A COMBINATORIAL APPROACH TO CHOW STABILITY OF UNIFORMLY K-STABLE TORIC SURFACES

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Abstract. In [LY24a], Lee and the author proved that any K-semistable polarized smooth toric variety (X, L) with the vanishing Futaki-Ono invariant (which is an obstruction for (X, L) to be asymptotically Chow *semistable*) is asymptotically Chow polytable. Hence, this gives a complete solution for proving [Yot18, Theorem 2.1] under an assumption that (X, L) is *K-semistable* which is a weaker concept than uniform K-stability. In contrast, this note aims to clarify the relation between uniform K-stability and asymptotic Chow polystability for polarized toric *surfaces* following the author's original view point (the Euler-Maclaurin formula) which was discussed in [Yot18].

1. Introduction

In this note, we shall give a proof of the following theorem using the Euler-Maclaurin formura.

Theorem 1.1. *Let (X, L) be a uniformly K-stable polarized toric surface with the vanishing Futaki-Ono invariants. Then, (X, L) is asymptotically Chow polystable.*

See Section 2 for more detailed description and terminology. As in Remark 3.2, the proof in this note (Section 3) has not been completed yet, so something more ingenious is required. However, the idea of a proof itself was developed in the nicest form through Zhou and the author's work in [YZ19], and hence I decide to write up the entire picture of the proof in this note, which should be retained for record.

We remark that the statement of Theorem 1.1 is correct and moreover, it can be generalized as follows.

Theorem 1.2 ([LY24a], Corollary 1.5)**.** *Let (X, L) be a K-semistable polarized toric manifold with the vanishing Futaki-Ono invariants. Then, (X, L) is asymptotically Chow polystable.*

In fact, uniform K-stability implies K-semistability by definitions. Hence, Lee and the author's result in Theorem 1.2 genralizes Theorem 1.1 into higher dimensional cases with a weaker assumption. A complete proof of Thereom 1.2 is given in [LY24a, Section 4], where we used a special type of triangulations (called a *type F traiangulation*) of the associated moment polytope Δ. In a sequel to [LY24a], we show the blow-up formula of the Chow weights of a polarized toric manifold exploiting the symplectic cuts of Δ [LY24b].

The key to prove Theorem 1.2 is that we find a way of estimating the integral of a convex function $u : \Delta \to \mathbb{R}$ by the discrete summands of the weights. More specifically, let p be a vertex of a lattice polytope Δ, and let

$$C(p) := \{\, p + t(\boldsymbol{x} - p) \mid \boldsymbol{x} \in \Delta, t \in \mathbb{R}_{\geqslant 0} \,\}$$

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be its *vertex cone*. Then we have the natural decomposition

$$\Delta = \bigcap_{j=1}^{R} C(p_j), \tag{1.1}$$

where $\mathcal{V}(\Delta) = \{ p_1, \dots, p_R \}$ denotes the set of vertices of Δ. Fo a positive integer i, the decomposition (1.1) naturally yields that

$$i\Delta = \bigcap_{j=1}^{R} C(ip_j)$$

with $C(p_j) = C(ip_j) - (i-1)p_j := \{ \boldsymbol{x} - (i-1)p_j \mid \boldsymbol{x} \in C(ip_j) \}$. Thus, each cone shares the same triangulation of Δ. Recall that a *simplex triangulation* of Δ is an integral triangulation of Δ such that every simplex is isomorphic to the standard simplex. We denote a simplex triangulation of $C(ip_j)$ by $T(C(ip_j))$.

Definition 1.3. A *type F triangulation* of $C(p_j)$ is a simplex triangulation $T(C(p_j))$ satisfying the following two conditions.

(i) $T(C(p_j)) = T(C(ip_j))$ for any positive integer i; and
(ii) there exist finitely many simplices $S_1, \dots, S_M$ such that for any simplex $S \in T(C(p_j))$, we have a vector $\boldsymbol{c} \in \mathbb{Z}^n$ satisfying

$$S - \boldsymbol{c} = S_k$$

for some $1 \leqslant k \leqslant M$.

Here and hereafter, we suppose that an n-dimensional lattice polytope Δ has a type F triangulation $T(C(p_j))$ of $C(p_j)$ for each vertex $p_j \in \mathcal{V}(\Delta)$.

Let C_{S_k} and $c_n^{S_k}$ be the constants determined in [LY24a, Corollary 2.13 (2.5)]. Then we define the constants C and c_n given by

$$C := \max_{1 \leq k \leq M} C_{S_k} \qquad \text{and} \qquad c_n := \max_{1 \leq k \leq M} c_n^{S_k} \tag{1.2}$$

Fixing a vertex $p_j \in \mathcal{V}(\Delta)$ and a positive integer $i \in \mathbb{Z}_{>0}$, we denote a type F triangulation of the vertex cone $C(ip_j)$ by $T(C(ip_j))$. Then we define the function $n_{j,i}$ by

$$\begin{array}{rccc} n_{j,i} : & C(kp_i) \cap \mathbb{Z}^n & \longrightarrow & \mathbb{Z} \\ & \cup\!\!\!\!\!\!\;\; & & \cup \\ & q & \longmapsto & \#\{ S \in T(C(ip_j)) \mid S \text{ is a simplex touching } q \}. \end{array}$$

We set $n_i(q) = \max_{1 \leq j \leq R} n_{j,i}(q)$. The following lemma was given in [LY24a, Lemma 2.18] for estimating the integration $\int_{i\Delta} u(\boldsymbol{x})\, dv$ by $\sum_{p \in i\Delta \cap \mathbb{Z}^n} u(p)$.

Lemma 1.4. *For a fixed positive integer i, let $u : i\Delta \to \mathbb{R}$ be a non-negative convex function. Then we have the inequality*

$$\int_{iD} u(\boldsymbol{x})\, dv \leqslant \sum_{p \in i\Delta \cap \mathbb{Z}^n} \frac{n_i(p) u(p)}{(n+1)!} + \frac{CRc_n}{i} \max_{1 \leq k \leq R} u(p_k),$$

where C and c_n are constants defined in (1.2)*.*

Meanwhile in this note, we consider a completely different approach for comparing the integral of a convex (piecewise linear) function and the discrete summands of the weights. Namely, for a two dimensional lattice polytope Δ, we use the Euler-Maclaurin formula

$$\sum_{\boldsymbol{a} \in \Delta \cap (\mathbb{Z}/i)^2} u(\boldsymbol{a}) = i^2 \int_{\Delta} u(\boldsymbol{x})\, dv + \frac{i}{2} \int_{\partial\Delta} u(\boldsymbol{x}) d\sigma + \alpha_0,$$

and estimate the constant α_0 in terms of $\int_\Delta u(\boldsymbol{x})\,dv$ and $\int_{\partial\Delta} u(\boldsymbol{x})\,d\sigma$. Here $\partial\Delta$ denotes the boundary of Δ and $d\sigma$ is the associated Lebesgue measure of $\partial\Delta$. See Lemma 3.1 and Section 4, for more precise statement and its proof.

This paper is organized as follows. Section 2 introduces notation and convention used in this paper. We give an (incomplete) proof of Theorem 1.1 in Section 3. In the appendix (Section 4), we shall prove Lemma 3.1 which gives explicit bounds on coefficients of the Ehrhart polynomial of a two dimensional lattice polytope.

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2. Preliminary

2.1. **Set up and Notation.** Let M be a free abelian group of rank n such that $M \cong \mathbb{Z}^n$. Let Δ be an n-dimensional convex lattice polytope in $M_\mathbb{R} := M \otimes \mathbb{R} \cong \mathbb{R}^n$. We denote the associated polarized toric variety by (X_Δ, L_Δ). See [CLS11] for a reference of toric varieties. We usually consider the discrete summands of weights given by

$$\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} \boldsymbol{a} = \frac{1}{i}\sum_{\boldsymbol{a}\in i\Delta\cap\mathbb{Z}^n} \boldsymbol{a}.$$

Let $E_\Delta(t)$ be the Ehrhart polynomial of Δ which is written in the form of

$$E_\Delta(t) = \mathrm{vol}(\Delta)t^n + \frac{\mathrm{vol}(\partial\Delta)}{2}t^{n-1} + O(t^{n-2}), \qquad \text{and}$$

$$E_\Delta(i) = \#\left(\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} \boldsymbol{a}\right) = \dim H^0(X_\Delta, L_\Delta^{\otimes i})$$

for any positive integer i. Let $\mathrm{Aut}^0(X_\Delta)$ denote the identity component of the automorphism group of X_Δ. Then there is a maximal torus $T = (\mathbb{C}^\times)^n < \mathrm{Aut}^0(X_\Delta)$ by Demazure's structure theorem. Denoting the normalizer of T in $\mathrm{Aut}^0(X_\Delta)$ by $N(T)$, we define the *Weyl group* $W(X_\Delta) := N(T)/T$.

2.2. **The Futaki-Ono invariant of** $(X_\Delta, L_\Delta^{\otimes i})$**.** Let (X_Δ, L_Δ) be a polarized toric variety with the moment polytope $\Delta \subset M_\mathbb{R}$. We fix any $i \in \mathbb{Z}_{>0}$. Let $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ be any T-equivariant test configuration of $(X_\Delta, L_\Delta^{\otimes i})$ (see, [Don02]).

Theorem 2.1 (Theorem 1.1 [Ono13], Corollary 2.7, [LLSW19])**.** *In the above, the Chow weight for the degeneration $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ is given by*

$$Q_\Delta(g;i) := E_\Delta(i)\int_\Delta g\,dv - \mathrm{vol}(\Delta)\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} g(\boldsymbol{a}) \tag{2.1}$$

where g is the corresponding rational piecewise linear concave function over Δ. In particular, $(X_\Delta, L_\Delta^{\otimes i})$ is Chow polystable iff $Q_\Delta(g;i) \geq 0$ holds for any Weyl group invariant concave piecewise linear function

$$g \in \mathrm{PL}(\Delta;i)^{W(X_\Delta)} = \{\, g \in \mathrm{PL}(\Delta;i) \mid g(w\cdot\boldsymbol{x}) = g(\boldsymbol{x}) \quad \forall w \in W \,\},$$

and equality holds when and only when g is an affine linear.

Applying (2.1) to linear functions, one can see the following.

Corollary 2.2 (Corollary 4.7 [Ono13]). *If $(X_\Delta, L_\Delta^{\otimes i})$ is Chow semistable for $i \in \mathbb{Z}_{>0}$, then*

$$\mathrm{FO}_\Delta(\boldsymbol{x}; i) := E_\Delta(i) \int_\Delta \boldsymbol{x}\, dv - \mathrm{vol}(\Delta) \sum_{\boldsymbol{a} \in \Delta \cap (\mathbb{Z}/i)^n} \boldsymbol{a} \tag{2.2}$$

vanishes identically. In short, the equality

$$\sum_{\boldsymbol{a} \in \Delta \cap (\mathbb{Z}/i)^n} \boldsymbol{a} = \frac{E_\Delta(i)}{\mathrm{vol}(\Delta)} \int_\Delta \boldsymbol{x}\, dv \tag{2.3}$$

holds.

We call $\mathrm{FO}_\Delta(\boldsymbol{x}; i)$ in (2.2) the *the Futaki-Ono invariant* of $(X_\Delta, L_\Delta^{\otimes i})$. By the equality (2.3), one can see that $Q_\Delta(g; i)$ is invariant when adding an affine linear function to g, and is homogeneous with respect to g.

Proof of Theorem 2.1. Since $Q_\Delta(g; i)$ is invariant under adding a constant, we may assume $g \geqslant 0$. Let $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ be a T-equivariant toric test configuration of $(X_\Delta, L_\Delta^{\otimes i})$ and g be the corresponding piecewise linear function. Hence $\mathscr{X}$ is an $(n+1)$-dimensional toric variety with the moment polytope

$$Q_g := \{\, (\boldsymbol{x}, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mid 0 \leq \lambda \leq g(\boldsymbol{x}) \,\}$$

We observe that

$$\mathrm{vol}(Q_g) = \int_\Delta g(x)\, dv, \qquad E_{Q_g}(i) - E_\Delta(i) = \sum_{\boldsymbol{a} \in \Delta \cap (\mathbb{Z}/i)^n} g(\boldsymbol{a}) \tag{2.4}$$

In the proof of Proposition 4.2.1 in [Don02], the weight of $\mathbb{C}^\times$-action on $\bigwedge^{E_\Delta(m)} H^0(\mathscr{X}_0, \mathscr{L}^{\otimes m}|_{\mathscr{X}_0})$ is given by

$$\begin{aligned} w_m &= \dim H^0(\mathscr{X}_{Q_g}, \mathscr{L}_{Q_g}^{\otimes m}) - \dim H^0(X_\Delta, L_\Delta^{\otimes m}) \\ &= E_{Q_g}(m) - E_\Delta(m) \\ &= a_{n+1}(i) m^{n+1} + a_n(i) m^n + \dots \end{aligned}$$

where

$$a_k(i) = a_{kn} i^n + a_{k,n-1} i^{n-1} + \dots$$

Note that there are asymptotic expansions

$$E_{Q_g}(m) = \mathrm{vol}(Q_g) m^{n+1} + \mathcal{O}(m^n), \qquad E_\Delta(m) = \mathrm{vol}(\Delta) m^n + \mathcal{O}(m^{n-1})$$

by the Ehrhart theorem. As in [RT07], the Chow weight for the degeneration $(\mathscr{X}, \mathscr{L}) \to \mathbb{P}^1$ is given by the *normalized* leading coefficient of $a_{n+1}(i)$, we compute

$$\begin{aligned}
& w_m - mE_\Delta(m)\frac{w_i}{E_\Delta(i)} \\
&= (E_{Q_g}(m) - E_\Delta(m)) - mE_\Delta(m)\frac{E_{Q_g}(i) - E_\Delta(i)}{E_\Delta(i)} \\
&= \mathrm{vol}(Q_g)m^{n+1} - \mathrm{vol}(\Delta)\frac{E_{Q_g}(i) - E_\Delta(i)}{E_\Delta(i)}m^{n+1} + \mathcal{O}(m^n) \\
&= m^{n+1}\left(\int_\Delta g\, dv - \frac{\mathrm{vol}(\Delta)}{E_\Delta(i)}\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} g(\boldsymbol{a})\right) + \mathcal{O}(m^n)
\end{aligned}$$

Here we used (2.4) in the last equality. The assertion is verified. □

2.3. **Uniform K-stability.** It is crucial to see the coercivity of the K-energy map when we consider the existence problem of constant scalar curvature Kähler metrics on a certain polarized manifold (X, L). It has been conjectured that the coercivity property of the K-energy map is corresponding to **uniform K-stability** of (X, L). In [His20], this conjecture was justified in the case where (X, L) is a polarized toric manifold. The toric reduction of uniform K-stability is the following.

Definition 2.3 (Hisamoto, [His20]). Let (X_Δ, L_Δ) be a polarized toric variety with the moment polytope $\Delta \subset M_\mathbb{R}$. For a rational piecewise linear convex function u over Δ, we define

$$\mathcal{L}_\Delta(u) := \int_{\partial\Delta} u\, d\sigma - \frac{\mathrm{vol}(\partial\Delta)}{\mathrm{vol}(\Delta)}\int_\Delta u\, dv.$$

Then (X_Δ, L_Δ) is said to be *uniformly K-stable in the toric sense* if there exists a constant $\delta_\Delta > 0$ such that

$$\mathcal{L}_\Delta(u) \geqslant \delta_\Delta \left\| u \right\|_J \tag{2.5}$$

where $\left\| u \right\|_J$ is the J-norm defined as

$$\left\| u \right\|_J := \inf_\ell \left\{ \frac{1}{\mathrm{vol}(\Delta)} \int_\Delta (u + \ell)\, dv - \min_\Delta \{ u + \ell \} \right\},$$

and ℓ runs over all the affine functions.

3. A convex analytical approach for proving Theorem 1.1 using the Euler-Maclaurin formula

3.1. **Approach.** One can see that $Q_\Delta(g; i) = 0$ for affine linear functions by our assumption in Theorem 1.1 and [LY24a, Proposition 4.3]. Hence it suffices to show that for $i \gg 0$, $Q_\Delta(g; i) > 0$ when $g \in PL(\Delta; i)^{W(X_\Delta)}$ is NOT affine linear, in order to prove Theorem 1.1.

3.2. **Proof of Theorem 1.1.** We prove Theorem 1.1 in this section. Since $Q_\Delta(g; i)$ is invariant when adding an affine linear function to g, we may assume that $u = -g$ is a rational piecewise linear convex function normalized at 0 in the sense that $\inf_{\boldsymbol{x}\in\Delta} u(\boldsymbol{x}) = u(\boldsymbol{0}) = 0$, and $\int_{\partial\Delta} u\, d\sigma = 1$.

The key lemma below is an improvement of Lemma 3.3 of [ZZ08], not only it has estimates on the coefficients but also it holds for general rational piecewise linear functions. The proof is presented in Appendix (Section 4).

Lemma 3.1 (Euler-Maclaurin Formula)**.** *Assume Δ is a two dimensional convex lattice polytope and u is a nonnegative continuous function on Δ. Then, we have*

$$\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^2} u(\boldsymbol{a}) = i^2\int_\Delta u(\boldsymbol{x})\,dv + \frac{i}{2}\int_{\partial\Delta} u(\boldsymbol{x})\,d\sigma + \alpha_0 \tag{3.1}$$

where

$$\alpha_0 \geqslant -C_\Delta\left(\int_{\partial\Delta} u(\boldsymbol{x})\,d\sigma + \int_\Delta u(\boldsymbol{x})\,dv\right) \tag{3.2}$$

for some constant $C_\Delta > 0$ depending only on Δ.

Proof of Theorem 1.1. Let

$$E_\Delta(i) = \sum_{k=0}^{n} e_k i^k.$$

By Jensen's inequality, we remark that

$$\int_\Delta u(\boldsymbol{x})\;dv \geqslant u\left(\int_\Delta \boldsymbol{x}\,dv\right) \geqslant \inf_\Delta u = 0.$$

Let Δ^* be the union of the interior of Δ and the interior of its co-dimension one faces. Denote

$$\mathcal{C}_1 = \left\{u \;\middle|\; u \text{ is convex on } \Delta^* \text{ and } \int_{\partial\Delta} u < \infty\right\}.$$

Observe that there is a constant $C > 0$ which is independent of u satisfying

$$\int_\Delta u\;dv \leqslant C\int_{\partial\Delta} u\;d\sigma, \quad u \in \tilde{\mathcal{C}_1} \tag{3.3}$$

where

$$\tilde{\mathcal{C}_1} = \left\{u \in \mathcal{C}_1 \;\middle|\; \inf_{\boldsymbol{x}\in\Delta} u(\boldsymbol{x}) = u(0) = 0\right\}.$$

By Lemma 3.1 and (3.3), we have

$$\begin{aligned}\alpha_k &\geqslant -C_{n,k;\Delta}\left(\int_{\partial\Delta} u(\boldsymbol{x})\;d\sigma + \int_\Delta u(\boldsymbol{x})\;dv\right)\\ &\geqslant -C'_{n,k;\Delta}\int_{\partial\Delta} u(\boldsymbol{x})\,d\sigma.\end{aligned} \tag{3.4}$$

Setting $u = -g$ and $\alpha_k = -\beta_k$ in Lemma 3.1, we have

$$\begin{aligned} Q_\Delta(i,g) &= E_\Delta(i)\int_\Delta g(\boldsymbol{x})\,dv - \mathrm{vol}(\Delta)\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} g(\boldsymbol{a})\\ &= \left(\mathrm{vol}(\Delta)i^n + \frac{\mathrm{vol}(\partial\Delta)}{2}i^{n-1} + \sum_{k=0}^{n-2} e_k i^k\right)\int_\Delta g(\boldsymbol{x})\;dv\\ &\qquad -\mathrm{vol}(\Delta)\left(i^n\int_\Delta g(\boldsymbol{x})\;dv + \frac{i^{n-1}}{2}\int_{\partial\Delta} g(\boldsymbol{x})\;d\sigma + \sum_{k=0}^{n-2}\beta_k i^k\right)\\ &= \frac{\mathrm{vol}(\Delta)}{2}\mathcal{L}_\Delta(u)i^{n-1} + \sum_{k=0}^{n-2}\left(\alpha_k\mathrm{vol}(\Delta) - e_k\int_\Delta u(\boldsymbol{x})\,dv\right)i^k.\end{aligned}$$

Since we assumed uniform K-polystability in the toric sense, (2.5) and (3.4) imply that

$$
\begin{aligned}
Q_\Delta(i,g) \;&\geqslant\; \frac{\mathrm{vol}(\Delta)}{2}\delta_\Delta \left\|u\right\|_J i^{n-1} \\
&\qquad -\sum_{k=0}^{n-2}\left(\mathrm{vol}(\Delta)C'_{n,k;\Delta}\int_{\partial\Delta} u(\boldsymbol{x})\,d\sigma + e_k\int_\Delta u(\boldsymbol{x})\,dv\right) i^k.
\end{aligned}
$$

Since we have the inequality

$$
\begin{aligned}
\int_\Delta (u(\boldsymbol{x})+\ell(\boldsymbol{x}))\,dv \;&\geqslant\; \int_\Delta \min_{\boldsymbol{x}\in\Delta}\{\,u(\boldsymbol{x})+\ell(\boldsymbol{x})\,\}\,dv \\
&= \mathrm{vol}(\Delta)\cdot\min_{\boldsymbol{x}\in\Delta}\{\,u(\boldsymbol{x})+\ell(\boldsymbol{x})\,\},
\end{aligned}
$$

this yields

$$
\begin{aligned}
&\int_\Delta (u+\ell)\,dv - \mathrm{vol}(\Delta)\cdot\min_\Delta\{\,u+\ell\,\} \\
&\qquad \geqslant\; \mathrm{vol}(\Delta)\cdot\min_\Delta\{\,u+\ell\,\} - \mathrm{vol}(\Delta)\cdot\min_\Delta\{\,u+\ell\,\} = 0.
\end{aligned}
$$

Then the following Lemma 3.3 shows that

$$
\int_\Delta (u+\ell)\,dv - \mathrm{vol}(\Delta)\cdot\min_\Delta\{\,u+\ell\,\} > 0.
$$

This yields $\left\|u\right\|_J > 0$ for any $u\in\tilde{\mathcal{C}}_1$. Remark that

$$
Q_\Delta(i,g) \;\sim\; \frac{\delta_\Delta \mathrm{vol}(\Delta)}{2}\left\|u\right\|_J
$$

for a sufficiently large $i\gg 0$. Therefore, there exists $i_0\in\mathbb{Z}_{>0}$, such that $Q_\Delta(i,g)>0$ when $i\geqslant i_0$. The theorem has been proved. □

Remark 3.2. However, the above integer i_0 in the proof depends on the choice of a concave function $g=-u$. Therefore, our proof is incomplete and something more ingenious is required. In fact, Chow polystability of $(X_\Delta, L_\Delta^{\otimes i})$ is determined by taking various concave functions g for a fixed i. In particular, the Euler-Maclaurin formula for an n-dimensional lattice polytope Δ,

$$
\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} u(\boldsymbol{a}) = i^n\int_\Delta u(\boldsymbol{x})\,dv + \frac{i^{n-1}}{2}\int_{\partial\Delta} u(\boldsymbol{x})\,d\sigma + \sum_{k=0}^{n-2}\alpha_k i^k,
$$

where

$$
\alpha_k \geqslant -C_k\left(\int_{\partial\Delta} u(\boldsymbol{x})\,d\sigma + \int_\Delta u(\boldsymbol{x})dv\right), \quad k=0,\dots,n-2
$$

does not have good invariance under scaling of domain (cf. [Yot18, Remark 3.4]). Thus, we need to estimate each coefficient α_k $(0\leqslant k\leqslant n-2)$ in the form of $\alpha_k\geqslant -\widetilde{C}_k$, where $\widetilde{C}_k$ doesn't depend on $\int_{\partial\Delta} u(\boldsymbol{x})d\sigma$.

Lemma 3.3. *We assume that* $\mathrm{FO}_\Delta(\boldsymbol{x};i)\equiv 0$ *for* $i\gg 0$. *Then*

$$
\int_\Delta (u+\ell)\,dv - \mathrm{vol}(\Delta)\cdot\min_\Delta\{\,u+\ell\,\} \neq 0.
$$

Proof. We shall show the contraposition of the statement. Then our assumption is

$$\int_\Delta (u+\ell)\,dv - \mathrm{vol}(\Delta)\cdot \min_\Delta \{\, u+\ell\,\} = 0. \tag{3.5}$$

In order to use the contradiction, we suppose $\mathrm{FO}_\Delta(\boldsymbol{x}; i) \equiv 0$. Using (3.5) and putting $u+\ell$ to be the coordinate function $\boldsymbol{x}$, we have

$$\int_\Delta \boldsymbol{x}\,dv = \int_\Delta (u+\ell)\,dv = \mathrm{vol}(\Delta)\cdot \min_\Delta \{\, u+\ell\,\} = C\mathrm{vol}(\Delta) \tag{3.6}$$

for some constant C. Then $\mathrm{FO}_\Delta(\boldsymbol{x}; i) \equiv 0$ and (3.6) yields

$$\begin{aligned} \mathrm{FO}_\Delta(\boldsymbol{x}; i) &= E_\Delta(i)\cdot C\mathrm{vol}(\Delta) - \mathrm{vol}(\Delta) \sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} \boldsymbol{a} \\ &= \mathrm{vol}(\Delta)\left\{ E_\Delta(i)\cdot C - \sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} \boldsymbol{a} \right\} \equiv 0. \end{aligned}$$

However this means that $\mathbb{R}$-valued degree n polynomial $E_\Delta(i)$ equals to $\mathbb{R}^n$-valued degree n polynomial $\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^n} \boldsymbol{a}$ up to constants, a contradiction. ☐

4. Appendix: Bounds on coefficients of Ehrhart polynomial and their applications

In this appendix, we discuss some application of the bounds on coefficients of Ehrhart polynomial. Some key idea was already written in [Yot18, Section 3], and this section aims to give a complete proof of the Main Lemma (Lemma 4.3) concerning a two dimensional convex lattice polygon Δ.

First, we recall some useful results on Ehrhart polynomial

$$E_\Delta(i) = \sum_{k=0}^{n} e_k i^k. \tag{4.1}$$

Recall that one has

$$e_n = \mathrm{vol}(\Delta),\ e_{n-1} = \frac{\mathrm{vol}(\partial\Delta)}{2}, e_0 = 1.$$

No geometric meaning is known for the rest coefficients. However, the upper and lower bounds for them have been established by [BM85, HT09], respectively. We conclude them as follows

Theorem 4.1. *Let Δ be an n-dimensional lattice polytope and e_k are given by (4.1). Then*

(1)

$$e_k \leqslant (-1)^{n-k}\mathfrak{s}(n,k)\mathrm{vol}(\Delta) + \frac{(-1)^{n-k-1}\mathfrak{s}(n,k+1)}{(n-1)!},$$

where $\mathfrak{s}_{n,k}$ denotes the Stirling numbers of the first kind which can be defined via the identity

$$\prod_{k=0}^{n-1}(z-k) = \sum_{k=1}^{n} \mathfrak{s}(n,k) z^k.$$

(2) *If $n \geqslant 3$, then for $k = 1, ..., n-1$, we have*

$$e_k \geqslant \frac{1}{n!}\left[(-1)^{n-k}\mathfrak{s}(n+1, k+1) + (n!\mathrm{vol}(\Delta) - 1)M_{k,n}\right].$$

Here $M_{k,n}$ is given by

$$M_{k,n} = \min\left\{ C^n_{k,j} : 1 \leqslant j \leqslant n-2 \right\},$$

where $C^n_{k,j}$ is the k-th coefficient of the polynomial

$$(z+j)(z+j-1)\cdots(z+j-(n-1))$$

with variable z.

The following fact will be frequently used later.

Lemma 4.2. *If Δ is an n-dimensional lattice polytope, then*

$$\mathrm{vol}(\Delta) \geqslant \frac{1}{n!}. \tag{4.2}$$

Now we prove the lemma used in Section 3. Since a general nonnegative continuous function can be approximated by nonnegative rational piecewise linear function, then Lemma 3.1 follows by an approximation argument and the following lemma.

Lemma 4.3. *Assume Δ is a two dimensional convex lattice polytope and u is a nonnegative rational piecewise linear function. Then we have the equality*

$$\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^2} u(\boldsymbol{a}) = i^2\int_\Delta u\, dv + \frac{i}{2}\int_{\partial\Delta} u\, d\sigma + \alpha_0, \tag{4.3}$$

where

$$\alpha_0 \geqslant -C_\Delta\left(\int_{\partial\Delta} u\, d\sigma + \int_\Delta u\, dv\right), \tag{4.4}$$

for some $C_\Delta > 0$ depending on Δ.

Proof. Assume $\Delta = \bigcup_{s=1}^p \Delta_s$ such that u is linear on Δ_s. We also assume that each Δ_s is lattice polytope. Otherwise, it suffices to consider a dilation of Δ.

Let $\{ F_r \}_{r=1}^q$ be all the one dimensional walls (line segments) which are obtained by the intersection $\Delta_s \cap \Delta_{s'}$ for $s \neq s'$. We also define inductively the set of all the lattice points $\{ \boldsymbol{v}_j \}_{j=1}^\ell$ which are obtained by the intersection of distinct walls $F_r \cap F_{r'} \cap F_{r''}$. Observe that

$$E_\Delta(i) = \sum_{s=1}^p E_{\Delta_s}(i) - \sum_{r=1}^q E_{F_r}(i) + \sum_{j=1}^\ell E_{\boldsymbol{v}_j}(i) \tag{4.5}$$

by the inclusion-exclusion principle. Let us denote

$$\begin{aligned}
\mathcal{D} &= \{ (x,t) \in \mathbb{R}^{n+1} \mid x \in \Delta, 0 \leqslant t \leqslant u(\boldsymbol{x}) \}, \\
\mathcal{D}_s &= \{ (x,t) \in \mathbb{R}^{n+1} \mid x \in \Delta_s, 0 \leqslant t \leqslant u(\boldsymbol{x}) \}, \quad s = 1, \dots, p, \\
\mathcal{F}_r &= \{ (x,t) \in \mathbb{R}^{n+1} \mid x \in F_r, 0 \leqslant t \leqslant u(\boldsymbol{x}) \}, \quad r = 1, \dots, q, \\
\mathcal{G}_j &= \{ (x,t) \in \mathbb{R}^{n+1} \mid 0 \leqslant t \leqslant u(\boldsymbol{v}_j) \}, \qquad j = 1, \dots, \ell.
\end{aligned}$$

Furthermore, we assume all $\mathcal{D}_s$ are lattice polytopes. Otherwise, we consider an $i_0 u$ for some $i_0 \in \mathbb{Z}$ since (4.3) is homogeneous with respect to u. Hence u is a rational linear function on each Δ_s. We further assume that u is an *integral* linear function on any Δ_s by taking $i_0' u$ for sufficiently large $i_0' \in \mathbb{Z}$. The straightforward computation with (4.5) shows that

$$
\begin{aligned}
\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^2} u(\boldsymbol{a}) &= \sum_{\boldsymbol{a}\in i\Delta\cap\mathbb{Z}^2} u(\boldsymbol{a}/i) \\
&= \frac{1}{i}(E_{\mathcal{D}}(i) - E_{\Delta}(i)) \\
&= \frac{1}{i}\left(\sum_{s=1}^{p} E_{\mathcal{D}_s}(i) - \sum_{r=1}^{q} E_{\mathcal{F}_r}(i) + \sum_{j=1}^{\ell} E_{\mathcal{G}_j}(i)\right) \\
&\qquad -\frac{1}{i}\left(\sum_{s=1}^{p} E_{\Delta_s}(i) - \sum_{r=1}^{q} E_{F_r}(i) + \sum_{j=1}^{\ell} E_{v_j}(i)\right) \\
&= \frac{1}{i}\left(\sum_{s=1}^{p}(E_{\mathcal{D}_s}(i) - E_{\Delta_s}(i)) - \sum_{r=1}^{q}(E_{\mathcal{F}_r}(i) - E_{F_r}(i))\right. \\
&\qquad \left. + \sum_{j=1}^{\ell}(E_{\mathcal{G}_j}(i) - E_{v_j}(i))\right). \qquad (4.6)
\end{aligned}
$$

Applying the Ehrhart theorem to each polytope $\mathcal{D}_s, \Delta_s, \mathcal{F}_r, F_r, \mathcal{G}_j$ and v_j, we have

$$
\begin{aligned}
E_{\mathcal{D}_s}(i) &= i^3 \int_{\Delta_s} u\, dv + \frac{i^2}{2}\left(\int_{\partial\Delta_s} u\, d\sigma + 2\mathrm{vol}(\Delta_s)\right) + \alpha_s i + 1, \\
E_{\Delta_s}(i) &= i^2 \mathrm{vol}(\Delta_s) + \frac{i}{2}\mathrm{vol}(\partial\Delta_s) + 1, \\
E_{\mathcal{F}_r}(i) &= i^2 \int_{F_r} u\, d\sigma + \tilde{\alpha}_r i + 1, \\
E_{F_r}(i) &= \mathrm{vol}(F_r) i + 1, \\
E_{\mathcal{G}_j}(i) &= u(v_j) i + 1, \qquad \text{and} \qquad E_{v_j}(i) \equiv 1.
\end{aligned}
$$

Substituting these into (4.6), we have

$$
\begin{aligned}
\sum_{\boldsymbol{a}\in\Delta\cap(\mathbb{Z}/i)^2} u(\boldsymbol{a}) &= \sum_{s=1}^{p}\left\{ i^2 \int_{\Delta_s} u\ dv + \frac{i}{2}\int_{\partial\Delta_s} u\ d\sigma + (\alpha_s - \frac{1}{2}\mathrm{vol}(\partial\Delta_s))\right\} \\
&\qquad - \sum_{r=1}^{q}\left\{ i \int_{F_r} u\ d\sigma + \tilde{\alpha}_r - \mathrm{vol}(F_r)\right\} + \sum_{j=1}^{\ell} u(v_j) \\
&= i^2 \sum_{s=1}^{p} \int_{\Delta_s} u\ dv + \frac{i}{2}\left(\sum_{s=1}^{p}\int_{\partial\Delta_s} u\ d\sigma - 2\sum_{r=1}^{q}\int_{F_r} u\ d\sigma\right) + \alpha_0 \\
&= i^2 \int_{\Delta} u\ dv + \frac{i}{2}\int_{\partial\Delta} u\ d\sigma + \alpha_0,
\end{aligned}
$$

where

$$
\alpha_0 = \sum_{s=1}^{p}(\alpha_s - \frac{1}{2}\mathrm{vol}(\partial\Delta_s)) - \sum_{r=1}^{q}(\tilde{\alpha}_r - \mathrm{vol}(F_r)) + \sum_{j=1}^{\ell} u(v_j).
$$

Now we estimate α_0. Firstly we remark that

$$\int_{\Delta_s} u\,dv \geqslant \frac{1}{3!}, \quad \int_{F_r} u\,d\sigma \geqslant \frac{1}{2!}, \quad \mathrm{vol}(\Delta_s) \geqslant \frac{1}{2!}, \quad \mathrm{vol}(F_r) \geqslant 1. \tag{4.7}$$

by Lemma 4.2. Then the inequality (2) in Theorem 4.1 and (4.7) yield

$$\alpha_s \geqslant \frac{1}{3!}\left\{\mathfrak{s}(4,2) + \left(3!\int_{\Delta_s} u\,dv - 1\right) M_{1,3}\right\}.$$

One can readily see that $\mathfrak{s}(4,2) = 11$ by definition of $\mathfrak{s}(n,k)$ in Theorem 4.1. Similarly the inequality (1) in Theorem 4.1 and (4.7) imply

$$\begin{aligned}
\frac{1}{2}\mathrm{vol}(\partial\Delta_s) &\leqslant (-1)\,\mathfrak{s}(2,1)\mathrm{vol}(\Delta_s) + \mathfrak{s}(2,2)\\
&= (-1)^2\mathrm{vol}(\Delta_s) + 1,\\
\frac{1}{2}\left\{\int_{\partial\Delta_s} u\,d\sigma + 2\mathrm{vol}(\Delta_s)\right\} &\leqslant (-1)\,\mathfrak{s}(3,2)\int_{\Delta_s} u\,dv + \frac{1}{2!}\mathfrak{s}(3,3)\\
&= 3\int_{\Delta_s} u\,dv + \frac{1}{2!} \leqslant 3\int_{\Delta_s} u\,dv + \mathrm{vol}(\Delta_s),\\
\tilde{\alpha}_r &\leqslant (-1)\,\mathfrak{s}(2,1)\mathrm{vol}(\mathcal{F}_r) + \frac{\mathfrak{s}(2,2)}{(2-1)!}\\
&= (-1)^2\mathrm{vol}(\mathcal{F}_r) + 1\\
&= \int_{F_t} u\ d\sigma + 1 \leqslant \int_{F_t} u\ d\sigma + \mathrm{vol}(F_r).
\end{aligned} \tag{4.8}$$

In particular,

$$\tilde{\alpha}_r - \mathrm{vol}(F_r) \leqslant \int_{F_r} u\ d\sigma$$

holds from the last inequality. Hence it follows

$$\begin{aligned}
&\sum_{s=1}^{p}(\alpha_s - \frac{1}{2}\mathrm{vol}(\partial\Delta_s))\\
\geqslant\ &\sum_{s=1}^{p}\left[\int_{\Delta_s} u\,dv + \frac{11 - M_{1,3}}{3!} - (\mathrm{vol}(\Delta_s) + 1)\right]\\
=\ &\sum_{s=1}^{p}\left[\int_{\Delta_s} u\,dv - \mathrm{vol}(\Delta_s) + \frac{5 - M_{1,3}}{6}\right]\\
=\ &\int_{\Delta} u\,dv - \mathrm{vol}(\Delta) + p
\end{aligned}$$

where we used $M_{1,3} = -1$ in the last equality. On the other hand, we see that the inequality (4.8) gives

$$\int_{\partial\Delta_s} u\,d\sigma \leqslant 6\int_{\Delta_s} u\,dv.$$

This yields

$$
\begin{aligned}
\sum_{r=1}^{q}(\tilde{\alpha}_r - \mathrm{vol}(F_r)) &\leqslant \sum_{r=1}^{q}\int_{F_r} u\, d\sigma \\
&= \frac{1}{2}\left(\sum_{s=1}^{p}\int_{\partial\Delta_s} u\, d\sigma - \int_{\partial\Delta} u\, d\sigma\right) \\
&\leqslant \frac{1}{2}\sum_{s=1}^{p}\left(\frac{n(n+1)}{2}\int_{\Delta_s} u\, dv + \frac{1}{n!} - \mathrm{vol}(\Delta_s)\right) - \frac{1}{2}\int_{\partial\Delta} u\, d\sigma \\
&\leqslant \frac{1}{2}\left\{6\sum_{s=1}^{p}\left(\int_{\Delta_s} u\, dv\right) - \int_{\partial\Delta} u\, d\sigma\right\} \\
&= 3\int_{\Delta} u\, dv - \frac{1}{2}\int_{\partial\Delta} u\, d\sigma.
\end{aligned}
$$

Since a function u is normalized at 0, we conclude that

$$
\begin{aligned}
\alpha_0 &\geqslant \sum_{s=1}^{p}(\alpha_s - \frac{1}{2}\mathrm{vol}(\partial\Delta_s)) - \sum_{r=1}^{q}(\tilde{\alpha}_r - \mathrm{vol}(F_r)) \\
&\geqslant \int_{\Delta} u\, dv - \mathrm{vol}(\Delta) + p - \left(3\int_{\Delta} u\, dv - \frac{1}{2}\int_{\partial\Delta} u\, d\sigma\right) \\
&= p - \mathrm{vol}(\Delta) - 2\int_{\Delta} u\, dv + \frac{1}{2}\int_{\partial\Delta} u\, d\sigma \\
&\geqslant -C_0\left(\int_{\partial\Delta} u\, d\sigma + \int_{\Delta} u\, dv\right),
\end{aligned}
$$

where $C_0 > 0$ is some constant only depending on Δ. The assertion is verified. □

Remark 4.4. It is natural to ask whether Theorem 1.1 can be generalized to higher dimensional case $n \geqslant 3$ after solving the problem in Remark 3.2. However, there is anothr technical difficulty to compute (4.5) in Lemma 4.3 by the inclusion-exclusion principle for general setting.

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