# Note on the homotopy stability of the space of non-resultant systems determined by a toric variety

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**Abstract**

For a fan $\Sigma$ in $\mathbb{R}^m$ and $r$-tuple $D = (d_1, \cdots, d_r)$ of positive integers, let $\mathrm{Poly}_n^D(\mathbb{F})$ denote the certain affine variety over $\mathbb{F}$ called *the space of non-resultant systems of bounded multiplicity of type* $(\Sigma, n)$, where $r$ is the number of one dimensional cones in $\Sigma$ and $X_\Sigma$ denotes the toric variety over $\mathbb{C}$ corresponds to the fan $\Sigma$. This space was first defined by B. Farb and J. Wolfson [10] when $X_\Sigma$ is the complex projective space, and it was originally studied for investigating the homological densities of algebraic cycles in a manifold [11]. In this note, we shall report about the recent results concerning the homotopy stability of the space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{C}$. This result is based on the joint works with A. Kozlowski [19].

## 1 Basic definitions and notations

Let $\mathbb{N}$ be a set of all positive integers. For connected spaces $X$ and $Y$, let $\mathrm{Map}(X, Y)$ denote the space consisting of all continuous maps $f : X \to Y$ with the compact open topology. Let $\mathrm{Map}^*(X, Y) \subset \mathrm{Map}(X, Y)$ be the subspace of all base point preserving maps $f : (X, *) \to (Y, *)$. For a based homotopy class $D \in \pi_0(\mathrm{Map}^*(X, Y)) = [X, Y]$, we denote by $\mathrm{Map}^*_D(X, Y) \subset \mathrm{Map}^*(X, Y)$ the path component containing the homotopy class $D$.

When $X$ and $Y$ are complex manifolds, let $\mathrm{Hol}^*_D(X, Y) \subset \mathrm{Map}^*_D(X, Y)$ denote the subspace consisting of all based holomorphic maps $f \in \mathrm{Map}^*_D(X, Y)$. Then we have the natural inclusion

$$i_D : \mathrm{Hol}^*_D(X, Y) \xrightarrow{\subset} \mathrm{Map}^*_D(X, Y). \tag{1.1}$$

Now recall several definitions and notations.

**Definition 1.1.** (i) *A convex rational polyhedral cone* in $\mathbb{R}^m$ is a subset of $\mathbb{R}^m$ of the form

$$\sigma = \mathrm{Cone}(S) = \mathrm{Cone}(\boldsymbol{m}_1, \cdots, \boldsymbol{m}_s) = \{\sum_{k=1}^{s} \lambda_k \boldsymbol{m}_k : \lambda_k \geq 0\} \tag{1.2}$$

for a finite set $S = \{\boldsymbol{m}_1, \cdots, \boldsymbol{m}_s\} \subset \mathbb{Z}^m$. The dimension of $\sigma$ is the dimension of the smallest subspace of $\mathbb{R}^m$ which contains $\sigma$.

(ii) A convex rational polyhedral cone $\sigma$ is called *strongly convex* if $\sigma \cap (-\sigma) = \{\mathbf{0}_m\}$, where we set $\mathbf{0}_m = \mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{R}^m$. *A face* $\tau$ of a convex rational polyhedral cone $\sigma$ is a subset $\tau \subset \sigma$ of the form $\tau = \sigma \cap \{\boldsymbol{x} \in \mathbb{R}^m : L(\boldsymbol{x}) = 0\}$ for some linear form $L$ on $\mathbb{R}^m$, such that $\sigma \subset \{\boldsymbol{x} \in \mathbb{R}^m : L(\boldsymbol{x}) \geq 0\}$.

If we set $\{k : 1 \leq k \leq s, L(\boldsymbol{m}_k) = 0\} = \{i_1, \cdots, i_t\}$, we easily see that $\tau = \text{Cone}(\boldsymbol{m}_{i_1}, \cdots, \boldsymbol{m}_{i_t})$. Hence, if $\sigma$ is a strongly convex rational polyhedral cone, so is any of its faces.[1]

(iii) Let $\Sigma$ be a finite collection of strongly convex rational polyhedral cones in $\mathbb{R}^m$. Then it is called *a fan* (in $\mathbb{R}^m$) if the following two conditions (1.2.1) and (1.2.2) are satisfied:

(1.2.1) Every face $\tau$ of $\sigma \in \Sigma$ belongs to $\Sigma$.

(1.2.2) If $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a common face of each $\sigma_k$ and $\sigma_1 \cap \sigma_2 \in \Sigma$.

(iv) An $m$ dimensional irreducible normal variety $X$ (over $\mathbb{C}$) is called *a toric variety* if it has a Zariski open subset $\mathbb{T}^m_{\mathbb{C}} = (\mathbb{C}^*)^m$ and the action of $\mathbb{T}^m_{\mathbb{C}}$ on itself extends to an action of $\mathbb{T}^m_{\mathbb{C}}$ on $X$. The most significant property of a toric variety is that it is characterized up to isomorphism entirely by its associated fan $\Sigma$. We denote by $X_\Sigma$ the toric variety associated to a fan $\Sigma$. □

**Definition 1.2.** Let $K$ be a simplicial complex on the index set $[r] = \{1, 2, \cdots, r\}$,[2] and let $(X, A)$ be a pairs of based spaces.

(i) Let $I(K)$ denote the collection of subsets $\sigma \subset [r]$ defined by

$$I(K) = \{\sigma \subset [r] : \sigma \notin K\}. \tag{1.3}$$

(ii) Define *the polyhedral product* $\mathcal{Z}_K(X, A)$ with respect to $K$ by

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma, \qquad \text{where} \tag{1.4}$$
$$(X, A)^\sigma = \{(x_1, \cdots, x_r) \in X^r : x_k \in A \text{ if } k \notin \sigma\}.$$

(iii) For each subset $\sigma = \{i_1, \cdots, i_s\} \subset [r]$, let $L_\sigma(\mathbb{C}^n)$ denote the subspace of $\mathbb{C}^{nr}$ defined by

$$L_\sigma(\mathbb{C}^n) = \{(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r) \in \mathbb{C}^{nr} : \boldsymbol{x}_i \in \mathbb{C}^n,\ \boldsymbol{x}_{i_1} = \cdots = \boldsymbol{x}_{i_s} = \mathbf{0}_n\} \tag{1.5}$$

[1] When $S$ is the emptyset $\emptyset$, we set $\text{Cone}(\emptyset) = \{\mathbf{0}_m\}$ and we may also regard it as one of strongly convex rational polyhedral cones in $\mathbb{R}^m$.

[2] Let $K$ be some set of subsets of $[r]$. Then the set $K$ is called *an abstract simplicial complex* on the index set $[r]$ if the following condition holds: if $\tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$. In this paper by a simplicial complex $K$ we always mean *an abstract simplicial complex*, and we always assume that a simplicial complex $K$ contains the empty set $\emptyset$.

and let $L_n(\Sigma)$ denote the subspace of $\mathbb{C}^{nr}$ defined by

$$L_n(\Sigma)=\bigcup_{\sigma\in I(K)} L_\sigma(\mathbb{C}^n)=\bigcup_{\sigma\subset[r],\sigma\notin K} L_\sigma(\mathbb{C}^n). \tag{1.6}$$

Then it is easy to see that

$$\mathcal{Z}_K(\mathbb{C}^n,(\mathbb{C}^n)^*)=\mathbb{C}^{nr}\setminus L_n(\Sigma),\quad \text{where } (\mathbb{C}^n)^*=\mathbb{C}^n\setminus\{\mathbf{0}_n\}. \tag{1.7}$$ □

**Remark 1.3.** It is well known that there are no holomorphic maps $\mathbb{C}\mathrm{P}^1=S^2\to\mathbb{T}^m_{\mathbb{C}}$ except the constant maps, and that the fan $\Sigma$ of $\mathbb{T}^m_{\mathbb{C}}$ is $\Sigma=\{\mathbf{0}_m\}$. Hence, without loss of generality we always assume that $X_\Sigma\neq\mathbb{T}^m_{\mathbb{C}}$ and that any fan $\Sigma$ in $\mathbb{R}^m$ satisfies the condition $\{\mathbf{0}_m\}\subsetneqq\Sigma$. □

**Definition 1.4.** Let $\Sigma$ be a fan in $\mathbb{R}^m$ such that $\{\mathbf{0}_m\}\subsetneqq\Sigma$, and let

$$\Sigma(1)=\{\rho_1,\cdots,\rho_r\} \tag{1.8}$$

denote the set of all one dimensional cones in $\Sigma$.

(i) For each $1\leq k\leq r$, we denote by $\boldsymbol{n}_k\in\mathbb{Z}^m$ *the primitive generator* of $\rho_k$, such that $\rho_k\cap\mathbb{Z}^m=\mathbb{Z}_{\geq 0}\cdot\boldsymbol{n}_k$. Note that $\rho_k=\mathrm{Cone}(\boldsymbol{n}_k)$.

(ii) Let $\mathcal{K}_\Sigma$ denote *the underlying simplicial complex of* $\Sigma$ defined by

$$\mathcal{K}_\Sigma=\Big\{\{i_1,\cdots,i_s\}\subset[r]:\boldsymbol{n}_{i_1},\boldsymbol{n}_{i_2},\cdots,\boldsymbol{n}_{i_s}\text{ span a cone in }\Sigma\Big\}. \tag{1.9}$$

It is easy to see that $\mathcal{K}_\Sigma$ is a simplicial complex on the index set $[r]$.

(iii) Define the subgroup $G_\Sigma\subset\mathbb{T}^r_{\mathbb{C}}=(\mathbb{C}^*)^r$ by

$$G_\Sigma=\{(\mu_1,\cdots,\mu_r)\in\mathbb{T}^r_{\mathbb{C}}:\prod_{k=1}^r(\mu_k)^{\langle\boldsymbol{n}_k,\boldsymbol{m}\rangle}=1\text{ for all }\boldsymbol{m}\in\mathbb{Z}^m\}, \tag{1.10}$$

where $\langle\boldsymbol{u},\boldsymbol{v}\rangle=\sum_{k=1}^m u_kv_k$ for $\boldsymbol{u}=(u_1,\cdots,u_m)$ and $\boldsymbol{v}=(v_1,\cdots,v_m)\in\mathbb{R}^m$.

(iv) Now consider the natural $G_\Sigma$-action on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n,(\mathbb{C}^n)^*)$ given by coordinate-wise multiplication, i.e.

$$(\mu_1,\cdots,\mu_r)\cdot(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_r)=(\mu_1\boldsymbol{x}_1,\cdots,\mu_r\boldsymbol{x}_r) \tag{1.11}$$

for $((\mu_1,\cdots,\mu_r),(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_r))\in G_\Sigma\times\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n,(\mathbb{C}^n)^*)$, where we set

$$\mu\boldsymbol{x}=(\mu x_1,\cdots,\mu x_n)\quad\text{if }(\mu,\boldsymbol{x})=(\mu,(x_1,\cdots,x_n))\in\mathbb{C}\times\mathbb{C}^n. \tag{1.12}$$

Then define the space $X_\Sigma(n)$ by the corresponding orbit space

$$X_\Sigma(n)=\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n,(\mathbb{C}^n)^*)/G_\Sigma. \tag{1.13}$$ □

**Remark 1.5.** (i) Let $\Sigma$ be a fan in $\mathbb{R}^m$ as in Definition 1.4. Then the fan $\Sigma$ is completely determined by the pair $(\mathcal{K}_\Sigma, \{\boldsymbol{n}_k\}_{k=1}^r)$ (see [16, Remark 2.3] in detail).

(ii) Note that the group $G_\Sigma$ acts on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ freely (cf. [25, Proposition 6.7]). Moreover, one can show that $X_\Sigma(n)$ is a toric variety. □

The following theorem plays a crucial role in the proof of the main result of this paper.

**Theorem 1.6** ([7], Theorem 2.1; [8], Theorem 3.1)**.** *Let $\Sigma$ be a fan in $\mathbb{R}^m$ as in Definition 1.4 and suppose that the set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^m$ (i.e. $\sum_{k=1}^r \mathbb{R}\cdot \boldsymbol{n}_k = \{\sum_{k=1}^r \lambda_k \boldsymbol{n}_k : \lambda_k \in \mathbb{R}\} = \mathbb{R}^m$).*

(i) *Then there is a natural isomorphism*

$$X_\Sigma \cong \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma(1). \tag{1.14}$$

(ii) *If $f : \mathbb{C}\mathrm{P}^s \to X_\Sigma$ is a holomorphic map, then there exists an $r$-tuple $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ of non-negative integers satisfying the condition $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$, and homogenous polynomials $f_i \in \mathbb{C}[z_0, \cdots, z_s]$ of degree $d_i$ $(i = 1, 2, \cdots, r)$ such that the polynomials $\{f_i\}_{i\in\sigma}$ have no common root except $\mathbf{0}_{s+1} \in \mathbb{C}^{s+1}$ for each $\sigma \in I(\mathcal{K}_\Sigma)$ and that the diagram*

$$\begin{array}{ccc} \mathbb{C}^{s+1} \setminus \{\mathbf{0}_{s+1}\} & \xrightarrow{(f_1,\cdots,f_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \\ \gamma_s \big\downarrow & & q_\Sigma \big\downarrow \\ \mathbb{C}\mathrm{P}^s & \xrightarrow{\ f\ } & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma \end{array} \tag{1.15}$$

*is commutative, where we identify $X_\Sigma = X_\Sigma(1)$ as in (1.14) and the two map $\gamma_s : \mathbb{C}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \to \mathbb{C}\mathrm{P}^s$ and $q_\Sigma : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \to X_\Sigma = X_\Sigma(1)$ denote the canonical Hopf fibering and the canonical projection induced from the identification (1.14), respectively. In this case, we call this holomorphic map $f$ a holomorphic map of degree $D = (d_1, \cdots, d_r)$ and we represent it as*

$$f = [f_1, \cdots, f_r]. \tag{1.16}$$

(iii) *If $g_i \in \mathbb{C}[z_0, \cdots, z_s]$ is a homogenous polynomial of degree $d_i$ $(1 \leq i \leq r)$ such that $f = [f_1, \cdots, f_r] = [g_1, \cdots, g_r]$, there exists some element $(\mu_1, \cdots, \mu_r) \in G_\Sigma$ such that $f_i = \mu_i \cdot g_i$ for each $1 \leq i \leq r$. Thus, the $r$-tuple $(f_1, \cdots, f_r)$ of homogenous polynomials representing a holomorphic map $f$ is determined uniquely up to $G_\Sigma$-action.* □

From now on, let $\Sigma$ be a fan in $\mathbb{R}^m$ as in Definition 1.4, and assume that $X_\Sigma$ is simply connected and non-singular. Moreover, we shall assume the following condition holds.

(1.16.1) There is an $r$-tuple $D_* = (d_1^*, \cdots, d_r^*) \in \mathbb{N}^r$ such that $\sum_{k=1}^r d_k^* \boldsymbol{n}_k = \mathbf{0}_m$.

**Remark 1.7.** It follows from [9, Theorem 12.1.10] that $X_\Sigma$ is simply connected if and only if the fan $\Sigma$ satisfies the following condition $(*)$:

$(*)$ The set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{Z}^m$ over $\mathbb{Z}$, i.e. $\sum_{k=1}^r \mathbb{Z}\cdot \boldsymbol{n}_k = \mathbb{Z}^m$.

Thus, one can easily see that the set $\{\boldsymbol{n}_k\}_{k=1}^r$ of all primitive generators spans $\mathbb{R}^m$ if $X_\Sigma$ is simply connected. In particular, we can see that $X_\Sigma$ is simply connected if $X_\Sigma$ is a compact smooth toric variety. □

## 2 The space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$

Let $X_\Sigma$ be a simply connected non-singular toric variety satisfying the condition (1.16.1), and from now on we identify $X_\Sigma = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)/G_\Sigma = X_\Sigma(1)$.

Now consider a base point preserving holomorphic map $f = [f_1,\cdots,f_r] : \mathbb{C}\mathrm{P}^s \to X_\Sigma$ for the case $s=1$. In this case, we make the identification $\mathbb{C}\mathrm{P}^1 = S^2 = \mathbb{C}\cup\infty$ and choose the points $\infty$ and $[1,1,\cdots,1]$ as the base points of $\mathbb{C}\mathrm{P}^1$ and $X_\Sigma$, respectively.

Then, by setting $z=\frac{z_0}{z_1}$, for each $1\leq k\leq r$, we can view $f_k$ as a monic polynomial $f_k(z)\in\mathbb{C}[z]$ of degree $d_k$ in the complex variable $z$. Now by using Theorem 1.6 we can define the space of holomorphic maps as follows.

**Definition 2.1.** (i) For a filed $\mathbb{F}$, let $\mathrm{P}^d_{\mathbb{F}}$ denote the space of all $\mathbb{F}$-coefficient monic polynomials $g(z)=z^d+a_1z^{d-1}+\cdots+a_{d-1}z+a_d\in\mathbb{F}[z]$ of degree $d$, and we set

$$\mathrm{P}^D_{\mathbb{F}} = \mathrm{P}^{d_1}_{\mathbb{F}}\times\cdots\times\mathrm{P}^{d_r}_{\mathbb{F}} \quad \text{if } D=(d_1,\cdots,d_r)\in\mathbb{N}^r. \tag{2.1}$$

Note that there is a natural homeomorphism $\phi:\mathrm{P}^d_{\mathbb{F}}\cong\mathbb{F}^d$ given by $\phi(z^d+\sum_{k=1}^d a_kz^{d-k}) = (a_1,\cdots,a_d)\in\mathbb{F}^d$. When $\mathbb{F}=\mathbb{C}$, we write $\mathrm{P}^D=\mathrm{P}^D_{\mathbb{C}}$, and $\mathrm{P}^d=\mathrm{P}^d_{\mathbb{C}}$.

(ii) For any $r$-tuple $D=(d_1,\cdots,d_r)\in\mathbb{N}^r$ satisfying the condition (1.16.1), we denote by $\mathrm{Hol}^*_D(S^2,X_\Sigma)$ the space consisting of all $r$-tuples $f=(f_1(z),\cdots,f_r(z))\in\mathrm{P}^D$ satisfying the following condition $(\dagger_\Sigma)$:

$(\dagger_\Sigma)$ For any $\sigma=\{i_1,\cdots,i_s\}\in I(\mathcal{K}_\Sigma)$, the polynomials $f_{i_1}(z),\cdots,f_{i_s}(z)$ have no common root, i.e. $(f_{i_1}(\alpha),\cdots,f_{i_s}(\alpha))\neq\mathbf{0}_s=(0,\cdots,0)$ for any $\alpha\in\mathbb{C}$.

By identifying $X_\Sigma=\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C},\mathbb{C}^*)/G_\Sigma$ and $\mathbb{C}\mathrm{P}^1=S^2=\mathbb{C}\cup\infty$, one can define the natural inclusion map $i_D:\mathrm{Hol}^*_D(S^2,X_\Sigma)\to\mathrm{Map}^*(S^2,X_\Sigma)=\Omega^2X_\Sigma$ by

$$i_D(f)(\alpha)=\begin{cases}[f_1(\alpha),f_2(\alpha),\cdots,f_r(\alpha)] & \text{if } \alpha\in\mathbb{C}\\ [1,1,\cdots,1] & \text{if } \alpha=\infty\end{cases} \tag{2.2}$$

for $f=(f_1(z),\cdots,f_r(z))\in\mathrm{Hol}^*_D(S^2,X_\Sigma)$, where we choose the points $\infty$ and $[1,1,\cdots,1]$ as the base points of $S^2$ and $X_\Sigma$.

Since the representation of polynomials in $\mathrm{P}^D=\mathrm{P}^D_{\mathbb{C}}$ representing a base point preserving holomorphic map of degree $D$ is uniquely determined, the space $\mathrm{Hol}^*_D(S^2,X_\Sigma)$ can be identified with *the space of base point preserving holomorphic maps of degree* $D$. Moreover, since $\mathrm{Hol}^*_D(S^2,X_\Sigma)$ is path-connected, the image of $i_D$ is contained in a certain path-component of $\Omega^2X_\Sigma$, which is denoted by $\Omega^2_DX_\Sigma$. Thus we have a natural inclusion

$$i_D:\mathrm{Hol}^*_D(S^2,X_\Sigma)\to\mathrm{Map}^*_D(S^2,X_\Sigma)=\Omega^2_DX_\Sigma. \qquad \Box \tag{2.3}$$

Now consider the space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ for $\mathbb{F}=\mathbb{C}$. For this purpose, we need the following notation.

**Definition 2.2.** For a monic polynomial $f(z) \in \mathrm{P}^d$ of degree $d$, let $F_n(f)(z)$ denote the $n$-tuple of monic polynomials of the same degree $d$ given by

$$F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \cdots, f(z) + f^{(n-1)}(z)). \tag{2.4}$$

Note that a monic polynomial $f(z) \in \mathrm{P}^d$ has a root $\alpha \in \mathbb{C}$ of multiplicity $\geq n$ iff $F_n(f)(\alpha) = \mathbf{0}_n \in \mathbb{C}^n$. □

**Definition 2.3.** Let $\mathbb{F}$ be a filed with its algebraic closure $\overline{\mathbb{F}}$.

(i) For each $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, $n \in \mathbb{N}$ and a fan $\Sigma$ in $\mathbb{R}^m$, let $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ denote the space of $r$-tuples $(f_1(z), \cdots, f_r(z)) \in \mathrm{P}^D_{\mathbb{F}}$ of $\mathbb{F}$-coefficients monic polynomials satisfying the following condition $(\dagger_{\Sigma,n})$:

$(\dagger_{\Sigma,n})$ For any $\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma)$, polynomials $f_{i_1}(z), \cdots, f_{i_s}(z)$ have no common root $\alpha \in \overline{\mathbb{F}}$ of multiplicity $\geq n$ (but they may have common roots of multiplicity $< n$).

Note that the condition $(\dagger_\Sigma)$ coincides with the condition $(\dagger_{\Sigma,n})$ if $(\mathbb{F}, n) = (\mathbb{C}, 1)$.

From now on, we only consider the case $\mathbb{F} = \mathbb{C}$.

(ii) When $\mathbb{F} = \mathbb{C}$ and $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_n$, define the map $i_D : \mathrm{Poly}_n^{D,\Sigma}(\mathbb{C}) \to \Omega^2 X_\Sigma(n)$ by

$$i_D(f)(\alpha) = \begin{cases} [F_n(f_1)(\alpha), F_n(f_2)(\alpha), \cdots, F_n(f_r)(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [\boldsymbol{e}, \boldsymbol{e}, \cdots, \boldsymbol{e}] & \text{if } \alpha = \infty \end{cases} \tag{2.5}$$

for $f = (f_1(z), \cdots, f_r(z)) \in \mathrm{Poly}_n^{D,\Sigma}(\mathbb{C})$ and $\alpha \in \mathbb{C} \cup \infty = S^2$, where the space $X_\Sigma(n)$ is the space defined as in (1.13) and we set $\boldsymbol{e} = (1, 1, \cdots, 1) \in \mathbb{C}^n$.

Since $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{C})$ is connected, the image of $i_D$ is contained some path-component of $\Omega^2 X_\Sigma(n)$, which is denoted by $\Omega^2_D X_\Sigma(n)$. Thus we have the map

$$i_D : \mathrm{Poly}_n^{D,\Sigma}(\mathbb{C}) \to \Omega^2_D X_\Sigma(n). \tag{2.6}$$ □

**Remark 2.4.** (i) By using the classical theory of resultants, one can show that $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ is an affine variety over $\mathbb{F}$ which is the complement of the set of solutions of a system of polynomial equations (called a generalised resultant) with integer coefficients. This is why we call it *the space of non-resultant systems of bounded multiplicity of type* $(\Sigma, n)$.

(ii) When $X_\Sigma$ is a simply connected non-singular toric variety (over $\mathbb{C}$) satisfying the condition (1.16.1), one can show that $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{C}) = \mathrm{Hol}^*_D(S^2, X_\Sigma)$ if $n = 1$ (see Definition 2.1 for the details). □

# 3 The space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{C})$

Before stating the results for the space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{C})$, we need to define the positive integers $r_{\min}(\Sigma)$ and $d(D; \Sigma, n)$.

**Definition 3.1.** We say that a set $S = \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\}$ is *a primitive* if $\mathrm{Cone}(S) \notin \Sigma$ and $\mathrm{Cone}(T) \in \Sigma$ for any proper subset $T \subsetneqq S$. Then we define $d(D, \Sigma, n)$ to be the positive integer given by

$$d(D; \Sigma, n) = (2nr_{\min}(\Sigma) - 3)\lfloor d_{\min}/n \rfloor - 2, \tag{3.1}$$

where $r_{\min}(\Sigma)$ and $d_{\min} = d_{\min}(D)$ are the positive integers given by

$$r_{\min}(\Sigma) = \min\{s \in \mathbb{N} : \{\boldsymbol{n}_{i_1}, \cdots, \boldsymbol{n}_{i_s}\} \text{ is primitive}\}, \tag{3.2}$$

$$d_{\min} = d_{\min}(D) = \min\{d_1, d_2, \cdots, d_r\}. \tag{3.3}$$

□

**Definition 3.2.** Let $f : X \to Y$ be a based continuous map, and let $N_0 \in \mathbb{N}$ be a fixed positive integer.

(i) The map $f$ is called *a homology* (*resp. homotopy*) *equivalence through dimension* $N_0$ if the induced homomorphism

$$f_* : H_k(X; \mathbb{Z}) \to H_k(Y; \mathbb{Z}) \quad (\text{resp. } f_* : \pi_k(X) \to \pi_k(Y)) \tag{3.4}$$

is an isomorphism for any $k \leq N_0$.

(ii) Similarly, the map $f$ is called *a homology* (*resp. homotopy*) *equivalence up to dimension* $N_0$ if the induced homomorphism $f_*$ (given by (3.4)) is an isomorphism for any $k < N_0$ and an epimorphism for $k = N_0$. □

For connected space $X$, let $\Omega^2_0 X$ denote the path-component of $\Omega^2 X$ which contains null-homotopic maps and recall the following result.

**Theorem 3.3** ([16]). *Let $X_\Sigma$ be an $m$ dimensional simply connected non-singular toric variety such that the condition* (*1.16.1*) *holds. Then if $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ and $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$, the inclusion map*

$$i_D : \mathrm{Hol}^*_D(S^2, X_\Sigma) \to \Omega^2_D X_\Sigma \simeq \Omega^2_0 X_\Sigma \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^2, S^1)$$

*is a homotopy equivalence through dimension $d(D; \Sigma, 1) = (2r_{\min}(\Sigma) - 3)d_{\min} - 2$ if $r_{\min}(\Sigma) \geq 3$ and a homology equivalence through dimension $d(D; \Sigma, 1) = d_{\min} - 2$ if $r_{\min}(\Sigma) = 2$.* □

The main result of this paper is a generalization of the above theorem (Theorem 3.3) to spaces of non-resultant systems of bounded multiplicity.

**Theorem 3.4** ([19]). *Let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, $n \geq 2$ and let $X_\Sigma$ be an $m$ dimensional simply connected non-singular toric variety such that the condition* (*1.16.1*) *holds.*

(i) *If $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$, then the natural map*

$$i_D : \mathrm{Poly}^{D,\Sigma}_n(\mathbb{C}) \to \Omega^2_D X_\Sigma(n) \simeq \Omega^2_0 X_\Sigma(n) \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

*is a homotopy equivalence through dimension $d(D; \Sigma, n)$.*

(ii) *If $\sum_{k=1}^r d_k \boldsymbol{n}_k \neq \mathbf{0}_m$, there is a map*

$$j_D : \mathrm{Poly}^{D,\Sigma}_n(\mathbb{C}) \to \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})$$

*which is a homotopy equivalence through dimension $d(D; \Sigma, n)$.* □

**Corollary 3.5** ([19]). *Let $n \geq 2$, $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, and let $X_\Sigma$ be an $m$ dimensional compact non-singular toric variety over $\mathbb{C}$ such that the condition (1.16.1) holds. Let $\Sigma(1)$ denote the set of all one dimensional cones in $\Sigma$, and let $\Sigma_1$ be any fan in $\mathbb{R}^m$ such that $\Sigma(1) \subset \Sigma_1 \subsetneqq \Sigma$.*

*Then $X_{\Sigma_1}$ is a non-compact non-singular toric subvariety of $X_\Sigma$ and the following two statements hold:*

(i) *If $\sum_{k=1}^r d_k \boldsymbol{n}_k = \mathbf{0}_m$, the map*

$$i_D : \mathrm{Poly}_n^{D,\Sigma_1}(\mathbb{C}) \to \Omega^2_D X_{\Sigma_1}(n) \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(D^{2n}, S^{2n-1})$$

*is a homotopy equivalence through the dimension $d(D; \Sigma_1, n)$.*

(ii) *If $\sum_{k=1}^r d_k \boldsymbol{n}_k \neq \mathbf{0}_m$, there is a map*

$$j_D : \mathrm{Poly}_n^{D,\Sigma_1}(\mathbb{C}) \to \Omega^2 \mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(D^{2n}, S^{2n-1})$$

*which is a homotopy equivalence through dimension $d(D; \Sigma_1, n)$.* □

**Remark 3.6.** We would like to study about the homotopy stability of the space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{F})$ for the case $\mathbb{F} = \mathbb{R}$ in our future paper. However, note that the homotopy type of the space $\mathrm{Poly}_n^{D,\Sigma}(\mathbb{R})$ was already studied when $X_\Sigma$ is a complex projective space in [20] (cf. [21], [26], [27]). We would like to generalize these results for more general toric varieties $X_\Sigma$.

**Acknowledgements.** The author was supported by JSPS KAKENHI Grant Number JP22K03283. This work was also supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

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