Gap Function Approach to Duality

— basic-model —

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Abstract

This paper discusses a class of pairs of quadratic optimization problem (primal) and its dual. The primal has a fixed initial state. We analyze both the problems through gap function method. A complete solution is given through characteristic equation. The model is based upon a complementary identity, which generates simultaneously a triplet of primal function, dual function and an equality condition.

1 Introduction

Recently, in [12–29] , S.Iwamoto, Y.Kimura, T.Fujita and A.Kira show that a duality for paired optimization problems through several methods such as (i) extended Lagrangean , (ii) plus-minus, (iii) inequality, (iv) identity, (v) complementary and others. As a historical background, see Bellman and others [1–7, 30], [9, 11, 32, 33] for dynamic optimization.

In this paper, we propose a method through gap function to show a duality between a primal problem and its dual problem. Section 2 considers a basic pair of n-variable minimization (primal) problem (P_n) and maximization (dual) problem (D_n). Then we define a gap function and discuss duality. In section 3, we give the optimal solution (point and value) of (P_n) and (D_n). Section 4 presents a pair of minimization problem and maximization problem for 4-variable.

2 Basic-model

In this section we assume that n is a natural number and c, ($\in R^1$) is a constant. c denotes an *initial state* at time 0 of a dynamic system.

As a basic pair of *primal* [1] and *dual*, we take n-variable optimization problems

$$
\mathrm{P}_n \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2+x_k^2\right]+(x_{n-1}-x_n)^2+x_n^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c \end{array}
$$

[1] Two nouns *primal* and *dual* mean *primal problem* and *dual problem*, respectively.

$$
\mathrm{D}_n \quad \begin{array}{l} \text{Maximize} \;\; 2c\mu_1 - \displaystyle\sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 - \mu_n^2 \\ \text{subject to} \;\; \text{(i)} \quad \mu \in R^n. \end{array}
$$

Let $f,\, g : R^n \to R^1$ be the respective objective functions of P_n, D_n:

$$
\begin{aligned}
f(x) &= \sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2\right] \\
g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - 2\mu_n^2.
\end{aligned}
$$

Note that $f(x)$ is convex and $g(\mu)$ is concave. Then it hods that

$$
f(x) \;\geq\; g(\mu) \qquad (x, \mu) \in R^n \times R^n. \tag{1}
$$

The sign of equality holds iff a linear system of $2n$-equation on $2n$-variable

$$
(\mathrm{EC}_1) \quad \begin{aligned}
c - x_1 &= \mu_1 & x_1 &= \mu_1 - \mu_2 & \\
x_{k-1} - x_k &= \mu_k & x_k &= \mu_k - \mu_{k+1} & 2 \leq k \leq n-1 \\
x_{n-1} - x_n &= \mu_n & x_n &= \mu_n &
\end{aligned}
$$

holds. (EC_1) is called an *equality condition* between P_n and D_n. Thus both problems are called *dual* of each other.

Lemma 1 *The equality condition* (EC_1) *yields a pair of linear systems of n-equation on n-variable:*

$$
(\mathrm{EQ}_x) \quad \begin{aligned}
3x_1 - x_2 &= c & \\
-x_{k-1} + 3x_k - x_{k+1} &= 0 & \qquad 2 \leq k \leq n-1 \\
-x_{n-1} + 2x_n &= 0, &
\end{aligned}
$$

$$
(\mathrm{EQ}_\mu) \quad \begin{aligned}
2\mu_1 - \mu_2 &= c & \\
-\mu_{k-1} + 3\mu_k - \mu_{k+1} &= 0 & \qquad 2 \leq k \leq n-1 \\
-\mu_{n-1} + 3\mu_n &= 0. &
\end{aligned}
$$

2.1 Gap function for basic-model

First we present an identity, which takes a fundamental role in analyzing respective pairs of primal and dual. Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$
(\mathrm{C}_1) \quad \sum_{k=1}^{n-1} [(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n \;=\; c\mu_1
$$

holds true. This identity is called *complementary.*

Now we derive both P_n and D_n through gap function. Let us define a *gap function* $h = h(x, \mu)$ between $x \in R^n$ and $\mu \in R^n$ by

$$h(x,\mu) = \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k - \mu_k)^2 + \{x_k - (\mu_k - \mu_{k+1})\}^2 \right] + \left[(x_{n-1} - x_n - \mu_n)^2 + (x_n - \mu_n)^2 \right]. \tag{2}$$

Thus $h(x, \mu)$ denotes a *total difference* between x and μ. It turns out that the quadratic function $h = h(x, \mu)$ is convex in (x, μ).

Lemma 2

$$\text{(i)} \quad f(x) - g(\mu) = h(x,\mu) \geq 0 \quad \forall (x,\mu) \in R^n \times R^n$$

$$\text{(ii)} \quad h(x,\mu) = 0 \implies (x,\mu) \textit{ satisfies } (\mathrm{EC}_1).$$

Theorem 1 (i) *It holds that*

$$f(x) \geq g(\mu) \quad \textit{on} \quad R^n \times R^n.$$

(ii) *It holds that*

$$f(x) = g(\mu) \iff (x,\mu) \textit{ satisfies } (\mathrm{EC}_1).$$

Then P_n *attains a minimum* $f(x)$*, while* D_n *attains a maximum* $g(\mu)$*.*

Hence a solution (x, μ) to (EC_1) yields a minimum point x for P_n and a maximum point μ for D_n.

Theorem 2 *Let* (x, μ) *satisfy* (EC_1)*. Then both sides become a common value with five expressions:*

$$(5\mathrm{V}_1) \quad \begin{aligned} f(x) &= c(c - x_1) \\ &= g(\mu) = \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + 2\mu_n^2 = c\mu_1. \end{aligned}$$

The primal P_n *has a minimum value*

$$m = f(x) = c(c - x_1)$$

at x*, while the dual* D_n *has a maximum value*

$$M = g(\mu) = \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + 2\mu_n^2 = c\mu_1$$

at μ*.*

Hence a solution (x, μ) to (EC_1) yields a minimum value $f(x) = c(c - x_1)$ for P_n and a maximum value $g(\mu) = c\mu_1$ for D_n. Thus *the first argument of optimal point characterizes the common optimum value.*

2.2 Characteristic equation for basic-model

Now let us solve the pair of linear systems (EQ_x) and (EQ_μ). We introduce a second-order linear difference equation

$$x_{n+2} - 3x_{n+1} + x_n \ = \ 0, \quad x_1 = 1, \ x_0 = 0 \tag{3}$$

Lemma 3 *The Eq (3) has a unique solution*

$$x_n \ = \ \frac{\beta^n - \alpha^n}{\beta - \alpha} \tag{4}$$

where $\alpha \, (<) \, \beta$ *are two positive solution*

$$\alpha \ = \ \frac{3 - \sqrt{5}}{2}, \quad \beta \ = \ \frac{3 + \sqrt{5}}{2} \tag{5}$$

to the associated characteristic equation

$$(\mathrm{CE}) \qquad t^2 - 3t + 1 \ = \ 0. \tag{6}$$

We note that

$$\alpha \ = \ \overline{\phi}^2, \qquad \beta \ = \ \phi^2 \tag{7}$$

where

$$\phi \ = \ \frac{1 + \sqrt{5}}{2}, \qquad \overline{\phi} \ = \ \frac{1 - \sqrt{5}}{2} \tag{8}$$

are positive and negative solutions to a quadratic equation

$$t^2 - t - 1 \ = \ 0.$$

Both ϕ and $\overline{\phi}$ are called the *Golden number* and its *conjugate*, respectively. It holds that

$$\phi + \overline{\phi} \ = \ 1, \quad \phi\overline{\phi} \ = \ -1,$$
$$\alpha + \beta \ = \ 3, \quad \alpha\beta \ = \ 1.$$

Definition 1 *Let us define the sequence* $\{K_n\}$ *by*

$$K_n \ = \ \frac{\beta^n - \alpha^n}{\beta - \alpha}. \tag{9}$$

We call $\{K_n\}$ a *Kibonacci* sequence. Thus $\{K_n\}$ satisfies a second-order linear difference equation

$$K_{n+1} \ = \ 3K_n - K_{n-1}, \quad K_1 = 1, \ K_0 = 0. \tag{10}$$

This has a unique solution (9). The solution — Kibonacci number — K_n turns out a *two-step Fibonacci* number.

Lemma 4

$$K_n \;=\; F_{2n}. \tag{11}$$

Proof. $\quad K_n = \dfrac{\beta^n - \alpha^n}{\beta - \alpha} = \dfrac{\phi^{2n} - \overline{\phi}^{2n}}{\phi^2 - \overline{\phi}^2} = \dfrac{\phi^{2n} - \overline{\phi}^{2n}}{\phi - \overline{\phi}} = F_{2n}.$ □

We remark that *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1,\ x_0 = 0. \tag{12}$$

Hence

$$F_n \;=\; \frac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}}. \tag{13}$$

n	$\cdots$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
F_n	$\cdots$	-1	1	0	1	1	2	3	5	8	13	21	34	55	$\cdots$

Table 1 Fibonacci sequence $\{F_n\}$

Lemma 5 *The system* (EQ_x) *has a unique solution*

$$x_k \;=\; c\,\frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} \;=\; c\,\frac{F_{2n+1-2k}}{F_{2n+1}} \qquad 0 \le k \le n$$

, while the system (EQ_μ) *has a unique solution*

$$\mu_k \;=\; c\,\frac{K_{n+1-k}}{2K_n - K_{n-1}} \;=\; c\,\frac{F_{2n+2-2k}}{F_{2n+1}} \qquad 1 \le k \le n.$$

We see that the x, μ satisfy the equality condition (EC_1).

Theorem 3 *The equality condition* (EC_1) *has a unique solution* $(x,\, \mu)$*;*

$$x_k \;=\; c\,\frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} \;=\; c\,\frac{F_{2n+1-2k}}{F_{2n+1}} \qquad 0 \le k \le n$$

$$\mu_k \;=\; c\,\frac{K_{n+1-k}}{2K_n - K_{n-1}} \;=\; c\,\frac{F_{2n+2-2k}}{F_{2n+1}}. \qquad 1 \le k \le n.$$

where

$$K_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha} \;=\; F_{2n}.$$

Hence the gap function h attains the zero minimum at $(x,\, \mu)$*.*

3 Primal vs dual for basic-model

Let us consider the paired n-variable problem:

$$\mathrm{P_n} \quad \begin{array}{l} \text{minimize} \quad f(x) \\ \text{subject to} \quad \text{(i)} \;\; x \in R^n, \;\; \text{(ii)} \;\; x_0 = c \end{array}$$

$$\mathrm{D_n} \quad \begin{array}{l} \text{Maximize} \quad g(\mu) \\ \text{subject to} \quad \text{(i)} \;\; \mu \in R^n \end{array}$$

Lemma 6 (i) *Let x be a minimum point for* P_n*. Then x satisfies*

Case $n = 1$

$$(\mathrm{EQ}_x) \qquad 2x_1 = c$$

where

$$f(x_1) = (c - x_1)^2 + x_1^2.$$

Case $n = 2$

$$(\mathrm{EQ}_x) \qquad \begin{aligned} 3x_1 - x_2 &= c \\ -x_1 + 2x_2 &= d \end{aligned}$$

where

$$f(x_1, x_2) = \left[(c - x_1)^2 + x_1^2\right] + \left[(x_1 - x_2)^2 + x_2^2\right].$$

Case $n \geq 3$

$$(\mathrm{EQ}_x) \qquad \begin{aligned} 3x_1 - x_2 &= c \\ -x_{k-1} + 3x_k - x_{k+1} &= 0 \qquad 2 \leq k \leq n-1 \\ -x_{n-1} + 2x_n &= 0, \end{aligned}$$

The minimum value $f(x)$ is given by $f(x) = c(c - x_1)$.

(ii) *Let μ be a maximum point for* D_n*. Then μ satisfies*

Case $n = 1$

$$(\mathrm{EQ}_\mu) \qquad 2\mu_1 = c \tag{14}$$

where

$$g(\mu_1) = 2c\mu_1 - 2\mu_1^2.$$

Case $n = 2$

$$(\mathrm{EQ}_\mu) \qquad \begin{aligned} 2\mu_1 - \mu_2 &= c \\ -\mu_1 + 3\mu_2 &= 0 \end{aligned} \tag{15}$$

where

$$g(\mu_1, \mu_2) \;=\; 2c\mu_1 - \left[\mu_1^2 + (\mu_1 - \mu_2)^2\right] - 2\mu_2^2.$$

Case $n \geq 3$

$$(\mathrm{EQ}_\mu) \qquad \begin{aligned} 2\mu_1 - \mu_2 &= c \\ -\mu_{k-1} + 3\mu_k - \mu_{k+1} &= 0 \qquad 2 \leq k \leq n-1 \\ -\mu_{n-1} + 3\mu_n &= 0 \end{aligned}$$

The maximum value $g(\mu)$ *is given by* $g(\mu) \;=\; \lambda c \mu_1$.

Theorem 4 *The primal* P_n *attains a minimum*

$$m \;=\; f(x) \;=\; c(c - x_1) \;=\; c^2\Big(1 - \frac{K_n - K_{n-1}}{K_{n+1} - K_n}\Big) \;=\; c^2\,\frac{F_{2n}}{F_{2n+1}}$$

at x*;*

$$x_k \;=\; c\,\frac{K_{n+1-k} - K_{n-k}}{K_{n+1} - K_n} \;=\; c\,\frac{F_{2n+1-2k}}{F_{2n+1}} \qquad 0 \leq k \leq n.$$

.

The dual D_n *attains a maximum*

$$M \;=\; g(\mu) \;=\; c\mu_1 \;=\; c^2\frac{K_n}{2K_n - K_{n-1}} \;=\; c^2\,\frac{F_{2n}}{F_{2n+1}}$$

at μ*;*

$$\mu_k \;=\; c\,\frac{K_{n+1-k}}{2K_n - K_{n-1}} \;=\; c\,\frac{F_{2n+2-2k}}{F_{2n+1}} \qquad 1 \leq k \leq n.$$

Both the optima are equal.

We remark that

$$K_{n+1} - 3K_n + K_{n-1} \;=\; 0$$

$$\text{i.e.} \qquad K_{n+1} - K_n - (K_n - K_{n-1}) \;=\; K_n$$

yields

$$1 - \frac{K_n - K_{n-1}}{K_{n+1} - K_n} \;=\; \frac{K_n}{2K_n - K_{n-1}} \;=\; \frac{F_{2n}}{F_{2n+1}}.$$

4 Four-variable pair for basic-model

The primal problem

$$\mathrm{P}_4 \qquad \begin{array}{ll} \text{minimize} & (c-x_1)^2 + x_1^2 + \left[(x_1-x_2)^2 + x_2^2\right] \\ & + \left[(x_2-x_3)^2 + x_3^2\right] + \left[(x_3-x_4)^2 + x_4^2\right] \\ \text{subject to} & \text{(i)} \ \ (x_1, x_2, x_3, x_4) \in R^4, \quad \text{(ii)} \ \ x_0 = c. \end{array}$$

has a minimum value

$$m_4 = c(c-\hat{x}_1) \ = \ \frac{21}{34}\,c^2$$

at a point $\hat{x} \ = \ (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$:

$$\hat{x}_1 \ = \ \frac{13}{34}\,c, \quad \hat{x}_2 \ = \ \frac{5}{34}\,c, \quad \hat{x}_3 \ = \ \frac{2}{34}\,c, \quad \hat{x}_4 \ = \ \frac{1}{34}\,c.$$

The dual problem

$$\mathrm{D}_4 \qquad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \left[\mu_1^2 + (\mu_1-\mu_2)^2\right] - \left[\mu_2^2 + (\mu_2-\mu_3)^2\right] \\ & \qquad - \left[\mu_3^2 + (\mu_3-\mu_4)^2\right] - 2\mu_4^2 \\ \text{subject to} & \text{(i)} \ \ (\mu_1, \mu_2, \mu_3, \mu_4) \in R^4 \end{array}$$

has a maximum value

$$M_4 = \mu_1^* c \ = \ \frac{\rho^3 + 4\rho^2 + 8\rho + 8}{34}\,c^2$$

at a point $\mu^* \ = \ (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$:

$$\mu_1^* \ = \ \frac{21}{34}\,c, \quad \mu_2^* \ = \ \frac{8}{34}\,c, \quad \mu_3^* \ = \ \frac{3}{34}\,c, \quad \mu_4^* \ = \ \frac{1}{34}\,c$$

Note that the equality condition EC_1 for 4-variable pair $\mathrm{P}_4, \mathrm{D}_4$ has a unique solution $(x,\, \mu)$;

$$x_k \ = \ c\,\frac{K_{5-k} - K_{4-k}}{2K_4 - K_3} \qquad 1 \le k \le 4$$

$$\mu_k \ = \ c\,\frac{K_{5-k}}{2K_4 - K_3} \qquad 1 \le k \le 4.$$

We show how both the minimum point $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$ and the maximum point $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$ are obtained. First we have

$$\begin{aligned} 2K_4 - K_3 \ &= \ 2 \cdot 21 - 8 \\ &= \ 34 \end{aligned}$$

that is

$$2K_4 - K_3 \;=\; 34. \tag{16}$$

Further

$$\begin{aligned} K_4 - K_3 &= 13 \\ K_3 - K_2 &= 5 \\ K_2 - K_1 &= 2 \\ K_1 - K_0 &= 1. \end{aligned}$$

Here

$$K_2 \;=\; 3, \quad K_3 \;=\; 8, \quad K_4 \;=\; 21.$$

Thus we obtain the desired minimum point :

$$\begin{aligned} \hat{x}_1 &= c\,\frac{K_4 - K_3}{2K_4 - K_3} \;=\; \frac{13}{34}\,c \\ \hat{x}_2 &= \frac{c}{\rho}\cdot\frac{K_3 - K_2}{2K_4 - K_3} \;=\; \frac{5}{34}\,c \\ \hat{x}_3 &= \frac{c}{\rho^2}\cdot\frac{K_2 - K_1}{2K_4 - K_3} \;=\; \frac{2}{34}\,c \\ \hat{x}_4 &= \frac{c}{\rho^3}\cdot\frac{K_1 - K_0}{2K_4 - K_3} \;=\; \frac{1}{34}\,c. \end{aligned}$$

In a similar way, the desired maximum point is obtained as follows.

$$\begin{aligned} \mu_1^* &= c\,\frac{K_4}{2K_4 - K_3} \;=\; \frac{21}{34}\,c \\ \mu_2^* &= \frac{c}{\rho}\cdot\frac{K_3}{2K_4 - K_3} \;=\; \frac{8}{34}\,c \\ \mu_3^* &= \frac{c}{\rho^2}\cdot\frac{K_2}{2K_4 - K_3} \;=\; \frac{3}{34}\,c \\ \mu_4^* &= \frac{c}{\rho^3}\cdot\frac{K_1}{2K_4 - K_3} \;=\; \frac{1}{34}\,c. \end{aligned}$$

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