Boundedness of bundle diffeomorphism groups over a circle — from a view point of attaching maps

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1 Introduction

Study of the automorphism group $\mathrm{Aut}(M, \mathfrak{s})$ of a manifold M with some structure $\mathfrak{s}$ is important to understand the geometry of the manifold $(M, \mathfrak{s})$. An extension of this study in one direction is the study of the automorphism group $\mathrm{Aut}(\xi)$ of a fiber bundle ξ. In this subject we are working on a systematic geometric group theoretic study of the group $\mathrm{Diff}^r_\pi(M)_0$ of bundle diffeomorphisms of a C^r (N, Γ) bundle $\pi : M \to B$.

In this exposition we are mainly concerned with the following two topics in the study of bundle diffeomorphism groups :

(1) Fragmentation, perfectness and relative simplicity of bundle diffeomorphism groups

(2) Boundedness of bundle diffeomorphism groups over a circle
— Description of the invariant k in term of the attaching map

In the second topic, for an (N, Γ) bundle $\pi : M \to S^1$ over a circle and $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, using the rotation angle in S^1, we can distinguish an integer $k = k(\pi, r) \in \mathbb{Z}_{\geq 0}$ and construct a function $\widehat{\nu} : \mathrm{Diff}^r_\pi(M)_0 \to \mathbb{R}_k$. We have shown that, when $k \geq 1$, the bundle diffeomorphism group $\mathrm{Diff}^r_\pi(M)_0$ is bounded and $cld\,\mathrm{Diff}^r_\pi(M)_0 \leq k + 3$, provided $\mathrm{Diff}^r_{pr,c}(\mathbb{R} \times N)_0$ is perfect for the product (N, Γ) bundle $pr : \mathbb{R} \times N \longrightarrow \mathbb{R}$. On the other hand, when $k = 0$, it is shown that $\widehat{\nu}$ is a unbounded quasimorphism, so that $\mathrm{Diff}^r_\pi(M)_0$ is unbounded and not uniformly perfect. In this exposition, we focus on a description of the invariant k in term of the attaching map associated to the bundle π. We can also mention a lower bound of $cld\,\mathrm{Diff}^r_\pi(M)_0$ in term of k. In the literature, until now there have been very few examples of bounded diffeomorphism groups with evaluations from below. Although a bundle diffeomorphism group is a slightly artificial object, it is of interest in that it gives an example of such an evaluation from below.

Main reference of this exposition is [7], while we also include some results obtained after this article. In a succeeding paper [8] we discuss boundedness, uniform perfectness and uniform relative simplicity of bundle diffeomorphism groups over higher dimensional base spaces This research is a joint work with Kazuhiko Fukui.

2 Preliminaries

This section includes a short survey for basic definitions and notations on bundle diffeomorphism groups. We refer to [7, Sections 2, 3] for more details.

2.1 Fiber bundles

Suppose $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, N is a C^r manifold and $\Gamma < \mathrm{Diff}^r(N)$. Here N is called a fiber and Γ is called a structure group on N. A C^r (N, Γ) bundle is a C^r map $\pi : M \to B$ between C^r manifolds endowed with a maximal Γ-atlas $\mathcal{U} = \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ of local trivializations of π. This means that each transition function takes values in Γ, that is, $(\varphi_\mu)_q (\varphi_\lambda)_q{}^{-1} \in \Gamma$ $(\lambda, \mu \in \Lambda, q \in U_\lambda \cap U_\mu)$. ($\mathcal{U}$ is regarded as a set rather than an indexed family.)

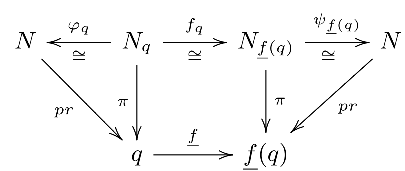
Example 2.1. (Principal bundles) Suppose G is a Lie group. Then a principal G bundle is exactly a (G, G_L)-bundle, where

$G_L := \{\varphi_a \mid a \in G\} < \mathrm{Diff}^\infty(G)$ and φ_a is the left translation on G by a.

2.2 Bundle diffeomorphism groups

Suppose $\pi : M \to B$ is a C^r (N, Γ) bundle with a maximal Γ-atlas $\mathcal{U}$.

Definition 2.1. A C^r bundle diffeomorphism $f : \pi \to \pi$ is a C^r diffeomorphism $f : M \to M$ such that



$(N_q := \pi^{-1}(q))$

(i) $\pi f = \underline{f}\, \pi$ for a (unique) $\underline{f} \in \mathrm{Diff}^r(B)$,

(ii) $\psi_{\underline{f}(q)}\, f_q\, (\varphi_q)^{-1} \in \Gamma$ $\quad (q \in B,\ q \in (U, \varphi) \in \mathcal{U},\ \underline{f}(q) \in (V, \psi) \in \mathcal{U})$.

This means that f maps each fiber onto some fiber and f is a fiberwise Γ-diffeomorphism

We use the symbols $\mathrm{Diff}^r_\pi(M)$ and $\mathrm{Isot}^r_\pi(M)$ to denote the groups of C^r bundle diffeomorphisms and C^r bundle isotopies of π respectively.
Here, a C^r bundle isotopy of π is a C^r isotopy $F = (F_t)_{t \in I}$ of M such that $F_t \in \mathrm{Diff}^r_\pi(M)$ $(t \in I \equiv [0, 1])$. The identity components of these groups are defined by

$\mathrm{Isot}^r_\pi(M)_0 := \{F \in \mathrm{Isot}^r_\pi(M) \mid F_0 = \mathrm{id}\}$ and $\mathrm{Diff}^r_\pi(M)_0 := \{F_1 \mid F \in \mathrm{Isot}^r_\pi(M)_0\}$.

The group homomorphism $P : \mathrm{Diff}^r_\pi(M) \to \mathrm{Diff}^r(B) : P(f) = \underline{f}$ induces en epimorphism $P : \mathrm{Diff}^r_\pi(M)_0 \to \mathrm{Diff}^r(B)_0$ by the isotopy lifting property of the fiber bundle π.

The notion of support for bundle diffeomorphisms/isotopies is very important. The base support of $f \in \mathrm{Diff}^r_\pi(M)$ is defined by $\mathrm{supp}_b\, f := cl_B\, \pi(\mathrm{supp}\, f) \subset B$. The support and base support of $F \in \mathrm{Isot}^r_\pi(M)$ are defined by

$\mathrm{supp}\, F = cl_M \bigcup_{t \in I} \mathrm{supp}\, F_t \subset M$ and $\mathrm{supp}_b\, F := cl_B\, \pi(\mathrm{supp}\, F) \subset B$.

We say that f and F have compact support with respect to π if their base supports are compact in B. The following notations are used with regard to suppot.

$\mathrm{Isot}^r_{\pi,c}(M)_0 := \{F \in \mathrm{Isot}^r_\pi(M)_0 \mid \mathrm{supp}_b\, F : \text{compact}\}$

$\mathrm{Isot}^r_\pi(M; \mathrm{supp}_b \Subset A)_0 := \{F \in \mathrm{Isot}^r_\pi(M)_0 \mid \mathrm{supp}_b\, F \subset \mathrm{Int}_B A\} \qquad (A \subset B)$

$\mathrm{Diff}^r_{\pi,c}(M)_0 := \{F_1 \mid F \in \mathrm{Isot}^r_{\pi,c}(M)_0\}$

$\mathrm{Diff}^r_\pi(M; \mathrm{supp}_b \Subset A)_0 := \{F_1 \mid F \in \mathrm{Isot}^r_\pi(M; \mathrm{supp}_b \Subset A)_0\}$.

Example 2.2. (Equivariant diffeomorphism groups)
Suppose $\varrho : G \curvearrowright M$ is a free C^∞ action of a compact Lie group G on a C^∞ manifold M. Then, the quotient map $\pi : M \to M/G$ is a principal G-bundle and
$\mathrm{Diff}^r_\pi(M) = \mathrm{Diff}^r_\varrho(M)$, the group of ϱ-equivariant diffeomorphisms of M.

2.3 Conjugation-invariant norms on the group $\mathrm{Diff}^r_\pi(M)_0$

We list three kind of conjugation-invariant norms, which are used to deduce boundedness, (uniform) perfectness and (uniform) relative simplicity of the group $\mathrm{Diff}^r_\pi(M)_0$ in Sections 3 and 4. On the other hand, unboundedness results are deduced from the existence of unbounded quasimorphisms. Suppose G is any (discrete) group.

[1] The commutator length $cl_G : G \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is a basic conjugation-invariant norm on G. It is based on the subset $G^c := \{[a,b] \mid a, b \in G\}$ and is defined by

$$cl_G\, g := \begin{cases} \min\{k \in \mathbb{Z}_{\geq 0} \mid g = g_1 \cdots g_k \ (g_1, \cdots, g_k \in G^c)\} & (g \in [G,G]), \\ \infty \qquad (g \in G - [G,G]). \end{cases}$$

[2] The conjugation-generated norm $\zeta_g : G \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ $(g \in G - \{e\})$ is based on the subset $C_g :=$ [the set of conjugates of $g^{\pm 1}$] and is defined by

$$\zeta_g(f) = \begin{cases} \min\{k \in \mathbb{Z}_{\geq 0} \mid f = g_1 \cdots g_k \ \ (g_1, \cdots, g_k \in C_g)\} & (f \in N(g)), \\ \infty \qquad (f \in G - N(g)). \end{cases}$$

Here, $N(g)$ is the normal subgroup of G generated by g.

[3] Suppose $\pi : M \to B$ is a C^r (N, Γ) bundle and B is a C^r n-manifold without boundary. Let $\mathcal{B}^r(B)$ denote the collection of C^r n-balls in B. For $D \in \mathcal{B}^r(B)$ any commutator $[g,h]$ $(g, h \in \mathrm{Diff}^r_\pi(M; \mathrm{supp}_b \Subset D)_0)$ is called a commutator supported in the ball D. The commutator length supported in balls in B, $clb_\pi : \mathrm{Diff}^r_\pi(M)_0 \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is based on the subset $\mathcal{S}_b := \bigcup \left\{\mathrm{Diff}^r_\pi(M; \mathrm{supp}_b \Subset D)^c_0 \mid D \in \mathcal{B}^r(B)\right\}$ of $\mathrm{Diff}^r_\pi(M)_0$ and is defined by

$$clb_\pi\, f = \begin{cases} \min\{k \in \mathbb{Z}_{\geq 0} \mid f = g_1 \cdots g_k \ \ (g_1, \cdots, g_k \in \mathcal{S}_b)\} & (f \in N(\mathcal{S}_b)), \\ \infty \qquad (f \in G - N(\mathcal{S}_b)). \end{cases}$$

Here, $N(\mathcal{S}_b)$ is the normal subgroup of G generated by $\mathcal{S}_b$.

For any conjugation-invariant norm $q : G \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$
the q-diameter of G is defined by $qd\, G := \sup \{q(g) \mid g \in G\}$.
Since the group $\mathrm{Diff}^r_\pi(M)_0$ contains $\mathrm{Ker}\, P$ as a normal subgroup, we need the notion of relative simplicity.

Definition 2.2.

(1) G is bounded. $\Longleftrightarrow$ Any conjugation-invariant norm $q : G \longrightarrow \mathbb{R}_{\geq 0}$ is bounded.

(2) G is perfect. $\Longleftrightarrow$ $G = [G,G]$ $\Longleftrightarrow$ $cl_G\, f < \infty$ $(f \in G)$
G is uniformly perfect. $\Longleftrightarrow$ cl_G is bounded. $\Longleftrightarrow$ $cld\, G < \infty$

(3) G is simple. $\iff G = N(g) \ \ (g \in G - \{e\}) \iff \zeta_g(f) < \infty \ \ (g \in G - \{e\}, f \in G)$
G is simple relative to a subset $S \subset G$
$\iff G = N(g) \ \ (g \in G - S) \iff \zeta_g(f) < \infty \ \ (g \in G - S, f \in G)$

(4) G is uniformly simple. $\iff \exists\, k \in \mathbb{Z}_{\geq 0}$ such that $\zeta_g \leq k \ \ (g \in G - \{e\})$
G is uniformly simple relative to a subset $S \subset G$.
$\iff \exists\, k \in \mathbb{Z}_{\geq 0}$ such that $\zeta_g \leq k \ \ (g \in G - S)$

Fact 2.1.

(1) If ζ_g is bounded for some $g \in G - \{e\}$, then G is bounded.

- ◦ If $g \in G - \{e\}$ and $\zeta_g \leq k$ for some $k \in \mathbb{Z}_{\geq 0}$, then
  $q \leq kq(g)$ for any conjugation-invariant norm q on G.

(2) Suppose N is a normal subgroup of G.
G is simple relative to N. $\iff [L \lhd G \ \curvearrowright \ L \subset N$ or $L = G]$
G/N is simple. $\iff [N \subset L \lhd G \ \curvearrowright \ L = N$ or $G]$

3 Perfectness and relative simplicity of bundle diffeomorphism groups

In this section we discuss basic properties of bundle diffeomorphism groups, such as fragmentation, perfectness and relative simplicity. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Suppose B is a connected C^r n-manifold without boundary and $\pi : M \to B$ is a C^r (N, Γ)-bundle over B.

3.1 Fragmentation of bundle diffeomorphisms

Definition 3.1. For $f \in \mathrm{Diff}^r_{\pi,c}(M)_0$ a fragmentation of f of length ℓ means a factorization
$f = f_1 \cdots f_\ell$ such that $f_i \in \mathrm{Diff}^r_\pi(M; \mathrm{supp}_b \Subset D_i)_0$ and $D_i \in \mathcal{B}^r(B)$ for each $i \in [\ell]$.

Theorem 3.1. Any $f \in \mathrm{Diff}^r_{\pi,c}(M)_0$ has a fragmentation.

Definition 3.2. The fragmentation norm on $\mathrm{Diff}^r_{\pi,c}(M)_0$ is defined by
$\nu : \mathrm{Diff}^r_{\pi,c}(M)_0 \longrightarrow \mathbb{Z}_{\geq 0} : \nu(f) :=$ the minimal length of fragmentation of f.

3.2 Perfectness of bundle diffeomorphism groups

Fragmentation theorem reduces the perfectness of the whole group $\mathrm{Diff}^r_{\pi,c}(M)_0$ to the following local assertion.

Condition $(*)$
The group $\mathrm{Diff}^r_{pr,c}(\mathbb{R}^n \times N)_0$ is perfect for the product (N, Γ)-bundle $pr : \mathbb{R}^n \times N \longrightarrow \mathbb{R}^n$.

Lemma 3.1. If the tuple (N, Γ, r, n) satisfies Condition $(*)$, then
$cld\, \mathcal{D} = clb_\pi d\, \mathcal{D} \leq 2$ for any of the following isomorphic groups :
$\mathcal{D} = \mathrm{Diff}^r_{pr,c}(\mathbb{R}^n \times N)_0$, $\mathrm{Diff}^r_{\pi,c}(\pi^{-1}(\mathrm{Int}\, D))_0$ and $\mathrm{Diff}^r_\pi(M; \mathrm{supp}_b \Subset D)_0 \ \ (D \in \mathcal{B}^r(B))$.

Theorem 3.2. If the tuple (N, Γ, r, n) satisfies Condition $(*)$, then

(i) $\mathrm{Diff}^r_{\pi,c}(M)_0$ is perfect and (ii) $cl\, f \leq clb_\pi\, f \leq 2\nu(f) < \infty$ for any $f \in \mathrm{Diff}^r_{\pi,c}(M)_0$.

At this moment, we have the following list of (N, Γ) bundles which satisfies Condition $(*)$.

Example 3.1. The group $\mathrm{Diff}^r_{\pi,c}(M)_0$ is perfect in the following cases ([7, Section 3.4]).

(1) (Principal bundle) $\pi : M \to B$ is a C^∞ principal G bundle, G is a compact Lie group, $n \geq 1$ and $r \neq n+1$.

(2) (Locally trivial bundle) N is a C^∞ closed manifold, $\Gamma = \mathrm{Diff}^\infty(N)$ and $r = \infty$.

The case (1) is proved by K. Abe and K. Fukui [1] in the context of equivariant diffeomorphism groups under free action of compact Lie groups.

The case (2) is reduced to the perfectness of leaf preserving diffeomorphism groups on foliated manifolds.

Theorem. ([11], [12]) Suppose X is a C^∞ manifold with a C^∞ foliation $\mathcal{F}$. Let $\mathrm{Diff}^\infty_c(\mathcal{F})$ denote the group of C^∞ diffeomorphisms of M with compact support which send each leaf L of $\mathcal{F}$ to L itself. Let $\mathrm{Diff}^\infty_c(\mathcal{F})_0$ denote the subgroup of $\mathrm{Diff}^\infty_c(\mathcal{F})$ consisting of $f \in \mathrm{Diff}^\infty_c(\mathcal{F})$ which is isotopic to id_M by a compactly supported isotopy F with $F_t \in \mathrm{Diff}^\infty_c(\mathcal{F})$ $(t \in I)$. Then, $\mathrm{Diff}^\infty_c(\mathcal{F})_0$ is perfect.

In the case (2) the total space M is foliated by its fibers and the group $\mathrm{Diff}^r_{\pi,c}(M)_0$ includes the perfect subgroup $(\mathrm{Ker}\, P_c)_0$ of fiber preserving diffeomorphisms, where

$P_c : \mathrm{Diff}_{\pi,c}(M)_0 \longrightarrow \mathrm{Diff}_c(B)_0$, $P_c(f) = \underline{f}$, is a surjective group homomorphism and

$(\mathrm{Ker}\, P_c)_0 := \{F_1 \mid F \in \mathrm{Isot}^r_{\pi,c}(M)_0,\ \underline{F} = \mathrm{id}_{B\times I}\} \lhd \mathrm{Ker}\, P_c$.

Then, the condition $(*)$ in the case (2) follows from the following general observations.

Lemma 3.2. Suppose $\pi : M \to B$ is a C^r (N, Γ) bundle.

(1) If both $\mathrm{Diff}^r_c(B)_0$ and $\mathrm{Ker}\, P_c$ are perfect, then $\mathrm{Diff}^r_{\pi,c}(M)_0$ is perfect.

(2) $\mathrm{Ker}\, P_c = (\mathrm{Ker}\, P_c)_0$ in the case of the product (N, Γ) bundle $pr : B \times N \longrightarrow B$.

Our task is to add more examples of tuples (N, Γ, r, n) to the above list that satisfy Condition $(*)$.

3.3 Relative simplicity of bundle diffeomorphism groups

Lemma 3.3. $\zeta_g(f) \leq 4\, clb_\pi\, f$ for any $g \in \mathrm{Diff}^r_{\pi,c}(M)_0 - \mathrm{Ker}\, P$ and $f \in \mathrm{Diff}^r_{\pi,c}(M)_0$.

Theorem 3.3. If the tuple (N, Γ, r, n) satisfies Condition $(*)$, then

(i) the group $\mathrm{Diff}^r_{\pi,c}(M)_0$ is simple relative to $\mathrm{Ker}\, P$ and

(ii) $\zeta_g(f) \leq 4\, clb_\pi\, f \leq 8\nu(f) < \infty$ for any $g \in \mathrm{Diff}^r_{\pi,c}(M)_0 - \mathrm{Ker}\, P$ and $f \in \mathrm{Diff}^r_{\pi,c}(M)_0$.

Remark 3.1. $\mathrm{Diff}^r_{\pi,c}(M)_0$ is uniformly simple relative to $\mathrm{Ker}\, P$ if $clb_\pi d\, \mathrm{Diff}^r_{\pi,c}(M)_0 < \infty$.

4 Bundle diffeomorphism groups over a circle

In Sections 4, 5 we discuss the case that the base space B is a circle.

Suppose $\pi : M \to S^1$ is a C^r (N, Γ) bundle, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

4.1 Rotation angle in S^1

We fix a universal cover $\pi_{S^1} : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ and a distinguished point $p \in S^1$. This implies that [the total angle (length) of S^1] = 1. Let $\mathcal{P}(S^1)$ denote the set of C^0 paths in S^1.

Definition 4.1.

(1) The rotation angle $\lambda(c) \in \mathbb{R}$ of a path $c \in \mathcal{P}(S^1)$ is defined by
$\lambda(c) := \widetilde{c}(1) - \widetilde{c}(0)$, where $\widetilde{c} \in \mathcal{P}(\mathbb{R})$ is any lift of c to $\mathbb{R}$.

(2) The rotation angle $\mu_p(G)$ of an isotopy $G \in \mathrm{Isot}^0(S^1)$ is defined by
$\mu_p(G) := \lambda(G(p,*))$, the rotation angle of the path $G(p,*)$ in S^1.

4.2 Invariant k and quasimorphism ν

Note that each $F \in \mathrm{Isot}^0_\pi(M)_0$ induces $\underline{F} \in \mathrm{Isot}^0(B)_0$ such that $\underline{F}_t = \underline{F_t}$ $(t \in I)$ and that the surjective group homomorphism $R : \mathrm{Isot}^r_\pi(M)_0 \longrightarrow \mathrm{Diff}^r_\pi(M)_0 : R(F) = F_1$ has the kernel $\mathrm{Isot}^r_\pi(M)_{\mathrm{id,id}} := \{F \in \mathrm{Isot}^r_\pi(M)_0 \mid F_1 = \mathrm{id}_M\}$.

Fact 4.1.

(1) The map $\nu : \mathrm{Isot}^0_\pi(M)_0 \longrightarrow \mathbb{R} : \nu(F) := \mu_p(\underline{F}) \equiv \lambda(\underline{F}(p,*))$
is a surjective quasimorphism of defect 1.

(2) The map ν restricts to a group homomorphism $\nu : \mathrm{Isot}^r_\pi(M)_{\mathrm{id,id}} \longrightarrow \mathbb{Z}$.
Hence, there exists a unique $k = k(\pi, r) \in \mathbb{Z}_{\geq 0}$ such that $\nu(\mathrm{Isot}^r_\pi(M)_{\mathrm{id,id}}) = k\mathbb{Z}$.

(3) The map ν induces a surjective map $\widehat{\nu} : \mathrm{Diff}^r_\pi(M)_0 \longrightarrow \mathbb{R}/k\mathbb{Z} : \widehat{\nu}(f) = [\nu(F)]$, where $F \in \mathrm{Isot}^0_\pi(M)_0$ with $F_1 = f$.

4.3 Boundedness of $\mathrm{Diff}^r_\pi(M)_0$

The boundedness of the group $\mathrm{Diff}^r_\pi(M)_0$ is distinguished by the invariant $k = k(\pi, r)$.
[**1**] The case that $k \geq 1$:
Consider the following condition on $(N, \varGamma, r)$ (see Section 3.2).

$(*)$ $\mathrm{Diff}^r_{pr,c}(\mathbb{R} \times N)_0$ is perfect for the product $(N, \varGamma)$-bundle $pr : \mathbb{R} \times N \longrightarrow \mathbb{R}$.

Proposition 4.1. If $(N, \varGamma, r)$ satisfies the condition $(*)$, then the following hold.

(1) If $F \in \mathrm{Isot}^r_\pi(M)_0$ and $|\nu(F)| < \ell \in \mathbb{Z}_{\geq 1}$, then $clb_\pi(F_1) \leq 2\ell + 1$.

(2) If $f \in \mathrm{Diff}^r_\pi(M)_0$ and $\widehat{\nu}(f) = [s] \in \mathbb{R}/k\mathbb{Z}$ $\big(s \in \big(-\frac{k}{2}, \frac{k}{2}\big]\big)$, then

$$\frac{1}{4}(\lfloor |s| \rfloor + 2) \leq cl\, f \leq clb_\pi f \leq 2\lfloor |s| \rfloor + 3$$

Theorem 4.1. If $(N, \varGamma, r)$ satisfies the condition $(*)$, then

$$\frac{1}{8}(k+2) \leq cld\, \mathrm{Diff}^r_\pi(M)_0 \leq clb_\pi d\, \mathrm{Diff}^r_\pi(M)_0 \leq k+3.$$

Hence, $\mathrm{Diff}^r_\pi(M)_0$ is uniformly simple rel. $\mathrm{Ker}\, P$, and so it is bounded and uniformly perfect.

[**2**] The case that $k = 0$:

Theorem 4.2.

(1) The map $\widehat{\nu} : \mathrm{Diff}^r_\pi(M)_0 \longrightarrow \mathbb{R}$ is a surjective quasimorphism of defect 1.

- $\widehat{\nu}$ restricts to a surjective group homomorphism $\widehat{\nu} : \mathrm{Ker}\, P \longrightarrow \mathbb{Z}$.

(2) $\mathrm{Diff}^r_\pi(M)_0$ is unbounded and not uniformly perfect.

5 Description of the invariant k in term of the attaching map

5.1 Mapping torus and its attaching map

Suppose N is a C^r manifold and $\Gamma < \mathrm{Diff}^r(N)$. We regard as $S^1 = \mathbb{R}/\mathbb{Z}$. Then, any $\varphi \in \Gamma$ determines the mapping torus $\pi_\varphi : M_\varphi \longrightarrow S^1$, which is a C^r (N, Γ) bundle obtained from $N \times [0,1]$ by attaching $N \times \{1\}$ to $N \times \{0\}$ by φ. More formally, M_φ is defined by $M_\varphi = (N \times \mathbb{R})/\sim_\varphi$ and $\pi_\varphi([x,s]) := [s]$ $([x,s] \in M_\varphi)$,

where $(x,s) \sim_\varphi (y,t) \iff (y,t) = (\varphi^{-n}(x), s+n)$ for some $n \in \mathbb{Z}$.

The diffeomorphism φ is called the attaching map of this mapping torus.

Any (N, Γ) bundle over S^1 is isomorphic to a mapping torus π_φ for some $\varphi \in \Gamma$ and this attaching map φ is unique up to isotopy and conjugation in Γ. In particular, if $\varphi \simeq \mathrm{id}_N$ in Γ, then π_φ is trivial.

When N is non-compact, standard topologies on $\mathrm{Diff}^r(N)$ do not suit our purpose. Instead we can use diffeological notion. For the notion of paths in Γ, this coincides exactly with the following usual convention for diffeomorphism groups : A C^r path α in Γ means a C^r isotopy $\alpha = (\alpha_t)_{t \in I}$ on N with $\alpha_t \in \Gamma$ $(t \in I)$ and a C^r path-homotopy $\eta = (\eta_s)_{s \in I}$ rel ends in Γ means a C^r isotopy of isotopies η_s $(s \in I)$ in Γ such that $\eta_s(0) = \eta_0(0)$ and $\eta_s(1) = \eta_0(1)$ $(s \in I)$.

5.2 Description of $k(\pi_\varphi, r)$ in term of the attaching map φ

Our goal of this section is to describe the invariant $k = k(\pi_\varphi, r) \in \mathbb{Z}_{\geq 0}$ in term of the attaching map φ. First we observe the following basic fact.

Proposition 5.1. If $\varphi, \psi \in \Gamma$ and (i) $\varphi \simeq \psi$ (C^r isotopic) in Γ or (ii) φ, ψ are conjugate in Γ, then $\pi_\varphi \cong \pi_\psi$ and $k(\pi_\varphi, r) = k(\pi_\psi, r)$. In particular, if $\varphi \simeq \mathrm{id}_N$, then $k(\pi_\varphi, r) = 1$.

The following is the main theorem of this section.

Theorem 5.1. Let $\ell \in \mathbb{Z}$.

$\ell \in k\mathbb{Z} \iff$ There exists a C^r path-homotopy $\eta = (\eta_s)_{s \in I}$ rel ends in Γ such that $\eta_s(0) = \mathrm{id}_N$, $\eta_s(1) = \varphi^\ell$ $(s \in I)$ and $\varphi\eta_1 = \eta_0\varphi$.

This result leads us to consider the mapping class $[\varphi]$ of φ and its order in the mapping class group Γ/Γ_0, where $\Gamma_0 := \{\gamma \in \Gamma \mid \gamma \simeq \mathrm{id}_N \text{ in } \Gamma\}$, the identity component of Γ. Consider the orders $\ell := \mathrm{ord}(\varphi, \Gamma)$ and $m := \mathrm{ord}([\varphi], \Gamma/\Gamma_0)$. We also use the following notation :

$$\widehat{k} := \left|\mathbb{Z}/k\mathbb{Z}\right| = \begin{cases} k & (k \geq 1) \\ \infty & (k = 0). \end{cases}$$

Proposition 5.2. (1) $[\varphi]^k = 1 \quad \therefore\ m \,|\, k \qquad$ (2) $\ell < \infty \implies \ell \in k\mathbb{Z}$

Corollary 5.1. (1) $m = \infty \implies k = 0 \quad (\widehat{k} = \infty)$

(2) $\ell < \infty \implies m \,|\, k,\ k \,|\, \ell$ in $\mathbb{Z}_{\geq 1}$ $\qquad$ (3) $\ell = m \implies \widehat{k} = \ell = m$

Example 5.1. Suppose Σ is an orientable closed surface and $(N, \mathit{\Gamma}) := (\Sigma, \mathrm{Diff}^r(\Sigma))$.

(1) $\widehat{k} = \mathrm{ord}\,[\varphi]$ in $\mathit{\Gamma}/\mathit{\Gamma}_0$ for any $\varphi \in \mathit{\Gamma}$
by Nielsen realization theorem, Proposition 5.1 and Corollary 5.1 (3).

(2) When $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (a torus) :

(i) Each $A \in GL(2, \mathbb{Z})$ defines a linear diffeomorphism $\varphi_A \in \mathit{\Gamma} \equiv \mathrm{Diff}^\infty(T^2)$, $\varphi_A([\boldsymbol{x}]) = [A\boldsymbol{x}]$. This correspondence yields the group isomorphism
$$GL(2, \mathbb{Z}) \cong \mathit{\Gamma}/\mathit{\Gamma}_0 :\ A \longmapsto [\varphi_A].$$

(ii) For the attaching map $\varphi_A \in \mathit{\Gamma}$ and $k := k(\pi_{\varphi_A}, r)$
$\widehat{k} = \mathrm{ord}\,A$ in $GL(2, \mathbb{Z})$ $\qquad$ since $\mathrm{ord}\,A = \mathrm{ord}\,\varphi_A = \mathrm{ord}\,[\varphi_A]$.

For fiber products of bundles, we have the following conclusion. Suppose $\pi_\varphi : M_\varphi \to S^1$ is an $(N, \mathit{\Gamma})$ bundle $(\varphi \in \mathit{\Gamma} < \mathrm{Diff}^r(N))$ and $\pi_\psi : M_\psi \to S^1$ is an (L, Λ) bundle $(\psi \in \Lambda < \mathrm{Diff}^r(L))$. Since each $(\alpha, \beta) \in \Gamma \times \Lambda$ defines the product $\alpha \times \beta \in \mathrm{Diff}^r(N \times L)$, we have the group monomorphism $\iota : \underset{(\alpha,\beta)}{\Gamma \times \Lambda} \cong \underset{\alpha \times \beta}{\iota(\Gamma \times \Lambda)} < \mathrm{Diff}^r(N \times L)$. Then, the attaching map $\varphi \times \psi$ determines an $(N \times L, \iota(\Gamma \times \Lambda))$ bundle $\pi_{\varphi\times\psi} : M_{\varphi\times\psi} \to S^1$.

Proposition 5.3.

(1) $k(\pi_{\varphi\times\psi}, r) \geq 1 \iff k(\pi_\varphi, r) \geq 1$ and $k(\pi_\psi, r) \geq 1$
In this case $\quad k(\pi_{\varphi\times\psi}, r) = \mathrm{lcm}(k(\pi_\varphi, r), k(\pi_\psi, r))$.

(2) $k(\pi_{\varphi\times\psi}, r) = 0 \iff k(\pi_\varphi, r) = 0$ or $k(\pi_\psi, r) = 0$

From Corollary 5.1 it is interesting to recognize the order of mapping classes $[\varphi]$ in the mappiing class group $\mathit{\Gamma}/\mathit{\Gamma}_0$ in various cases, for example, the groups of symplectomorphisms or contactomorphisms, etc.

5.3 Principal bundle case

In the case of a principal G bundle, all results in §5.2 are translated into terms of G itself. In more details, for a Lie group G, a principal G bundle is exactly a (G, G_L) bundle, where $G_L \equiv \{\varphi_a \mid a \in G\} < \mathrm{Diff}^\infty(G)$ and φ_a is the left translation by a on G. Since the canonical isomorphism $G \cong G_L$ is also a C^∞ diffeomorphism in the diffeological sense, it follows that C^r isotopies in G_L reduce to C^r paths in G and that for $a \in G$ all statements on $\varphi_a \in G_L$, π_{φ_a} and $k_a := k(\pi_{\varphi_a}, r)$ are translated into terms of a itself in G. They are summarized as follows. The symbol G_0 denotes the component of the unit element e in G.

Proposition 5.4. For $a, b \in G$
$k_a = k_b$ if (i) a, b are conjugate in G or (ii) there exists a path from a to b in G.
In particular, $k_a = 1$ if $a \in G_0$.

Theorem 5.2. For $\ell \in \mathbb{Z}$

$\ell \in k_a\mathbb{Z} \iff$ There exists a path γ in G from e to a^ℓ such that $\gamma \simeq_* a^{-1}\gamma a$ in G.

Here, the symbol $\simeq_*$ denotes a path-homotopy rel ends.

Corollary 5.2. Let $\ell \equiv \ell_a := \mathrm{ord}\,(a, G)$ and $m \equiv m_a := \mathrm{ord}\,([a], G/G_0)$.

(1) $a^{k_a} \in G_0$ and $m \,|\, k_a$ (2) If $m = \infty$, then $k_a = 0$ (or $\widehat{k}_a = \infty$).

(3) If $\ell < \infty$, then $m \,|\, k_a$, $k_a \,|\, \ell$ in $\mathbb{Z}_{\geq 1}$ (4) If $\ell = m$, then $\widehat{k}_a = \ell = m$.

Example 5.2.

(1) $k_a = 1 \iff a \in G_0$

- If G is connected, then $k_a = 1$.
  (ex. $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $SL(n, \mathbb{R})$, $U(n)$, $SU(n)$, $SO(n)$, $\mathbb{R}^n$, T^n, etc)

(2) $k_a = 2$ for $G = GL(n, \mathbb{R})$, $O(n)$ and $a \in G^- := \{c \in G \mid \det c < 0\}$

Note that there exists $c \in G^-$ with $c^2 = e$ and that $G^- = c\,G_0$, $G = G_0 \cup c\,G_0$.

(3) $\widehat{k}_a = m_a$ in the following cases :

(i) G is commutative (ii) G_0 is simply connected

(4) If G is discrete ($G_0 = \{e\}$), then $\widehat{k}_a = \ell_a = m_a$

(i) If G is a finite group, then $k_a = \ell_a$

(ii) If $G = \mathbb{Z}$ and $a \in \mathbb{Z} - \{0\}$, then $k_a = 0$ ($\widehat{k}_a = \infty$).

Example 5.3.

Suppose G, H are Lie groups, $a \in G$, $k_a := k(\pi_{\varphi_a}, r)$ and $b \in H$, $\ell_b := k(\pi_{\varphi_b}, r)$

(1) Consider the product $G \times H$. For $(a, b) \in G \times H$ it is seen that

$$\varphi_{(a,b)} = \varphi_a \times \varphi_b \in (G \times H)_L, \text{ where } \varphi_a \in G_L \text{ and } \varphi_b \in H_L.$$

(i) $k_{(a,b)} \equiv k(\varphi_{(a,b)}, r) \geq 1 \iff k_a \geq 1$ and $\ell_b \geq 1$
In this case, $k_{(a,b)} = \mathrm{lcm}(k_a, \ell_b)$.

(ii) $k_{(a,b)} = 0 \iff k_a = 0$ or $\ell_b = 0$.

(2) Suppose $f : G \to H$ is a Lie group homomorphism and $b = f(a)$. Then,

(i) $k_a \in \ell_b\mathbb{Z}$,

(ii) $k_a = \ell_b$ if f is surjective, $\mathrm{Ker}\, f$ is path-connected and the inclusion $\iota : \mathrm{Ker}\, f \subset G$ induces the zero homomorphism $\iota_* = 0 : \pi_1(\mathrm{Ker}\, f, e) \longrightarrow \pi_1(G, e)$.

At the moment the following conjecture is still open.

Question. Is it true that $k_a \geq 1$ for any compact Lie group G and any $a \in G$?

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