Parallel Minority Games Using the Block Norm

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1. Introduction

The El Farol Bar problem, introduced by W. Brian Arthur in 1994, is a classic coordination game that illustrates how individuals make decisions based on limited information and collective behavior [1]. The scenario involves a bar in Santa Fe, New Mexico, that offers live music on Thursday nights. The bar is enjoyable only if it is not too crowded, creating a dilemma for individuals deciding whether to attend. If more than 60% of the group goes, the bar becomes overcrowded, making the experience unpleasant. Conversely, if fewer than 60% attend, those who do go have a good time.

This situation highlights a paradox: while each person wants to maximize their enjoyment by attending when the bar is not too crowded, the lack of communication and coordination among individuals makes it hard to predict attendance. As a result, attendance fluctuates as people adapt their strategies based on past outcomes and expectations. The El Farol Bar problem demonstrates how agents with limited information and no central control can exhibit complex collective behavior, providing valuable insights into decentralized decision-making.

Agents in the El Farol Bar problem often use adaptive heuristics to predict attendance, relying on simple rules based on previous trends [4]. These dynamics have inspired research across various fields, including economics, game theory, and complex systems, emphasizing the challenges of achieving equilibrium in decentralized settings. Social dynamics such as imitation also influence outcomes: agents who share information locally may form groups that act collectively, creating disparities in resource distribution [9]. Interestingly, the effectiveness of such coordination depends on the probability of imitation, revealing a trade-off between social tensions and global efficiency.

Adding further complexity, incomplete strategies, where agents lack information about some of their choices, require them to adapt dynamically [10]. This randomness can expand the set of uncorrelated strategies, improving overall performance by achieving stable outcomes with fewer computational steps. Additionally, introducing variable payoffs—where fewer participants result in higher individual rewards demonstrates that while initial dynamics change, the Minority Game remains resilient, eventually returning to patterns similar to the standard model after a transition phase [7].

Another intriguing extension involves constrained information flow. In real-life scenarios, not all agents have equal access to information [2]. For instance, only those who attended the bar might know its actual attendance, while others might rely on incomplete or delayed information from social networks. This creates opportunities for arbitrage, where better-informed agents gain an advantage, adding new layers to the game's dynamics.

Despite its simplicity, the Minority Game is a powerful model for understanding systems where agents strive to be in the minority [6]. The interplay of crowds and anti-crowds explains fluctuations in attendance both intuitively and quantitatively. The model is also applicable to broader contexts, such as traffic congestion, where agents' decisions to switch lanes resemble a Minority Game, and the resulting network efficiency depends on both individual behavior and infrastructure.

In markets, the Minority Game models interactions among producers, speculators, and noise traders, where the flow of information is key. Producers rely on fundamental data, while speculators exploit arbitrage opportunities, adapting to changing conditions. These dynamics underscore the trade-offs and challenges in decision-making under uncertainty.

A more complex variation arises when agents face multiple choices rather than a binary decision [5]. In such cases, agents may only consider a subset of options at a time, leading to parallel, coupled Minority Games. A stochastic strategy often performs well here, balancing resource allocation and maximizing utility. For example, this model has been applied to population movements during pandemics, explaining how individuals relocate between regions to optimize outcomes.

Building on these principles, this study extends the Minority Game to a multi-choice framework, where agents independently select from numerous destinations, each offering distinct, variable payoffs. Unlike traditional models with binary choices, this setup reflects real-world scenarios like market trading, traffic routing, and epidemic response strategies, where agents face a wider array of options.

The multi-choice Minority Game introduces new complexities in resource allocation and payoff optimization. Agents must avoid overcrowded destinations while dynamically adjusting their strategies to fluctuating benefits. To manage these decisions, agents use stochastic strategies, which probabilistically balance distribution and maximize individual gains.

Through simulations, such as job-hunting movements, the model demonstrates how agents optimize relocations to reduce risks and improve well-being [8]. These findings offer insights into efficient resource utilization and decision-making in decentralized, dynamic systems, providing a framework for understanding and optimizing complex environments.

2 Mathematical Model

In this section, we present a model where agents choose destinations based on payoffs that decrease with crowding. Agents use predictive strategies to estimate payoffs, adapt these strategies through reinforcement learning, and operate under information constraints, often making decisions with incomplete data. Simulations, such as job-hunting scenarios, evaluate agent distribution, payoff variability, and resource efficiency.

Agents and Decisions

Let N be the number of agents in the system, indexed by $i = 1, 2, \ldots, N$. Each agent i has to decide at each time step t which destination d to go to from a set of possible destinations $D = d_1, d_2, \ldots, d_m$.

Payoffs

Each destination $d \in D$ has a variable payoff $P_d(t)$ at time t, which depends on the number of agents $n_d(t)$ who choose that destination at time t.

$$P_d(t) = f(n_d(t))$$

The function f can take various forms depending on the specific problem, but it typically decreases with $n_d(t)$ due to congestion effects. For simplicity, let's assume:

$$P_d(t) = \frac{A_d}{(1 + n_d(t))}$$

where A_d is a constant representing the maximum payoff for destination d when it is not crowded.

Decision Strategy

Each agent i maintains a set of strategies S_i to predict the payoffs for each destination. These strategies could be based on past attendance and payoffs. At each time step t, agent i uses one of its strategies to estimate the expected payoff $\widehat{P}d_{i*}(t)$ for each destination d.

Agent i then chooses the destination $d_{i*}(t)$ that maximizes their expected payoff:

$$d_{i*}(t) = arg \max_{d \in D} \widehat{P}d_i(t)$$

Adaptive Strategy Updates

Agents adapt their strategies based on the observed payoffs. After choosing a destination and observing the actual payoff, each agent updates their strategies to improve future predictions. This can be modeled using a reinforcement learning approach, where the expected payoff is updated as follows:

$$\widehat{P}_{i,d(t+1)} = \widehat{P}_{i,d(t)} + \alpha(P_{d(t)} - \widehat{P}_{i,d(t)})$$

where α is a learning rate.

Information Constraints

To incorporate information constraints, assume that not all agents have complete information about past attendances and payoffs. Let ρ be the probability that an agent receives accurate information about the outcomes at each destination. Agents can then make decisions based on either complete or incomplete information, adding another layer of realism to the model.

Simulation and Analysis

The model's behavior can be analyzed through simulations. For instance, consider a scenario where agents must optimize their relocations during a job-hunting process to maximize their career prospects and overall satisfaction. The simulation would involve:

1. **Initializing the number of agents (N) and job opportunities (D)**:
   Agents represent job seekers, and destinations represent cities or companies offering job opportunities.
2. **Assigning initial payoffs (Ad) for each destination**:
   Payoffs could reflect factors like average salaries, job stability, work-life balance, or cost of living in a particular destination.
3. **Running the model for a number of time steps**:
   Agents would evaluate job opportunities based on their strategies (e.g., prioritizing salary over work-life balance), choose destinations, receive payoffs based on their chosen jobs, and update their strategies based on feedback (e.g., job satisfaction or peer performance).
4. **Analyzing outcomes**:
   Metrics to evaluate include the distribution of agents across destinations, variability in payoffs (e.g., income inequality), and efficiency of resource allocation (e.g., whether all jobs are filled or some destinations are overcrowded while others are neglected).

3 Game Mechanics:

In this section, we present a 2D Minority Game where agents move within a bounded plane using predefined strategies. Agents select from 8 movement options, updating positions unless moving out of bounds. After moves, agents in the minority region earn a win, while strategies are updated based on results. This model introduces asymmetrical efficiency between regions.

1. **Initial Setup**:
   - **Agents** start at random locations in the 2D plane $[-1,1]\times[-1,1]$
   - Each agent selects **3 out of 8 possible strategies**, where each strategy corresponds to a vector with a **fixed radius** R and an angle θ (cardinal directions, including 45° angles).
2. **Strategy Selection**:
   - The agents have 8 possible movement directions: North, South, East, West, and the 45° angles (NE, NW, SE, SW).
   - At each time step t, the agent selects one of their 3 strategies (movement vectors).

3. **Movement**:
    - If the selected vector keeps the agent inside the [−1,1]×[−1,1] boundary, they update their position by moving along the vector.
    - If the vector moves the agent outside the boundary, they remain in their current position and are **not counted** in the minority game for that round, but still incur a loss for the chosen strategy.
4. **Minority Game**:
    - After agents make their moves, they are counted in either the **positive** or **negative** region of the 2D plane, based on their position.
    - Agents in the minority region (either positive or negative) receive a **win** (1), and those in the majority get a **loss** (0).
5. **Score Update**:
    - The score for each strategy is updated based on the result of the minority game: score($t+1$)=score(t)+δ×result
    - The **result** is 1 for a win and 0 for a loss.
    - If an agent is not counted (due to being out of bounds), the strategy still incurs a loss.
6. **Efficiency**
    - Unlike standard minority game that has a symmetric efficiency between the 2 areas, in this model, we need to represent each areas' own efficiency:

$$E_i(t) = \frac{1}{N}\frac{1}{t}\sum_{\tau=1}^{t}(N_i(\tau) - \frac{N}{4})^2$$

$E_i(t)$: efficiency of the region i at time t

$N_i(t)$: number of agents in the region i at time t

N: total number of agents

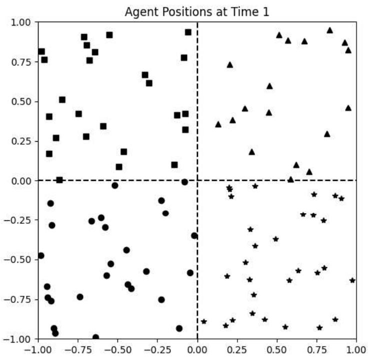
Considerations:

- **Exploration**: Over time, agents may start favoring strategies that have historically placed them in the minority, leading to an evolution of strategy preferences.
- **Edge Behavior**: Agents near the boundaries may need to adjust their strategies more cautiously, as selecting an out-of-bound move results in a guaranteed loss.

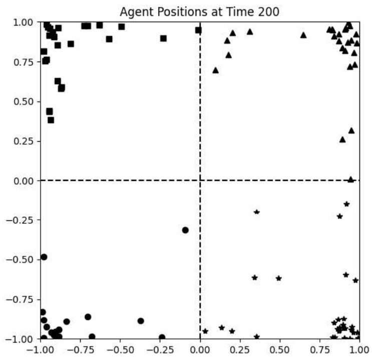
Computation of the simulation

Control

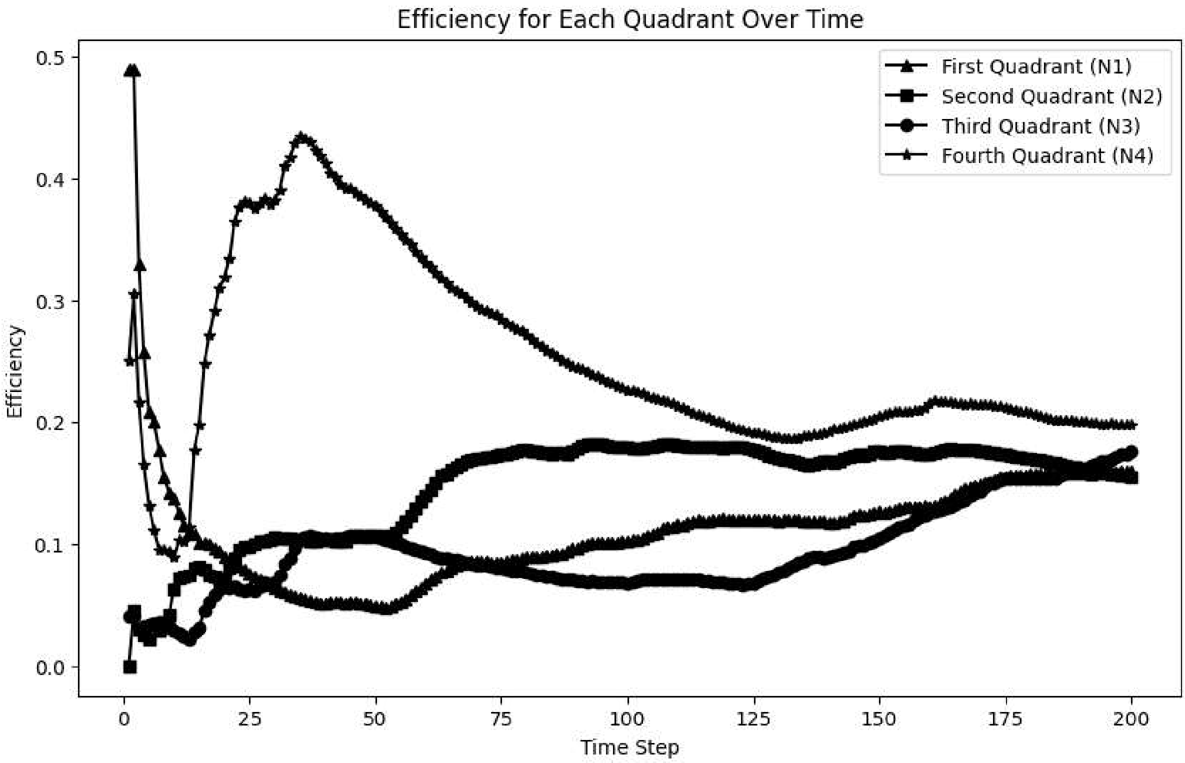
The first and simplest way of representing our model will be by using 2 lines along the orthonormal axis intersecting at (0,0). Thus having 4 equal areas where the agents will thrive with their 3 given strategies the main parameters used are 100 agents, a movement radius (speed) of 0.2, a δ to update the strategies' score of 0.1 and we observe the model for 200 rounds.



(1a) initial position of agents



(1b) final position of agents



(1c) efficiency of the quadrants (control)

Observations control

Figure (1a) and (1b) represent the location of the agents initially and after 200 iterations. Figure (1c) is the efficiency over time for each quadrant, the lower the value, the more evenly spread the agents are. Most of the agents tend to flee the center and gather in the corners. There are 2 types of agents: the **passive agents** that prefer to stay in a specific area (usually the corner) relying on the **flexible agents** oscillating between 2 or more areas, leading to actual changes in the minority games. The quadrants are behaving by pairs and after a few rounds increasing the efficiency, there's a decreasing tendency.

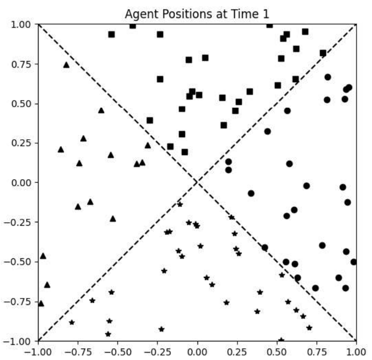
Analysis control

Agents gravitate toward the corners because they avoid the competition and congestion at the center, which likely has higher agent density at the beginning. Corners act as "natural boundaries," preventing further movement and creating a stable zone for "passive agents." The presence of flexible agents, who oscillate between areas, helps adjust the local densities, but the inherent competition at the center makes the corners more attractive for stability.

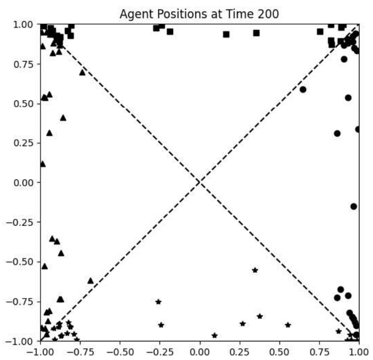
Quadrant pairing emerges because agents on opposite sides adjust to each other's presence, creating a feedback loop that equalizes the density across paired quadrants. The efficiency decrease after an initial rise reflects diminishing returns on strategic adjustments as agents settle into patterns that no longer exploit the minority conditions optimally.

Diagonal

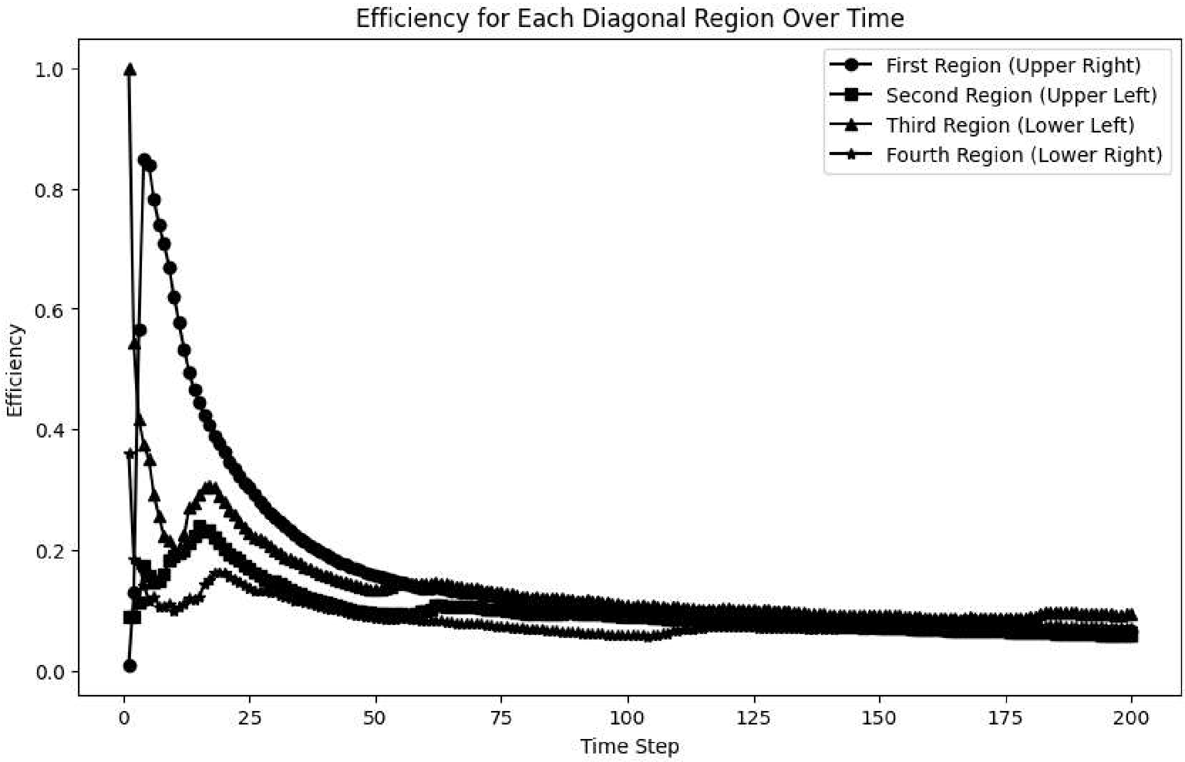
The second model will use 2 lines along the diagonal of the orthonormal axis intersecting at (0,0). Thus having 4 equal areas where the agents will thrive with their 3 given strategies the main parameters used are 100 agents, a movement radius (speed) of 0.2, a δ to update the strategies' score of 0.1 and we observe the model for 200 rounds.



(2a) initial position of agents



(2b) final position of agents



(2c) efficiency of the quadrants (diagonal)

Observations cross

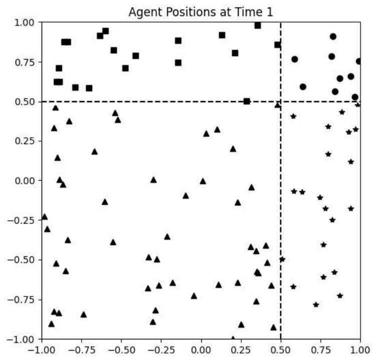
Figure (2a) and (2b) represent the location of the agents initially and after 200 iterations for the cross experiment. On the cross section, we still have the paired quadrants, the corner effect and the 2 types of agents. On figure (2c) we can observe that the transition phase is different but still tends to decrease.

Analysis cross

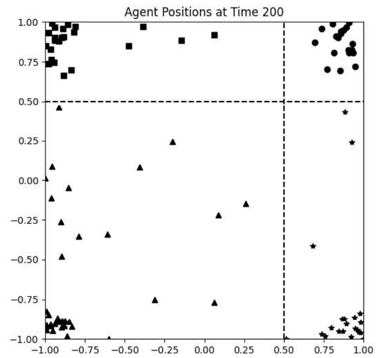
Even with diagonal partitions, the corner effect remains due to the natural constraint provided by boundaries, which stabilize agent distributions. Pairing of quadrants persists because agents adjust to minority conditions dynamically, maintaining an equilibrium between paired regions despite the new diagonal orientation.
The transition phase shows a different trajectory due to the altered geometry of the regions, which affects how quickly agents can identify and exploit minority conditions. While the system eventually reaches a similar efficiency trajectory, the initial dynamics are influenced by the new constraints introduced by diagonal lines.

Shifted center

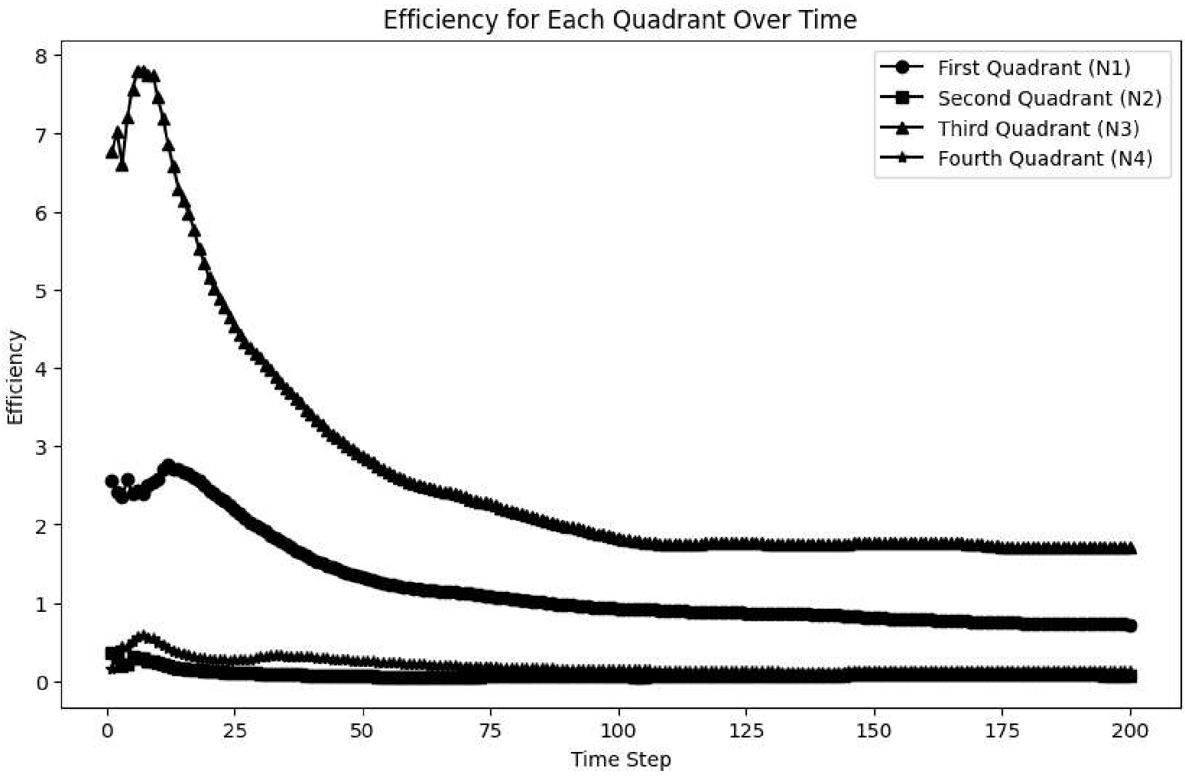
The third model will use 2 lines along the orthonormal axis intersecting at (0.5, 0.5). Thus having a big, a small and 2 equal areas where the agents will thrive with their 3 given strategies the main parameters used are 100 agents, a movement radius (speed) of 0.2, a δ to update the strategies' score of 0.1 and we observe the model for 200 rounds.



(3a) initial position of agents



(3b) final position of agents



(3c) efficiency of the quadrants (shifted center)

Observations shifted center

Figure (3a) and (3b) represent the location of the agents initially and after 200 iterations for the shifted center experiment. According to Figure (3c), the maximum efficiency value is about 10 times greater than usual but this model has the best decreasing curbs, even though the minimal value obtained is still high compared to the previous models. Here the pairing is most likely to be between the opposite quadrants probably due to the surface area.

Analysis shifted center

The shifted center creates regions of unequal size, causing agents to redistribute unevenly. Larger regions have more capacity to absorb agents, which increases efficiency values. Pairing between opposite quadrants is likely driven by surface area disparities, as agents are drawn to larger regions while still responding to local minority dynamics.
The asymmetry in quadrant sizes reduces the scope for extreme minority conditions, which keeps the efficiency values relatively higher. Smaller regions are less likely to attract a disproportionate number of agents, leading to more balanced distributions and limiting efficiency dips.

General Observations Across Models

Corner Clustering: Corners provide a predictable environment with less competition compared to open spaces or crowded central areas.

Quadrant Pairing: Feedback mechanisms between adjacent or opposite regions ensure that agent distributions dynamically stabilize, reflecting the minority-seeking behavior.

Decreasing Efficiency: Over time, agents settle into quasi-stable patterns that reduce the potential for exploiting minority conditions, leading to lower efficiency in resource utilization.

Conclusion

This study has demonstrated the adaptability and robustness of the Minority Game framework when extended to multi-choice and spatially distributed scenarios. By simulating agent behavior in 2D planes with varying partitioning schemes, we have observed distinct dynamics in agent distribution and efficiency across regions. The model highlights the interplay between agent strategy selection, resource allocation, and spatial constraints, revealing nuanced behaviors such as corner clustering and quadrant pairing. These findings underline the Minority Game's utility in exploring decentralized decision-making and resource optimization in complex environments. Future research could extend these results by incorporating more dynamic payoff structures, agent heterogeneity, and real-world applications such as urban planning and market dynamics.

References

[1] W. B. Arthur, “Inductive reasoning and bounded rationality”, *American Economic Review* 84 (1994) 406.
[2] B. C. Lustosa and D. O. Cajueiro, “Constrained Information Minority Game: How Was the Night at El Farol?”, *Physica A* 389 (2010) 1230–1238
[3] J. Linde, J. Sonnemans and J. Tuinstra, “Strategies and Evolution in the Minority Game: A Multi-Round strategy experiment”, *Games and Economic Behavior* 86 (2014) 77–95
[4] F. Slanina, “Social Organization in the Minority Game Model”, *Physica A* 286 (2000) 367–376
[5] S. Biswas, A. K. Mandal, “Parallel Minority Game and its application in movement optimization during an epidemic”, *Physica A* 561 (2021) 125271
[6] D. Challet, M. Marsili and Yi-Cheng Zhang, ”Modeling Market Mechanism with Minority Game”, *Physica A* 246, 407 (1997)
[7] Y. Li, A. VanDeemen and R.Savit, “The Minority Game with Variable Payoffs”, *Physica A* 284 (2000) 461–477
[8] S. Gourley and N. F. Johnson, “Effects of decision-making on the transport costs across complex networks”, *Physica A* 363 (2006) 82–88
[9] M. Harta, P. Jefferies, N.F. Johnson, P.M. Hui, “Crowd–Anticrowd Theory of the Minority Game”, *Physica A* 298 (2001) 537–544
[10] C. Yang, S. Sun “The Minority Game with Incomplete Strategies.” *Physica A* 379 (2007) 645–653

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