ON SOME GENERALIZATIONS OF CONTRACTIONS IN PROBABILISTIC METRIC SPACES AND IN FUZZY METRIC SPACES

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1. Introduction and Preliminaries

Probabilistic metric spaces were introduced in 1942 by Menger [20]. The notion of distance between two points x and y is replaced by a distribution function $F_{x,y}$. Sehgal, in his Ph.D. Thesis [33], extended the notion of a contraction mapping to the setting of the Menger probabilistic metric spaces. The probabilistic version of the classical Banach Contraction Principle was first studied in 1972 by Sehgal and Bharucha-Reid [34]. After that many authors have obtained fixed point theorems for probabilistic φ-contractions under the assumption that φ is nondecreasing and such that $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for any $t > 0$ (see, e.g., [6] and the references in [5]). Ćirić [5] consider the more weak conditions and Jachymski [18] correctly defined the conditions and give tha following Theorem.

Theorem 1. *(See Jachymski [18].) Let (X, F, Δ) be a complete Menger probabilistic metric space with a continuous t-norm Δ of H-type, and let $\varphi : R_+ \to R_+$ be a function satisfying conditions:*

$$0 < \varphi(t) < t \text{ and } \lim_{n\to\infty} \varphi(t) = 0 \text{ for all } t > 0.$$

If $T : X \to X$ is a probabilistic φ-contraction, then T has a unique fixed point $x^* \in X$, and $\{T^n(x_0)\}$ converges to x^* for each $x_0 \in X$.

Let R denote the real number and $R_+ = \{x \in R \mid x > 0\}$. A mapping $F : R \to R_+$ is called a distribution if it is non-decreasing left-continuous with $\sup_{t\in R} F(t) = 1$ and $\inf_{t\in R} F(t) = 0$. The set of all distribution functions is denoted by $\mathcal{D}$, and $\mathcal{D}_+ = \{F | F \in \mathcal{D}, F(0) = 0\}$. A special element H of $\mathcal{D}_+$ is defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:

(i) $\Delta(a, 1) = a$;
(ii) $\Delta(a, b) = \Delta(b, a)$;
(iii) $a \geq b$, $c \geq d$ implies $\Delta(a, c) \leq \Delta(b, d)$;
(iv) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

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Definition 2. (Menger [20]., Schweizer and Sklar [35]). A triplet (X, F, Δ) is called a Menger probabilistic metric space (for short, a Menger space) if X is a non-empty set, Δ is a t-norm and F is a mapping from $X \times X$ into $\mathcal{D}$ satisfying the following conditions (for $x, y \in X$, we denote $\mathcal{F}(x, y)$ by $F_{x,y}$):

(1) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;
(2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;
(3) $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $s, t > 0$.

Schweizer et al. [31, 32] point out that if the t-norm Δ of a Menger PM-space (X, F, Δ) satisfies the condition

$$\sup_{0<t<1} \Delta(t, t) = 1,$$

then (X, F, Δ) is a Hausdorff topological space in the (ε, λ)-topology τ, i.e., the family of sets

$$\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1]\}(x \in X)$$

is a basis of neighborhoods of point x for T, where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$. By virtue of this topology T, a sequence $\{x_n\}$ in (X, F, Δ) is said to be τ-convergent (simply convergent) to $x \in X$ (we write $x_n \to x$) if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a τ-Cauchy (simply Cauchy) sequence in (X, F, Δ) if for any given $\varepsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) \geq 1 - \lambda$, whenever $n, m \geq N$; (X, F, Δ) is said to be τ-complete (simply comp), if each τ-Cauchy sequence in X is τ-convergent to some point in X. In what follows, we will always assume that (X, F, Δ) is a Menger space with the (ε, Δ)-topology

Lemma 3. *(Sehgal and Bharucha-Reid [34]). Let (X, d) be a metric space. Define a mapping $F : X \times X \to \mathcal{D}$ by*

(1.1) $$F_{x,y}(t) = H(t - d(x, y)) \text{ for any } x, y \in X \text{ and } t > 0.$$

Then $(X, F, \min)$ is a Menger space. and it is called the induced Menger space by (X, d), and it is complete if (X, d) is complete.

Definition 4. (Hadžíc [11], Hadžíc and Pap [14]). A t-norm Δ is said to be of H-type (Hadžič̃ type) if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where $\Delta^1(t) = \Delta(t, t)$,

$$\Delta^m(t) = \Delta(t, \Delta^{m-1}(t)), m = 1, 2, ..., t \in [0, 1].$$

The t-norm $\Delta_M = min$ is a trivial example of t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_M$ (see, e.g., [13]).

Definition 5. Let (X, F, Δ) be a menger metric space. A mapping $T : X \to X$ is called asymptotic regular if for every $\varepsilon > 0$ and every $\lambda > 0$, there exists an integer $M_{\varepsilon,\lambda}$ such that

$$F_{T^n x, T^{n+1} x}(\varepsilon) > 1 - \lambda$$

whenever $n > M_{\varepsilon,\lambda}$. In this case we write $\lim_{n\to\infty} F_{x_n,x_{n+1}}(\varepsilon) = 1$.

Next we define the φ-K contractions and φ_n-K contractions in Menger sapces.

Definition 6. Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with continuous t-norm Δ and a mapping $\varphi : R_+ \to R_+$ satisfying for any $t > 0$ there exists $t_1, t_2 > 0$, $0 \le K < \infty$ and $r \ge t$ such that $0 \le \varphi(r) + K(t_1 + t_2) < t$. Then $T : X \to X$ is a probabilistic φ-K contraction if T satisfy the following inequality:

$$F_{Tx,Ty}(\varphi(r) + K(t_1 + t_2)) \ge \Delta\left(F_{x,y}(t),\, F_{x,Tx}(Kt_1),\, F_{y,Ty}(Kt_2)\right) \tag{1.2}$$

for any $K \ge 0$.

Definition 7. Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with continuous t-norm Δ and mappings $\varphi_n : R_+ \to R_+$ satisfying for any $t > 0$ there exists $t_1, t_2 > 0$, $0 \le K < \infty$ and $r \ge t$ such that $0 \le \sum_{n=1}^{\infty} \varphi_n(r) + K(t_1 + t_2) < t$. Then $T : X \to X$ is a probabilistic φ_n-K contraction if T satisfy the following inequality:

(1.3)
$$F_{T^n x,T^n y}(\varphi_n(r) + K(t_1 + t_2)) \ge \Delta\left(F_{x,y}(t),\, F_{T^{n-1}x,T^n x}(Kt_1),\, F_{T^{n-1}y,T^n y}(Kt_2)\right)$$

for any $K \ge 0$.

.

2. Main result

We give the following Theorem.

Theorem 8. *Let (X, F, Δ) be a complete Menger space such that Δ is a continuous triangular norm of Hadžić type. Let $\varphi : R_+ \to R_+$ be a mapping such that for any $t > 0$, there exist $0 \le K < \infty$, t_1, $t_2 > 0$ and $r \ge t$ such that $\varphi(r) + Kt_1 + Kt_2 < t$ and a mapping $T : X \to X$ be asymptotic regular. If T is a probabilistic φ-K contraction, then T has a unique fixed point x^*, and for any $x_0 \in X$ $\lim_{n\to\infty} T^n x_0 = x^*$.*

Proof. Let $x_0 \in X$ and $x_n := Tx_{n-1}$ for any $n \in N$. Since T is asymptotic regular, we have

$$\lim_{n\to\infty} F_{x_n,Tx_n}(t) = 1 \tag{2.1}$$

for any $t > 0$.

Now let $n \in N$ and $t > 0$, then there exist $0 \le K < \infty$, t_1, $t_2 > 0$ and $r \ge t$ such that $\varphi(r) + Kt_1 + Kt_2 < t$. We show by induction that, for any $k \in N$,

$$\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) \ge \Delta^k(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \varphi(t)). \tag{2.2}$$

Since the mapping T is asymptotic regular, we have

$$\lim_{n\to\infty} F_{y_n,y_{n+1}}(Kt_1) = 1,\ \lim_{n\to\infty} F_{y_{n+k},y_{n+k+1}}(Kt_2) = 1.$$

Putting $\psi(t) = \varphi(r(t)) + K(t_1 + t_2)$, note that

$$F_{Tx,Ty}(\psi(t)) \ge \Delta(F_{x,y}(t), F_{x,Tx}(Kt_1), F_{y,Ty}(Kt_2)),$$

then we have

$$
\begin{aligned}
&F_{x_n,x_{n+k+1}}(t)\\
&= F_{x_n,x_{n+k+1}}(t-\psi(t)+\psi(t))\\
&= \Delta(F_{x_n,x_{n+1}}(t-\psi(t)), F_{x_{n+1},x_{n+k+1}}(\psi(t)))\\
&\geq \Delta\left(F_{x_n,x_{n+1}}(t-\psi(r)), \Delta\left(F_{x_n,x_{n+k}}(t),\, F_{x_n,x_{n+1}}(Kt_1),\, F_{x_{n+kr},x_{n+k+1}}(Kt_2)\right)\right).
\end{aligned}
$$

In this case note that we have

$$
\begin{aligned}
&\Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+k}}(t),\, \lim_{n\to\infty} F_{x_n,x_{n+1}}(Kt_1),\, \lim_{n\to\infty} F_{x_{n+kr},x_{n+k+1}}(Kt_2)\right)\\
&= \Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+k}}(t),\, 1,\, 1\right)\\
&= \Delta^k\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t-\psi(t))\right).
\end{aligned}
$$

Since Δ is continuos, we have

$$
\begin{aligned}
&\lim_{n\to\infty} F_{x_n,x_{n+k+1}}(t)\\
&= \Delta\left(\lim_{n\to\infty} F_{x_n,x_{n+k+1}}(t-\psi(t)), \Delta^k\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t-\psi(t))\right)\right)\\
&= \Delta^{k+1}\left(\lim_{n\to\infty} F_{x_n,x_{n+1}}(t-\psi(t))\right).
\end{aligned}
$$

We show that sequence $\{x_n\}$ is Cauchy, that is,

$$\lim_{m,n\to\infty} F_{x_n,x_m}(t) = 1 \text{ for any } t > 0.$$

Let $t > 0$ and $\varepsilon > 0$. By hypothesis, $\{\Delta^n \mid n \in N\}$ is equicontinuous at 1 and $\Delta^n(1) = 1$, so there is $\delta > 0$ such that

$$\text{if } s \in (1-\delta, 1],\ \text{then } \Delta^n(s) \geq 1-\varepsilon \text{ for all } n \in N. \tag{2.3}$$

Since T is asymtotic regular, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t-\psi(r)) = 1.$$

Then there exists $n_0 \in N$ such that, for any $n \geq n_0$,

$$F_{x_n,x_{n+1}}(t-\psi(r)) \in (1-\delta, 1].$$

Hence, by (3.3) and (3.7), we get $F_{x_n,x_{n+k}}(t) > 1-\varepsilon$ for any $k \in N \cup \{0\}$. This proves the Cauchy condition for $\{x_n\}$. By completeness, $\{x_n\}$ converges to some $p \in S$, that is, $\lim_{n\to\infty} F_{x_n,p}(t) = 1$. for any $t > 0$. We show that p is a fixed point of T. By monotonicity and continuity of Δ, we get

$$
\begin{aligned}
F_{p,Tp}(t) &\geq \Delta(F_{p,x_{n+1}}(t-\psi(t)), F_{Tp,Tx_n}(\psi(t)))\\
&\geq \Delta(F_{p,x_{n+1}}(t-\psi(t)), F_{p,x_n}(t)).
\end{aligned}
$$

Then we have $\lim_{n\to\infty} \Delta(F_{p,x_{n+1}}(t-\psi(t)), F_{p,x_n}(t)) = \Delta(1,1) = 1$. This yields $F_{p,Tp}(t) = 1$ for any $t > 0$, and hence $p = Tp$.

Finally, we show the uniqueness of a fixed point. Let p and q be fixed point of mapping T with $p \neq q$. Then $F_{p,q}(t) < 1$ for any $t > 0$. Since T is asymptotic regular, sequence $\{x_n\}$ in X satisies $\lim_{n\to\infty} F_{Tx_n,Tx_{n+1}}(t) = 1$ for any $t > 0$. In this case it can be prove that $\{Tx_n\}$ is Cauchy and converges in X. Then for the p, q, any $t > 0$ we have $\lim_{n\to\infty} F_{Tx_n,p}(t) = 1$ and $\lim_{n\to\infty} F_{Tx_{n+1},q}(t) = 1$. Then

$$1 > F_{p,q}(t) \geq \Delta(F_{p,Tx_n}(t/3), F_{Tx_n,Tx_{n+1}}(t/3), F_{Tx_{n+1},q}(t/3)) \to \Delta(1,1,1) = 1$$

for any $t > 0$. This is contaradiction. Therefore $p = q$. □

Same analogy we can give the following Theorem.

Theorem 9. *Let $(X, \mathcal{F}, \Delta)$ be a complete probabilistic metric space such that Δ is a continuous triangular norm of Hadžić type. Let $\varphi_j : R_+ \to R_+$ $(j = 1, 2, \ldots)$ be a mapping such that for any $t > 0$, $n \in N$, there exist $r \geq t$, $0 \leq K < \infty$, $t_1, t_2 > 0$ such that $0 \leq \sum_{j=1}^{\infty} \varphi_j(r) + Kt_1 + Kt_2 < t$ and a mapping T be asymptotic regular. If $T : X \to X$ is a probabilistic φ_n-K contraction, then T has a unique fixed point x^*, and, for any $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x^*$.*

Remark 10. In [5] and [18], in the proof of the main theorem, in order to have the existence of fixed point, they prove the asymptotic regular of T using the condition $\lim_{n\to\infty} \varphi^n(t) = 0$ and and also the uniquness of fixed point, they use the condition $\lim_{n\to\infty} \varphi^n(t) = 0$.

3. Fixed point theorems for generalized ψ contraction in fuzzy metric spaces

In this section, we shall apply the results in Section 3 to obtain the corresponding fixed point theorems for generalized ψ-contraction in KM-fuzzy metric spaces.

Definition 11. (cf. Kramosil and Michálek [13].) A fuzzy metric space in the sense of Kramosil and Michálek (briefly, a KM-fuzzy metric space) is a triple (X, M, Δ) where X is a nonempty set, Δ is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(FM-1) $M(x, y, 0) = 0$;
(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$;
(FM-3) $M(x, y, t) = M(y, x, t)$;
(FM-4) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$;
(FM-5) $M(x, y, \cdot) : R_+ \to [0, 1]$ is left continuous.

Remark 12. A slight difference between Definition 11 and the original definition in [13] is that in [13], Δ is continuous. From $(FM-4)$ and $(FM-2)$, it is easy to show that $M(x, y, t)$ is nondecreasing for all $x, y \in X$ (see [8]). So, by Definition 2.2 and Definition 2.3, it is easy to obtain the following lemma.

Lemma 13. *If (X, M, Δ) is a KM-fuzzy metric space satisfying the condition*
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ *for all $x, y \in X$,*
then (X, F, Δ) is a Menger space, where F is defined by

$$F_{x,y}(t) = \begin{cases} M(x, y, t), t \geq 0, \\ 0, t < 0. \end{cases} \tag{3.1}$$

On the other hand, if (X, F, Δ) is a Menger space, then (X, M, Δ) is a KM-fuzzy space with $(FM - 6)$, where M is defined by $M(x, y, t) = F_{x,y}(t)$ for $t \geq 0$.

Definition 14. (See George and Veeramani [9], Mihet [21].) Let (X, M, Δ) be a KM-fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, t) = 0 \text{ for all } t > 0.$$

A sequence $\{x_n\}$ in X is said to be M-Cauchy sequence, if for each $\varepsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $m, n \geq n_0$. A fuzzy metric space is called complete if every M-Cauchy sequence is convergent in X.

From the theorem in [7], we have the following Theorem.

Theorem 15. *Let (X, M, Δ) be a KM-fuzzy metric spacen, where the t-norm Δ is continuous at (1, 1). Suppose that there exist $x_0, x_1 \in X$ such that*

$$\lim_{t\to\infty} M(x_0, x_1, t) = 1.$$

Define

$$Y_0 = \{y \in X | \lim_{t\to\infty} M(x_0, y, t) = 1\}.$$

Then (Y_0, F, Δ) is a Menger space, where F is defined by (3.1). If (X, M, Δ) is complete, then (Y_0, F, Δ) is also a complete Menger space.

Proof. The proof is same as that of .Lemma 4.1 in [7], □

We assume that $M(x_n, y_n, t)$ is asymptotic regular if for any $t > 0$,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$$

holds. In this case we have the following Theorem,

Theorem 16. *Let (X, M, Δ) be a complete KM-fuzzy metric space with a t-norm Δ of H-type. Let $\varphi : R_+ \to R_+$ be a mapping such that, for any $t > 0$, $n \in N$, there exists $r \geq t$, $0 \leq K < \infty$, t_1, $t_2 > 0$ such that $0 \leq \varphi(r) + Kt_1 + Kt_2 < t$. Let $T : X \to X$ be a asymptotic regular mapping such that for $K \geq 0$,*

(3.2)
$$M(Tx, Ty, \varphi(r) + K(t_1 + Kt_2)) \geq \Delta\left(M(x, y, t),\, M(x, Tx, Kt_1),\, M(y, Ty, Kt_2)\right),$$

and assume that Kt_1 or $Kt_2 \to \infty$, if $t \to \infty$. Suppose that there exists some $x_0 \in X$ such that

$$\lim_{t\to\infty} M(x_0, Tx_0, t) = 1.$$

If T is orbitally continuous, then T has a unique fixed point $x^* \in Y_0$ where

$$Y_0 = \{y \in X \mid \lim_{t\to\infty} M(x_0, y, t) = 1\},$$

and $\{T^n(y_0)\}$ converges to x^* for each $y_0 \in Y_0$. In particular, $\{T^n(x_0)\}$ converges to x^*

Proof. We define a mapping $F : Y_0 \times Y_0 \to D_t$ by (2.2). Since (X, M, Δ) is complete KM-fuzzy metric space and there exists some $x_0 \in X$ such that

$$\lim_{t\to\infty} M(x_0, Tx_0, t) = 1.$$

By Theorem 15 we know that (Y_0, F, Δ) is a complete Menger space. Let $x_0 \in X$ and define sequence $\{x_n\}$ with $x_n := T^n x_0$ for $n \in N$. Then $\{x_n\}$ is Cauchy sequence in M. As in the proof of Theorem 8, we can prove that

$$\lim_{n\to\infty} F_{x_n, x_{n+k}}(t) \geq \Delta^k(\lim_{n\to\infty} F_{x_n, x_{n+1}}(t - \varphi(t)). \tag{3.3}$$

$$\lim_{n\to\infty} M(x_n, x_{n+k}, t) \geq \Delta^k(\lim_{n\to\infty} M(x_n, x_{n+1}, t - \varphi(t))). \tag{3.4}$$

Firstly we have

(3.5)

$$
\begin{aligned}
&M(x_n, x_{n+k+1}, t)\\
&= M(x_n, x_{n+k+1}, t - (\varphi(r) + K(t_1 + t_2)) + \varphi(r) + K(t_1 + t_2))\\
&\geq \Delta(M(x_n, x_{n+1}, t - (\varphi(r) + K(t_1 + t_2)), M(x_{n+1}, x_{n+k+1}, \varphi(r) + K(t_1 + t_2))
\end{aligned}
$$

and by (3.2) we have

$$
\begin{aligned}
&M(x_{n+1}, x_{n+k+1}, \varphi(r) + K(t_1 + t_2))\\
&\geq \Delta(M(x_n, x_{n+k}, t), M(x_n, x_{n+1}, Kt_1), M(x_{n+k}, x_{n+k+1}, Kt_2)).
\end{aligned}
$$

Since T is asymptotic regular we have

$$
\begin{aligned}
&\lim_{n\to\infty} M(x_{n-1}, Tx_{n-1}, Kt_1) = 1,\\
&\lim_{m\to\infty} M(x_{m-1}, Tx_{m-1}, Kt_2) = 1,
\end{aligned}
$$

and continuity of Δ and $\Delta(1, a) = a$, we have

$$
\begin{aligned}
&\lim_{n\to\infty} M(x_n, x_{n+k+1}, t)\\
&\geq \Delta\left(\Delta^k(\lim_{n\to\infty} M(x_n, x_{n+1}, t - \psi(t)), 1, 1)\right) \qquad (3.6)\\
&= \Delta^{k+1}(\lim_{n\to\infty}(M(x_n, x_{n+1}, t - \psi(t))),
\end{aligned}
$$

where $\psi(t) = \varphi(t) + K(t_1 + t_2)$. We show that sequence $\{x_n\}$ is Cauchy, that is,

$$\lim_{m,n\to\infty} M(x_n, x_m, t) = 1 \text{ for any } t > 0..$$

Let $t > 0$ and $\varepsilon > 0$. By hypothesis, $\{\Delta^n \mid n \in N\}$ is equicontinuous at 1 and $\Delta^n(1) = 1$, so there is $\delta > 0$ such that

$$\text{if } s \in (1 - \delta, 1], \text{ then } \Delta^n(s) \geq 1 - \varepsilon \text{ for all } n \in N. \qquad (3.7)$$

Since T is asymtotic regular , we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t - \psi(t)) = 1.$$

By (3.4) and (3.7)

$$M(x_n, x_{n+k}, t) > 1 - \varepsilon.$$

for any $k \in N$. This proves that the Cauchy condition for $\{x_n\}$. By completeness, $\{x_n\}$ converges to some $p \in Y_0$, that is,

$$\lim_{n\to\infty} M(x_n, p, t) = 1.$$

for any $t > 0$.

For this $p \in Y_0$, since φ satisfies that for any $t > 0$, $n \in N$, there exists $r \geq t$, $0 \leq K < \infty$, t_1, $t_2 > 0$ such that $0 \leq \varphi(r) + K(t_1 + t_2) < t$. Since

$M(x_0^*, Tx_n, t)$
$\geq \Delta\left(M\left(x_0^*, Tx_0^*, \frac{t}{2}\right), M\left(Tx_0^*, Tx_n, \frac{t}{2}\right)\right)$
$\geq \Delta\left(M\left(x_0^*, Tx_0^*, \frac{t}{2}\right), \Delta\left(M\left(x_0^*, x_n, \frac{t}{6}\right), M\left(x_0^*, Tx_0^*, Kt_1\left(\frac{t}{6}\right)\right) M\left(x_n, Tx_n, Kt_2(\frac{t}{6})\right)\right)\right),$
$M\left(x_0^*, Tp, t\right) = \lim_{n\to\infty} M(x_0^*, Tx_n, t), M\left(x_0^*, p, \frac{t}{6}\right) = \lim_{n\to\infty} M\left(x_0^*, x_n, \frac{t}{6}\right) = 1,$ and
$\lim_{n\to\infty} M\left(x_n, Tx_n, Kt_2\left(\frac{t}{6}\right)\right) = 1,$

we have

$$M\left(x_0^*, Tp, t\right) \geq \Delta\left(M\left(x_0^*, Tx_0^*, \frac{t}{2}\right), \Delta\left(M\left(x_0^*, p, \frac{t}{6}\right), M\left(x^*, Tx^*, Kt_1\left(\frac{t}{6}\right)\right), 1\right)\right) \tag{3.8}$$

We also by assumption, $Kt_1 \to \infty$ as $t \to \infty$, we have

(3.9)
$\lim_{t\to\infty} M(x_0^*, Tp, t)$
$\geq \Delta\left(\lim_{t\to\infty} M\left(x_0^*, Tx_0^*, \frac{t}{2}\right), \lim_{t\to\infty} M\left(x_0^*, p, \frac{t}{6}\right), \lim_{t\to\infty} M\left(x^*, Tx^*, Kt_1\left(\frac{t}{6}\right)\right), 1\right)$
$= \Delta\left(1, 1, 1, 1\right) = 1$

Then we have $\lim_{t\to\infty} M(x_0^*, Tp, t) = 1$. Then $Tp \in Y_0$. This shows that T is a mapping of Y_0 into itself. Then the assumptions of theorem implies that $\psi : R_+ \to R_+$ satisfying for any $t > 0$ there exists $t_1, t_2 > 0$, $0 \leq K < \infty$ and $r \geq t$ such that $0 \leq \varphi(r) + K(t_1 + t_2) < t$. Also the mapping $T : X \to X$ satisfy the following inequality:
By (3.2),

$$F_{Tx,Ty}(\psi(r) + K(t_1 + Kt_2)) \geq \Delta\left(F_{x,y}(t),\, F_{x,Tx}(Kt_1),\, F_{y,Ty}(Kt_2)\right) \tag{3.10}$$

for $K \geq 0$. Thiese show that T is a probabilistic ψ-contraction in (Y_0, F, Δ). Thus, by Theorem 3.1, we conclude that T has a unique fixed point $x^* \in Y_0$, and $\{T^n(y_0)\}$ converges to x^* for each $y_0 \in Y_0$. In particular, $\{T^n(y_0)\}$ converges to x^*. This completes the proof. □

Same analogy we have the following Theorem.

Theorem 17. *Let (X, M, Δ) be a complete KM-fuzzy metric space with t-norm of H-type. Let $\varphi_n : R_+ \to R_+$ be a mappings such that, for each $t > 0$, $n \in N$, there exists $r \geq t$, $0 \leq K < \infty$, t_1, $t_2 > 0$ such that $0 \leq \sum_{j=1}^{\infty} \varphi_j(r) + Kt_1 + Kt_2 < t$. Let $T : X \to X$ be a asymptotic regular mapping such that for $K \geq 0$, we have*

$$\begin{aligned} &M(T^n x, T^n y, \varphi_n(r) + K(t_1 + Kt_2)) \\ &\geq \Delta\left(M(x, y, t),\, M(T^{n-1}x, T^n x, Kt_1),\, M(T^{n-1}y, T^n y, Kt_2)\right). \end{aligned} \tag{3.11}$$

and assume that Kt_1 or $Kt_2 \to \infty$, if $t \to \infty$. Suppose that there exists some $x_0 \in X$ such that

$$\lim_{t\to\infty} M(T^{n-1}x_0, T^n x_0, t) = 1.$$

Then T has a unique fixed point $x^* \in Y_0$ where

$$Y_0 = \{y \in X \mid \lim_{t\to\infty} M(x_0, y, t) = 1\},$$

and $\{T^n y_0\}$ converges to x^* for each $y_0 \in Y_0$. In particular, $\{T^n y_0\}$ converges to x^*

From the above proof, it is easy to see that Theorem 3.1 implies Theorem 4.1. On the other hand, by Lemma 2.2 we know that if (X, F, Δ) is a complete Menger space, then (X, M, Δ) is a complete KM-fuzzy metric space with (FM-6), where M is defined by $M(x, y, t) = F_{x,y}(t)$ for $t \geq 0$. So, it is not difficult to prove that Theorem 4.1 implies Theorem 3.1. This shows that Theorem 4.1 is equivalent to Theorem 3.1. That is to say, Theorem 4.1 is an equivalent type of Theorem 3.1 in KM-fuzzy metric spaces. In the same way, from Theorem 3.2, Corollary 3.2 and Theorem 3.3,we can prove the following theorems, respectively

Example 18. We consider the example of ψ-K contaractive mapping. Suppose that $X = [0, M]$, where $M \sim 1.02517$ is a solution of $\frac{x}{1+x} + \cos x = x$. And $|\frac{x}{1+x} + \cos x - x| \leq 1$ for any $x \in X$. Δ is minimum, that is $\Delta(a, b) = \min\{a, b\}$. Then Δ is a t-norm of H-type. We define $F_{x,y} : X \times X \to \mathcal{D}_+$ by

$$F_{x,y}(t) = \begin{cases} e^{-\frac{|x-y|}{t}} e \text{ if } t < |x-y| \\ 1, \quad \text{if } t \geq |x-y| \end{cases}$$

for all $x, y \in X$. In this case it is easy to see that (X, F, Δ) is menger space. It is also clear that (X, F, Δ) is complete.

Next let $Tx = \frac{1}{4}\left(\frac{x}{1+x} + cosx\right)$, where $x \in R_+$, $\varphi(t) = \frac{1}{4}\left(\frac{t}{1+t}\right)$, then we have the following;

$$F_{Tx,Ty}(\varphi(t) + K(t_1 + t_2)) \geq \Delta\{e^{-\frac{|x-y|}{t}}, e^{-\frac{|x-Tx|}{Kt_1}}, e^{-\frac{|y-Ty|}{Kt_2}}\}, \tag{3.12}$$

that is, T is φ-K contaractive mapping.

Let $0 \leq K = \frac{1}{4}$, $t_1 = |x - Tx|t$, and $t_2 = |y - Ty|t$. Then $\varphi(t) + Kt_1 + Kt_2 = \frac{1}{4}\frac{t}{1+t} + \frac{1}{4}(|x - Tx| + |x - Tx|)\, t$. In ths case since $\frac{1}{4}\frac{t}{1+t} + \frac{1}{4}(|x - Tx| + |x - Tx|)\, t < \frac{3t}{4} < t$, we have $0 \leq \varphi(t) + Kt_1 + Kt_2 < t$. If $\varphi(t) + Kt_1 + Kt_2 < |Tx - Ty|$, then

$$\begin{aligned} \varphi(t) &< |Tx - Ty| - (Kt_1 + Kt_2) \\ &< \varphi(|x-y|) + \frac{1}{4}(|x - Tx| + |y - Ty|) - \frac{1}{4}(|x - Tx| + |y - Ty|) \\ &= \varphi(|x-y|). \end{aligned}$$

Since the function $f(x) = \frac{x}{1+x}$, $x \geq 0$, is strictly increasing, we have $t < |x - y|$. In this case $\frac{1+t}{1+|x-y|} < 1$ and $\frac{\varphi(|x-y|)}{\varphi(t)} = \frac{|x-y|}{1+|x-y|}\frac{1+t}{t} < \frac{|x-y|}{t}$ so we have $e^{-\frac{\varphi(|x-y|)}{\varphi(t)}} \geq$

$e^{-\frac{|x-y|}{t}}$. Therefore we have

$$\begin{aligned}
&F_{Tx,Ty}(\varphi(t)+K(t_1+t_2))\\
&= e^{-\frac{|Tx-Ty|}{\varphi(t)+Kt_1+Kt_2}}\\
&\geq e^{-\frac{|Tx-Ty|}{\varphi(t)+Kt_1+Kt_2}}\\
&\geq e^{-\frac{(\varphi(|x-y|)+|x-Tx|+|y-Ty|)}{\varphi(t)+Kt_1+Kt_2}}\\
&\geq e^{-\frac{\varphi(|x-y|)}{\varphi(t)+Kt_1+Kt_2}}e^{-\frac{|x-Tx|}{\varphi(t)+Kt_1+Kt_2}}e^{-\frac{|y-Ty|}{\varphi(t)+Kt_1+Kt_2}}\\
&\geq e^{-\frac{\varphi(|x-y|)}{\varphi(t)}}e^{-\frac{|x-Tx|}{Kt_1}}e^{-\frac{|y-Ty|}{Kt_2}}\\
&\geq e^{-\frac{|x-y|}{t}}e^{-\frac{|x-Tx|}{Kt_1}}e^{-\frac{|y-Ty|}{Kt_2}}\\
&\geq \Delta\{e^{-\frac{|x-y|}{t}}, e^{-\frac{|x-Tx|}{Kt_1}}, e^{-\frac{|y-Ty|}{Kt_2}}\}
\end{aligned}$$

where Δ is minimum, $\Delta(a,\Delta(b,c)) = \min\{a.b,c\}$.

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