_3)^{\alpha_3} (tz_4)^{\alpha_4}.$$

For this integrand, $\Phi_{(2,1,1)}(z; \alpha; \gamma)$ can be considered as a generalization from a classical Beta function. On the other hand, if $\det(z_1, z_3) = 0$, then $\chi_{(2,1,1)}(tz; \alpha)$, for some constant $s_0 \in \mathbb{C}$, degenerates into

$$\chi_{(2,1,1)}(tz; \alpha) = s_0^{\alpha_3} (tz_1)^{\alpha_1 + \alpha_3} e^{\alpha_2 \frac{tz_2}{tz_1}} (tz_4)^{\alpha_4},$$

so $\Phi_{(2,1,1)}(z; \alpha; \gamma)$ for this integrand can be seen as a generalized form of a classical Gamma function.

The function $\Phi_{(2,1,1)}(z; \alpha; \gamma)$ also satisfies a contiguity relation. The next theorem, given by Kimura, Haraoka, and Takano[4], is a special case that $(r, N) = (2, 4)$ and $\lambda = (2, 1, 1)$.

Theorem 2.5. *The differential operator* $L_{E_{\varepsilon^{(k)} - \varepsilon^{(l)}}}$ *gives the contiguity relation for the function* $\Phi_{(2,1,1)}(z; \alpha; \gamma)$ *as*

$$L_{E_{\varepsilon^{(k)} - \varepsilon^{(l)}}} \Phi_{(2,1,1)}(z; \alpha; \gamma) = \alpha^{(l)}_{n_l - 1} \Phi_{(2,1,1)}(z; \alpha + \varepsilon^{(k)} - \varepsilon^{(l)}; \gamma).$$

3 The extended system

The system $M_{(2,1,1)}(\alpha)$ defined in Section 2.2.1 was made by the scalar operators $L_X - \alpha(X)$, $M_Y + \mathrm{Tr}(Y)$, and $\Box_{p,q}$. In this section, we take operators for vector valued functions and define a system which can be considered as a generalization of $M_{(2,1,1)}(\alpha)$.

3.1 The extended system $\hat{M}^{[2]}_{(2,1,1)}(\alpha,\beta)$

As the extended operators from $L_X - \alpha(X)$, $M_Y + \mathrm{Tr}(Y)$, and $\Box_{p,q}$, we define the followings.

Definition 3.1. Let us define the three differential operators by

$$\hat{L}^{[2]}_X(\alpha,\beta) = \begin{pmatrix} L_X - \alpha(X) & -\beta(X) \\ & L_X - \alpha(X) \end{pmatrix},$$
$$\hat{M}^{[2]}_Y = \begin{pmatrix} M_Y + \mathrm{Tr}(Y) & \\ & M_Y + \mathrm{Tr}(Y) \end{pmatrix},$$
$$\hat{\Box}^{[2]}_{p,q} = \begin{pmatrix} \Box_{p,q} & \\ & \Box_{p,q} \end{pmatrix},$$

where $\beta \in \mathfrak{h}^*_{(2,1,1)}$, with the condition of $\beta_1 + \beta_3 + \beta_4 = 0$, is a linear map defined by

$$\begin{aligned}\beta(X) &= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 \\ &= \beta^{(1)}_0 X^{(1)}_0 + \beta^{(1)}_1 X^{(1)}_1 + \beta^{(2)}_0 X^{(2)}_0 + \beta^{(3)}_0 X^{(3)}_0, \quad \beta_j \in \mathbb{C}, 1 \le j \le 4\end{aligned}$$

for a square matrix of size 4,

$$X = \left(\begin{array}{cc|c|c} X_1 & X_2 & & \\ & X_1 & & \\ \hline & & X_3 & \\ \hline & & & X_4 \end{array}\right) = \left(\begin{array}{cc|c|c} X^{(1)}_0 & X^{(1)}_1 & & \\ & X^{(1)}_0 & & \\ \hline & & X^{(2)}_0 & \\ \hline & & & X^{(3)}_0 \end{array}\right) \in \mathfrak{h}_{(2,1,1)}.$$

The superscript [2] of the operators in Definition 3.1 means that the size is 2 as a square matrix. By these operators, an extended system from $M_{(2,1,1)}(\alpha)$ is defined.

Definition 3.2. Let $\alpha, \beta \in \mathfrak{h}^*_{(2,1,1)}$ be linear maps and let $\hat{F}^{[2]}(z)$, $z \in \mathrm{Mat}(2,4)$ be a vector-valued function in the form of a column vector of size 2. We define the system of differential equations $\hat{M}^{[2]}_{(2,1,1)}(\alpha, \beta)$ by

$$\begin{array}{ccccl} \hat{L}^{[2]}_X(\alpha,\beta) & \hat{F}^{[2]} & = & 0 & (X \in \mathfrak{h}_{(2,1,1)}), \\ \hat{M}^{[2]}_Y & \hat{F}^{[2]} & = & 0 & (\,Y \in \mathfrak{gl}(2)\ ), \\ \hat{\Box}^{[2]}_{p,q} & \hat{F}^{[2]} & = & 0 & (1 \le p, q \le 4). \end{array}$$

Let us denote by $\hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha, \beta)$ the set of the solutions in a neighborhood of a fixed point z°.

In Definition 3.2, we have defined a new system. For this extended one, we are going to define the two operators $\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ and $\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ to give contiguity relations. While the operator $L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ defined in Section 2.2.2 was made only from a simultaneous eigenvector $E_{\varepsilon^{(k)}-\varepsilon^{(l)}}$, in this section, a generalized simultaneous eigenvector $\tilde{E}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ is also used to make $\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$. However, another one $\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ is made only from an eigenvector. The superscripts [2] means that these operators have the form of a square matrix of size 2.

The generalized simultaneous eigenvectors are the square matrices of size 4 which span a generalized simultaneous eigenspace of $\mathfrak{gl}(4)$ with respect to $\{\mathrm{ad}_X\}_{X \in \mathfrak{h}_{(2,1,1)}}$, $\hat{\mathfrak{g}}_0$ or $\hat{\mathfrak{g}}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$, where the space $\hat{\mathfrak{g}}_0$ corresponds to the space $\mathfrak{g}_0$, and $\hat{\mathfrak{g}}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ corresponds to $\mathfrak{g}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$. The direct sum of those spaces is equal to the whole of $\mathfrak{gl}(4)$ as a vector space:

$$\hat{\mathfrak{g}}_0 + \sum_{1 \le k \ne l \le 3} \hat{\mathfrak{g}}_{\varepsilon^{(k)}-\varepsilon^{(l)}} = \mathfrak{gl}(4).$$

The generalized eigenvectors $\tilde{E}_{\varepsilon^{(k)}-\varepsilon^{(l)}} \in \hat{\mathfrak{g}}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ have the form

$$\tilde{E}_{\varepsilon^{(1)}-\varepsilon^{(2)}} = \left(\begin{array}{cc|c|c} & & 0 & \\ & & 1 & \\ \hline & & & \\ \hline & & & \end{array}\right), \quad \tilde{E}_{\varepsilon^{(1)}-\varepsilon^{(3)}} = \left(\begin{array}{cc|c|c} & & & 0 \\ & & & 1 \\ \hline & & & \\ \hline & & & \end{array}\right),$$

$$\tilde{E}_{\varepsilon^{(2)}-\varepsilon^{(1)}} = \left(\begin{array}{cc|c|c} & & & \\ & & & \\ \hline 1 & 0 & & \\ \hline & & & \end{array}\right), \quad \tilde{E}_{\varepsilon^{(3)}-\varepsilon^{(1)}} = \left(\begin{array}{cc|c|c} & & & \\ & & & \\ \hline & & & \\ \hline 1 & 0 & & \end{array}\right).$$

By the preparation above, the extended operators are

$$
\begin{aligned}
\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} &= \begin{pmatrix} L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} & L_{\tilde{E}_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \\ & L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \end{pmatrix}, \quad (k,l) = (1,2),(1,3),(2,1),(3,1), \\
\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} &= \begin{pmatrix} L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} & \\ & L_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \end{pmatrix}, \qquad\qquad 1 \le k \ne l \le 3.
\end{aligned}
$$

By these operators, the contiguity relations for the solution space $\hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta)$ are given as the next theorem.

Theorem 3.3. *The differential operators* $\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ *and* $\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ *give the contiguity relations for the solution space*

$$
\begin{aligned}
\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} &: \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta) \to \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha+\varepsilon^{(k)}-\varepsilon^{(l)}, \beta+\varepsilon^{(1)}_1), \quad (k,l) = (1,2),(1,3), \\
\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} &: \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta) \to \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha+\varepsilon^{(k)}-\varepsilon^{(l)}, \beta-\varepsilon^{(1)}_1), \quad (k,l) = (2,1),(3,1), \\
\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} &: \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta) \to \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha+\varepsilon^{(k)}-\varepsilon^{(l)}, \beta), \qquad 1 \le k \ne l \le 3.
\end{aligned}
$$

Moreover, if the parameters satisfy the condition (2.8)*, these linear maps have their inverses*

$$
\begin{aligned}
(\hat{C}^{[2]}_{(k,l)})^{-1}\hat{L}^{[2]}_{E_{\varepsilon^{(l)}-\varepsilon^{(k)}}} &: \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta) \to \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha+\varepsilon^{(l)}-\varepsilon^{(k)}, \beta-\varepsilon^{(1)}_1), (k,l) = (1,2),(1,3), \\
(\hat{C}^{[2]}_{(k,l)})^{-1}\hat{L}^{[2]}_{E_{\varepsilon^{(l)}-\varepsilon^{(k)}}} &: \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta) \to \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha+\varepsilon^{(l)}-\varepsilon^{(k)}, \beta+\varepsilon^{(1)}_1), (k,l) = (2,1),(3,1), \\
(\hat{C}^{[2],\mathrm{eigen}}_{(k,l)})^{-1}\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(l)}-\varepsilon^{(k)}}} &: \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha,\beta) \to \hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha+\varepsilon^{(l)}-\varepsilon^{(k)}, \beta), \qquad 1 \le k \ne l \le 3.
\end{aligned}
$$

and so are isomorphic, where $\hat{C}^{[2]}_{(k,l)}$, $(k,l) = (1,2),(1,3),(2,1),(3,1)$ *and* $\hat{C}^{[2],\mathrm{eigen}}_{(k,l)}$, $1 \le k \ne l \le 3$ *are square matrices of size* 2 *defined by*

$$
\begin{aligned}
\hat{C}^{[2]}_{(1,2)} &= \begin{pmatrix} \alpha_2\alpha_3 & \alpha_2\beta_3+(\alpha_1+\beta_2+2)\alpha_3 \\ & \alpha_2\alpha_3 \end{pmatrix}, &
\hat{C}^{[2]}_{(2,1)} &= \begin{pmatrix} (\alpha_3+1)\alpha_2 & (\alpha_3+1)(\alpha_1+\beta_2)+\beta_3\alpha_2 \\ & (\alpha_3+1)\alpha_2 \end{pmatrix}, \\
\hat{C}^{[2]}_{(1,3)} &= \begin{pmatrix} \alpha_2\alpha_4 & \alpha_2\beta_4+(\alpha_1+\beta_2+2)\alpha_4 \\ & \alpha_2\alpha_4 \end{pmatrix}, &
\hat{C}^{[2]}_{(3,1)} &= \begin{pmatrix} (\alpha_4+1)\alpha_2 & (\alpha_4+1)(\alpha_1+\beta_2)+\beta_4\alpha_2 \\ & (\alpha_4+1)\alpha_2 \end{pmatrix}, \\
\hat{C}^{[2],\mathrm{eigen}}_{(1,2)} &= \begin{pmatrix} \alpha_2\alpha_3 & \alpha_2\beta_3+\beta_2\alpha_3 \\ & \alpha_2\alpha_3 \end{pmatrix}, &
\hat{C}^{[2],\mathrm{eigen}}_{(2,1)} &= \begin{pmatrix} (\alpha_3+1)\alpha_2 & (\alpha_3+1)\beta_2+\beta_3\alpha_2 \\ & (\alpha_3+1)\alpha_2 \end{pmatrix}, \\
\hat{C}^{[2],\mathrm{eigen}}_{(1,3)} &= \begin{pmatrix} \alpha_2\alpha_4 & \alpha_2\beta_4+\beta_2\alpha_4 \\ & \alpha_2\alpha_4 \end{pmatrix}, &
\hat{C}^{[2],\mathrm{eigen}}_{(3,1)} &= \begin{pmatrix} (\alpha_4+1)\alpha_2 & (\alpha_4+1)\beta_2+\beta_4\alpha_2 \\ & (\alpha_4+1)\alpha_2 \end{pmatrix}, \\
\hat{C}^{[2],\mathrm{eigen}}_{(2,3)} &= \begin{pmatrix} (\alpha_3+1)\alpha_4 & (\alpha_3+1)\beta_4+\beta_3\alpha_4 \\ & (\alpha_3+1)\alpha_4 \end{pmatrix}, &
\hat{C}^{[2],\mathrm{eigen}}_{(3,2)} &= \begin{pmatrix} (\alpha_4+1)\alpha_3 & (\alpha_4+1)\beta_3+\beta_4\alpha_3 \\ & (\alpha_4+1)\alpha_3 \end{pmatrix}.
\end{aligned}
$$

3.2 The extended function

In Section 3.1, we defined the system $\hat{M}^{[2]}_{(2,1,1)}(\alpha,\beta)$ as an extended one from $M_{(2,1,1)}(\alpha)$. In this section, we give a function satisfying $\hat{M}^{[2]}_{(2,1,1)}(\alpha,\beta)$ which can be seen as an extension from the generalized confluent hypergeometric function of Kummer type $\Phi_{(2,1,1)}(z;\alpha;\gamma)$.

Definition 3.4. Let us define a function by

$$\tilde{\Phi}_{(2,1,1)}(z;\alpha,\beta;\gamma) := \int_\gamma \chi_{(2,1,1)}(tz;\alpha)\log \chi_{(2,1,1)}(tz;\beta)(t_0dt_1 - t_1dt_0),$$

where the parameter $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ satisfies $\beta_1 + \beta_3 + \beta_4 = 0$.

Definition 3.5. A vector-valued function of size 2, $\hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha,\beta;\gamma)$, is defined by

$$\hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha,\beta;\gamma) := \begin{pmatrix} \tilde{\Phi}_{(2,1,1)}(z;\alpha,\beta;\gamma) \\ \Phi_{(2,1,1)}(z;\alpha;\gamma) \end{pmatrix}.$$

The superscript [2] indicates that the size of the column vector is 2.

Theorem 3.6. *The operators* $\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ *and* $\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$ *give the contiguity relations for the function* $\hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha,\beta;\gamma)$*:*

$$\begin{aligned}
\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha,\beta;\gamma) &= \hat{C}^{[2],(k,l)} \hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha+\varepsilon^{(k)}-\varepsilon^{(l)},\beta+\varepsilon^{(1)}_1;\gamma), (k,l)=(1,2),(1,3),\\
\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha,\beta;\gamma) &= \hat{C}^{[2],(k,l)} \hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha+\varepsilon^{(k)}-\varepsilon^{(l)},\beta-\varepsilon^{(1)}_1;\gamma), (k,l)=(2,1),(3,1),\\
\hat{L}^{[2],\mathrm{eigen}}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}} \hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha,\beta;\gamma) &= \hat{C}^{[2],\mathrm{eigen},(k,l)} \hat{\Phi}^{[2]}_{(2,1,1)}(z;\alpha+\varepsilon^{(k)}-\varepsilon^{(l)},\beta;\gamma), \qquad 1 \le k \ne l \le 3,
\end{aligned}$$

where $\hat{C}^{[2],(k,l)}, (k,l) = (1,2),(1,3),(2,1),(3,1)$ *and* $\hat{C}^{[2],\mathrm{eigen},(k,l)}, 1 \le k \ne l \le 3$ *are the matrices given by the parameters as*

$$\begin{aligned}
\hat{C}^{[2],(1,2)} &= \begin{pmatrix} \alpha_3 & \beta_3 \\ & \alpha_3 \end{pmatrix}, & \hat{C}^{[2],(1,3)} &= \begin{pmatrix} \alpha_4 & \beta_4 \\ & \alpha_4 \end{pmatrix},\\
\hat{C}^{[2],(2,1)} &= \begin{pmatrix} \alpha_2 & \alpha_1+\beta_2 \\ & \alpha_2 \end{pmatrix}, & \hat{C}^{[2],(3,1)} &= \begin{pmatrix} \alpha_2 & \alpha_1+\beta_2 \\ & \alpha_2 \end{pmatrix},\\
\hat{C}^{[2],\mathrm{eigen},(1,2)} &= \begin{pmatrix} \alpha_3 & \beta_3 \\ & \alpha_3 \end{pmatrix}, & \hat{C}^{[2],\mathrm{eigen},(2,1)} &= \begin{pmatrix} \alpha_2 & \beta_2 \\ & \alpha_2 \end{pmatrix},\\
\hat{C}^{[2],\mathrm{eigen},(1,3)} &= \begin{pmatrix} \alpha_4 & \beta_4 \\ & \alpha_4 \end{pmatrix}, & \hat{C}^{[2],\mathrm{eigen},(3,1)} &= \begin{pmatrix} \alpha_2 & \beta_2 \\ & \alpha_2 \end{pmatrix},\\
\hat{C}^{[2],\mathrm{eigen},(2,3)} &= \begin{pmatrix} \alpha_4 & \beta_4 \\ & \alpha_4 \end{pmatrix}, & \hat{C}^{[2],\mathrm{eigen},(3,2)} &= \begin{pmatrix} \alpha_3 & \beta_3 \\ & \alpha_3 \end{pmatrix}.
\end{aligned}$$

4 Summary

In this article, we gave extended contiguity relations for the solution space $\hat{\mathcal{S}}^{[2]}_{(2,1,1)}(\alpha, \beta)$ and for the function $\hat{\Phi}^{[2]}_{(2,1,1)}(z; \alpha, \beta; \gamma)$. In order to construct the contiguity operators $\hat{L}^{[2]}_{E_{\varepsilon^{(k)}-\varepsilon^{(l)}}}$, we used both of the simultaneous eigenvectors and the simultaneous generalized eigenvectors in the simultaneous generalized eigenspaces $\hat{\mathfrak{g}}_{\varepsilon^{(k)}-\varepsilon^{(l)}}$ of the vector space $\mathfrak{gl}(4)$ with respect to the family of the commutative linear maps $\{\mathrm{ad}_X\}_{X \in \mathfrak{h}_{(2,1,1)}}$.

Thus, all the linearly independent vectors in the non-diagonal blocks of the vector space $\mathfrak{gl}(4)$ with respect to the partition of the natural number 4, $(2, 1, 1)$ were made use of producing the contiguity operators. However, generalized eigenvectors in the $(1, 1)$ block, vectors in the space $\hat{\mathfrak{g}}_0$, were not used.

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