Fully conservative convection schemes for turbulence simulations

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1 Background

The fully conservative convection schemes are the finite difference schemes for the convection term which preserve primary and secondary conservation quantities simultaneously. Therefore, they are now recognized as useful tools for unsteady turbulence simulations like direct numerical simulation (DNS) and large eddy simulation (LES). However, the original motivation of the scheme construction came from the latter.

The amplitude of the subgrid scale stress of the LES is estimated as the order of the squared filter width from the Taylor expansion. The subgrid scale stress is rooted in the convection term, therefore, at least 4th-order accuracy is advisable for the convection scheme so that the truncation error of the scheme does not overcome the subgrid scale stress. The standard 2nd-order accurate convection scheme in a staggered grid [1] which is suitable for incompressible flow simulations is fully conservative, however, its result is not fine. On the other hand, an existing 4th-order accurate convection scheme [2] was not fully conservative, and made unphysical results at high Reynolds numbers. Therefore, constructing high-order and fully conservative convection schemes was one of important studies for LES in 1990s.

In 1995 [3], the author finally found the 4th- and higher-order accurate fully conservative convection schemes in a staggered grid for incompressible flows. After that, the schemes are extended to those for compressible flows and flows on moving grid.

In this presentation, fully conservative convection schemes for incompressible flows [3, 4], for compressible flows [5], and for flows on moving grid [6] are introduced.

2 Full conservative convection schemes for incompressible flows

The governing equations for incompressible flows are the continuity and the Navier-Stokes equations.

$$(Cont.^{I}) \equiv \frac{\partial u_j}{\partial x_j} = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_j u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i, \tag{2}$$

where u_i is the x_i component of velocity, p pressure, ρ density, ν kinematic viscosity, and f_i is the x_i component of body force. For the incompressible flow, ρ is constant. In the NS equation (Eq.(2)), the convection term is written in conservative form. This form is also called divergence form. On the other hand, the convection term is sometimes written in different forms.

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i, \tag{3}$$

$$\frac{\partial u_i}{\partial t} + \frac{1}{2}\frac{\partial u_j u_i}{\partial x_j} + \frac{1}{2} u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i, \tag{4}$$

$$\frac{\partial u_i}{\partial t} + u_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2}\frac{\partial u_j u_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i \tag{5}$$

The first variant is non-conservative form, or also called advection form. The average of the divergence and advection forms is called skew-symmetric form. The last variant is rotation form or also called Lam form, which is written by using vorticity and dynamic pressure.

2.1 Forms of convection term in incompressible flow equation

The variants of the convection term are divergence, advection, skew-symmetric, and rotation forms, respectively, defined as follows:

$$(Div.^{I})_i \equiv \frac{\partial u_j u_i}{\partial x_j}, \tag{6}$$

$$(Adv.^{I})_i \equiv u_j \frac{\partial u_i}{\partial x_j}, \tag{7}$$

$$(Skew.^{I})_i \equiv \frac{1}{2}\left(\frac{\partial u_j u_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} \right), \tag{8}$$

$$(Rot.^I)_i \equiv u_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial u_j u_j}{\partial x_i} \tag{9}$$

These forms are commutable with each other with the Leibniz rule of calculus ($\frac{\partial(\phi\psi)}{\partial x_j} = \phi\frac{\partial\psi}{\partial x_j} + \psi\frac{\partial\phi}{\partial x_j}$) and with the aid of the continuity of Eq.(1).

$$(Div.^I)_i = (Adv.^I)_i + u_i(Cont.^I), \tag{10}$$

$$\begin{aligned}(Skew.^I)_i &= \frac{1}{2}(Div.^I)_i + \frac{1}{2}(Adv.^I)_i \\ &= (Adv.^I)_i + \frac{1}{2}u_i(Cont.^I) = (Div.^I)_i - \frac{1}{2}u_i(Cont.^I),\end{aligned} \tag{11}$$

$$(Rot.^I)_i = (Adv.^I)_i \tag{12}$$

Note that the rotation form is just a rewritten form of the advection form and omitted in the following discussions. In the variants of the convection forms, the divergence form is primary conservative a priori, and the skew-symmetric form is secondary conservative a priori. The secondary conservation property of the skew-symmetric form is demonstrated as follows:

$$u_\alpha\left[\frac{\partial u_\alpha}{\partial t} + (Skew.^I)_\alpha\right] = \frac{\partial\, u_\alpha^2/2}{\partial t} + \frac{\partial\, u_j u_\alpha^2/2}{\partial x_j} \tag{13}$$

The commutability and conservation properties are used for the guiding principle for constructing fully conservative convection schemes in this study.

2.2 Discrete operators

In order to construct the convection schemes, some discrete operators are defined at the beginning. In this study, the discrete points are distributed uniformly for simplicity. Therefore, $(x_1)_{\rm I} = {\rm I}\Delta x_1$, $(x_2)_{\rm J} = {\rm J}\Delta x_2$, $(x_3)_{\rm K} = {\rm K}\Delta x_3$, and $t^{\rm N} = {\rm N}\Delta t$, where Δx_1, Δx_2, and Δx_3 are spatial increments, and Δt is the time increment. The discrete values are represented as $\phi((x_1)_{\rm I}, (x_2)_{\rm J}, (x_3)_{\rm K}, t^{\rm N}) = \phi^{\rm N}_{\rm I,J,K}$.

Then, spatial discrete operators for x_1 direction with stencil width m are defined as follows:

$$\left.\frac{\delta_{\rm m}\phi}{\delta_{\rm m}x_1}\right|_{\rm I,J,K} \equiv \frac{\phi_{\rm I+m/2,J,K} - \phi_{\rm I-m/2,J,K}}{{\rm m}\Delta x_1}, \tag{14}$$

$$\left.\overline{\phi}^{{\rm m}x_1}\right|_{\rm I,J,K} \equiv \frac{\phi_{\rm I+m/2,J,K} + \phi_{\rm I-m/2,J,K}}{2}, \tag{15}$$

$$\left.\widetilde{\phi\,\psi}^{{\rm m}x_1}\right|_{\rm I,J,K} \equiv \frac{\phi_{\rm I+m/2,J,K}\,\psi_{\rm I-m/2,J,K} + \psi_{\rm I+m/2,J,K}\,\phi_{\rm I-m/2,J,K}}{2} \tag{16}$$

Operators in x_2 and x_3 are also defined in the same manner.

In addition to the definitions, two application rules are prescribed for the spatial discrete operators. First, the direction index j in the finite difference operator of $\delta_{\rm m}/\delta_{\rm m}x_j$ is physical and follows the summation convention. Second, the direction indices j in the interpolation of $\overline{\phi}^{{\rm m}x_j}$ and the permanent product of $\widetilde{\phi\psi}^{{\rm m}x_j}$ are numerical and do not follow the summation convention. Instead, these numerical indices take the same value as the same physical index in the same term.

Temporal discrete operators with stencil width 1 are also defined in this study as follows.

$$\left.\frac{\delta_1 \phi}{\delta_1 t}\right|^{\rm N+1/2} \equiv \frac{\phi^{\rm N+1} - \phi^{\rm N}}{\Delta t}, \tag{17}$$

$$\left.\overline{\phi}^{1t}\right|^{\rm N+1/2} \equiv \frac{\phi^{\rm N+1} + \phi^{\rm N}}{2} \tag{18}$$

These temporal operators are used in spatio-temporal discretized schemes.

From the definitions, the following identities are satisfied among the discrete operators.

$$\frac{\delta_{\rm m}\, \overline{\phi}^{{\rm n}x_i}}{\delta_{\rm m} x_j} = \overline{\frac{\delta_{\rm m}\phi}{\delta_{\rm m} x_j}}^{{\rm n}x_i}, \tag{19}$$

$$\frac{\delta_{\rm m}\psi\, \overline{\phi}^{{\rm m}x_j}}{\delta_{\rm m} x_j} = \psi\, \overline{\frac{\delta_{\rm m}\phi}{\delta_{\rm m} x_j}}^{{\rm m}x_j} + \phi\, \frac{\delta_{\rm m}\psi}{\delta_{\rm m} x_j}, \tag{20}$$

$$\phi\left(\frac{\delta_{\rm m}\psi\, \overline{\phi}^{{\rm m}x_j}}{\delta_{\rm m} x_j} + \psi\, \overline{\frac{\delta_{\rm m}\phi}{\delta_{\rm m} x_j}}^{{\rm m}x_j}\right) = \frac{\delta_{\rm m}\psi\, \widetilde{\phi\,\phi}^{{\rm m}x_j}}{\delta_{\rm m} x_j}, \tag{21}$$

$$\frac{\delta_1\, \overline{\phi}^{1t}}{\delta_1 t} = \overline{\frac{\delta_1\phi}{\delta_1 t}}^{1t}, \tag{22}$$

$$\frac{\delta_1 \phi\psi}{\delta_1 t} = \overline{\psi}^{1t}\frac{\delta_1\phi}{\delta_1 t} + \overline{\phi}^{1t}\frac{\delta_1\psi}{\delta_1 t}, \tag{23}$$

$$\overline{\phi}^{1t}\, \frac{\delta_1\phi}{\delta_1 t} = \frac{1}{2}\frac{\delta_1\phi^2}{\delta_1 t} \tag{24}$$

The identity of Eq.(19) is the commutability between the spatial finite difference and interpolation operators. The identity of Eq.(20) is a discrete analogue of the Leibniz rule. The identity of Eq.(21) is used for the proof of the secondary conservation property of the skew-symmetric form. Eqs.(22)-(24) are identities for the temporal discrete operators. Commutabilities between the temporal and the spatial operators are also satisfied.

2.3 Fully conservative convection schemes for incompressible flows

The next stage is constructing the fully conservative convection schemes for incompressible flows. From the analytical relations, the guiding principle of the scheme construction is decided as follows: 1) Find proper set of convection schemes which satisfy the commutability and the conservation properties in a discrete sense. 2) Extend them to higher order ones with keeping the commutability and the properties.

In this study, a staggered grid arrangement [1] is used for incompressible flows. In this arrangement, the velocity components are located at the cell surfaces, while the pressure is located at the cell center. In addition, the components of the Navier-Stokes equation are discretized at the velocity points, while the continuity is discretized at the pressure point. The staggered grid is preferred for proper coupling of the Navier-Stokes and continuity equations.

First of all, existing second-order accurate convection schemes are assessed based on the guiding principle. The continuity is discretized in the staggered grid arrangement as follows.

$$(Cont.^I - S2) \equiv \frac{\delta_1 u_j}{\delta_1 x_j} = 0 \tag{25}$$

The standard convection scheme in the staggered grid by Harlow & Welch [1] is the divergence form and primary conservative.

$$(Div.^I - S2)_i \equiv \frac{\delta_1\, \overline{u_j}^{1x_i} \overline{u_i}^{1x_j}}{\delta_1 x_j} \tag{26}$$

Corresponding advection form was proposed by Kajishima [7].

$$(Adv.^I - S2)_i \equiv \overline{\overline{u_j}^{1x_i} \frac{\delta_1 u_i}{\delta_1 x_j}}^{1x_j} \tag{27}$$

The skew-symmetric form which is the average of the divergence and advection forms was proposed by Piacsek & Williams [8].

$$(Skew.^I - S2)_i \equiv \frac{1}{2}(Div.^I - S2)_i + \frac{1}{2}(Adv.^I - S2)_i \tag{28}$$

The divergence and advection forms are commutable with the aid of the discrete continuity of Eq.(25).

$$(Div.^I - S2)_i = (Adv.^I - S2)_i + u_i\, \overline{(Cont.^I - S2)}^{1x_i} \tag{29}$$

This commutability is demonstrated by using the identities of Eqs.(20) and (19). The secondary conservation property of the skew-symmetric form is demonstrated as follows.

$$u_\alpha \left[\frac{du_\alpha}{dt} + (Skew.^I - S2)_\alpha \right] = \frac{d\, u_\alpha^2/2}{dt} + \frac{\delta_1\, \overline{u_j}^{1x_\alpha} \widetilde{u_\alpha u_\alpha}^{1x_j}/2}{\delta_1 x_j}, \tag{30}$$

where no summation of α is taken for the staggered grid arrangement. The secondary conservation property is demonstrated by using the identity of Eq.(21). Therefore, the convection forms of $(Div.^I - S2)_i$, $(Adv.^I - S2)_i$, and $(Skew.^I - S2)_i$ are commutable and fully conservative with the aid of the continuity of Eq.(25), and compose the proper set of second-order accurate convection schemes for incompressible flows.

On the other hand, existing fourth-order accurate convection schemes [2, 7, 9] were not fully conservative. There were no fourth-order accurate fully conservative convection schemes for incompressible flows until 1995 at least in literatures.

In 1995, the author finally found the fourth-order accurate fully conservative convection schemes for incompressible flows. The continuity is discretized with the fourth-order accurate finite difference in the staggered grid as follows.

$$(Cont.^I - S4) \equiv \frac{9}{8}\frac{\delta_1 u_j}{\delta_1 x_j} - \frac{1}{8}\frac{\delta_3 u_j}{\delta_3 x_j} = 0 \tag{31}$$

The members of the proper set of fourth-order accurate convection scheme for incompressible flows are defined as follows.

$$(Div.^I - S4)_i \equiv \frac{9}{8}\frac{\delta_1\, \overline{u_j}^{4th\_x_i}\overline{u_i}^{1x_j}}{\delta_1 x_j} - \frac{1}{8}\frac{\delta_3\, \overline{u_j}^{4th\_x_i}\overline{u_i}^{3x_j}}{\delta_3 x_j}, \tag{32}$$

$$(Adv.^I - S4)_i \equiv \frac{9}{8}\overline{\overline{u_j}^{4th\_x_i}\frac{\delta_1 u_i}{\delta_1 x_j}}^{1x_j} - \frac{1}{8}\overline{\overline{u_j}^{4th\_x_i}\frac{\delta_3 u_i}{\delta_3 x_j}}^{3x_j}, \tag{33}$$

$$(Skew.^I - S4)_i \equiv \frac{1}{2}(Div.^I - S4)_i + \frac{1}{2}(Adv.^I - S4)_i, \tag{34}$$

where the forth-order interpolation of the convection velocity is defined as $\overline{u_j}^{4th\_x_i} \equiv \frac{9}{8}\overline{u_j}^{1x_i} - \frac{1}{8}\overline{u_j}^{3x_i}$. The commutability and the secondary conservation property of the skew-symmetric forms are expressed as follows.

$$(Div.^I - S4)_i = (Adv.^I - S4)_i + u_i\,\overline{(Cont.^I - S4)}^{4th\_x_i} \tag{35}$$

$$u_\alpha\left[\frac{du_\alpha}{dt} + (Skew.^I - S4)_\alpha\right] = \frac{d\,u_\alpha^2/2}{dt} + \frac{9}{8}\frac{\delta_1\, \overline{u_j}^{4th\_x_\alpha}\widetilde{u_\alpha u_\alpha}^{1x_j}/2}{\delta_1 x_j} - \frac{1}{8}\frac{\delta_3\, \overline{u_j}^{4th\_x_\alpha}\widetilde{u_\alpha u_\alpha}^{3x_j}/2}{\delta_3 x_j} \tag{36}$$

Here, the recipe of constructing high-order fully conservative convection scheme is: (1) The same high-order interpolation is used for the convection velocity. (2) The same stencil width is used for the finite difference and interpolation operators in a term except for the convection velocity. This recipe makes possible to use the identities of Eqs.(19)-(21) even for high-order discretization.

Once we know the recipe, higher-order fully conservative convection schemes are easily obtained. Sixth-order accurate fully conservative convection schemes with the continuity discretization are defined as follows [4].

$$(Cont.^I - S6) \equiv \frac{150}{128}\frac{\delta_1 u_j}{\delta_1 x_j} - \frac{25}{128}\frac{\delta_3 u_j}{\delta_3 x_j} + \frac{3}{128}\frac{\delta_5 u_j}{\delta_5 x_j} = 0, \tag{37}$$

$$(Div.^I - S6)_i \equiv \frac{150}{128}\frac{\delta_1\, \overline{u_j}^{6th\_x_i}\overline{u_i}^{1x_j}}{\delta_1 x_j} - \frac{25}{128}\frac{\delta_3\, \overline{u_j}^{6th\_x_i}\overline{u_i}^{3x_j}}{\delta_3 x_j} + \frac{3}{128}\frac{\delta_5\, \overline{u_j}^{6th\_x_i}\overline{u_i}^{5x_j}}{\delta_5 x_j}, \tag{38}$$

$$(Adv.^I - S6)_i \equiv \frac{150}{128}\overline{\overline{u_j}^{6th\_x_i}\frac{\delta_1 u_i}{\delta_1 x_j}}^{1x_j} - \frac{25}{128}\overline{\overline{u_j}^{6th\_x_i}\frac{\delta_3 u_i}{\delta_3 x_j}}^{3x_j} + \frac{3}{128}\overline{\overline{u_j}^{6th\_x_i}\frac{\delta_5 u_i}{\delta_5 x_j}}^{5x_j}, \tag{39}$$

$$(Skew.^I - S6)_i \equiv \frac{1}{2}(Div.^I - S6)_i + \frac{1}{2}(Adv.^I - S6)_i, \tag{40}$$

where the sixth-order interpolation of the convection velocity is defined as $\overline{u_j}^{6th\_x_i} \equiv \frac{150}{128}\overline{u_j}^{1x_i} - \frac{25}{128}\overline{u_j}^{3x_i} + \frac{3}{128}\overline{u_j}^{5x_i}$. Higher-order accurate fully conservative convection schemes can be made in the same manner.

2.4 Spatio-temporal discretization schemes

In the discussions so far, the temporal discretization is ignored. Therefore, the conservation properties are suffered from time marching error. In what follows, spatio-temporal discretization is considered. Here, spatio-temporal staggered grid is used for incompressible flows.

The governing equations for incompressible flows are again the continuity and the Navier-Stokes equations of Eqs.(1) and (2). Fully (spatio-temporally) discretized equations with the implicit midpoint time marching method are as follows.

$$(Cont.^I - FS2) \equiv \frac{\delta_1 u_j}{\delta_1 x_j} = 0, \tag{41}$$

$$\frac{\delta_1 u_i}{\delta_1 t} + \frac{\delta_1\, \overline{\overline{u_j}^{1t}}^{1x_i}\overline{\overline{u_i}^{1t}}^{1x_j}}{\delta_1 x_j} = -\frac{1}{\rho}\frac{\delta_1 p}{\delta_1 x_i} + \nu\frac{\delta_1}{\delta_1 x_j}\frac{\delta_1 \overline{u_i}^{1t}}{\delta_1 x_j} + f_i \tag{42}$$

Note that the discrete operators with stencil width of 1 are used, therefore, the discrete Navier-Stokes equation is second-order accurate in space and time. Since the spatial accuracy can properly be increased by the method explained so far, the second-order accurate spatial discretization is considered hereafter.

In the fully discretized Navier-Stokes equation, the divergence form is used for convection.

$$(Div.^I - FS2)_i \equiv \frac{\delta_1 \overline{\overline{u_j}^{1t}}^{1x_i} \overline{\overline{u_i}^{1t}}^{1x_j}}{\delta_1 x_j} \tag{43}$$

Corresponding advection and skew-symmetric forms are defined as follows.

$$(Adv.^I - FS2)_i \equiv \overline{\overline{\overline{u_j}^{1t}}^{1x_i} \frac{\delta_1 \overline{u_i}^{1t}}{\delta_1 x_j}}^{1x_j}, \tag{44}$$

$$(Skew.^I - FS2)_i \equiv \frac{1}{2}(Div.^I - FS2)_i + \frac{1}{2}(Adv.^I - FS2)_i \tag{45}$$

The commutability between the divergence and advection forms is demonstrated by using the identities of Eqs.(20) and (19).

$$(Div.^I - FS2)_i = (Adv.^I - FS2)_i + \overline{u_i}^{1t} \overline{\overline{(Cont.^I - FS2)}^{1t}}^{1x_i} \tag{46}$$

The secondary conservation property of the skew-symmetric form is demonstrated by using the identities of Eqs.(24) and (21).

$$\overline{u_\alpha}^{1t} \left[\frac{du_\alpha}{dt} + (Skew.^I - FS2)_\alpha \right] = \frac{d\, u_\alpha^2/2}{dt} + \frac{\delta_1 \overline{\overline{u_j}^{1t}}^{1x_\alpha} \widetilde{\overline{u_\alpha}^{1t} \overline{u_\alpha}^{1t}}^{1x_j}/2}{\delta_1 x_j} \tag{47}$$

Therefore, these convection schemes are fully discretized and fully conservative with the aid of the corresponding discrete continuity.

Note that for the fully discretized schemes, the conservation properties are satisfied in the order of round-off error of computer. Therefore, the corresponding time marching method is absolute stable as long as the discrete system is solved. The present implicit method is particularly useful for the turbulence simulations with the streamwise grid spacings are extremely small.

3 Full conservative convection schemes for compressible flows

Now, we move on the topic for compressible flows. The governing equations for compressible flows are the continuity, momentum, and internal energy equations as well as the equation of state.

$$(Cont.) \equiv \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \tag{48}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i, \tag{49}$$

$$\frac{\partial \rho e}{\partial t} + \frac{\partial \rho u_j e}{\partial x_j} = -p\frac{\partial u_i}{\partial x_i} + \tau_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho S_e, \tag{50}$$

$$p = p(\rho, e) \quad \text{or} \quad \rho = \rho(p, e) \quad \text{or} \quad e = e(\rho, p), \tag{51}$$

where e is the internal energy, S_e the heat source, τ_{ij} the viscos stress, and q_j the heat flux. The internal energy equation can be replaced by one of energy equations, for instance, the total energy ($E = e + u_j u_j/2$) and enthalpy ($h = e + p/\rho$) equations.

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_j E}{\partial x_j} + \frac{\partial u_i p}{\partial x_i} = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho u_i f_i + \rho S_e, \tag{52}$$

$$\frac{\partial \rho h}{\partial t} + \frac{\partial \rho u_j h}{\partial x_j} - \frac{Dp}{Dx_i} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho S_e, \tag{53}$$

where $D/Dt \equiv \partial/\partial t + u_j \partial/\partial x_j$ is the material derivative.

3.1 Forms of convection term in compressible flow equation

The momentum and energy equations for compressible flows are written in the general form of transport equation.

$$\frac{\partial \rho \phi}{\partial t} + \frac{\partial \rho u_j \phi}{\partial x_j} = \frac{\partial F_{\phi j}}{\partial x_j} + \rho S_\phi \tag{54}$$

The conservative form above is usually preferred for the compressible flow analysis. The left-hand side corresponds to the divergence form of convection.

On the other hand, the left-hand side of the general transport equation can be rewritten into the non-conservative form.

$$\rho\frac{\partial \phi}{\partial t} + \rho u_j \frac{\partial \phi}{\partial x_j} = \frac{\partial F_{\phi j}}{\partial x_j} + \rho S_\phi \tag{55}$$

The left-hand side corresponds to the advection form of convection.

With regard to the skew-symmetric form, some skew-symmetric like forms, that is, quasi-skew-symmetric forms were used in literatures ([10, 11, 12]). However, they are not secondary conservative. The skew-symmetric form which is secondary conservative a priori was not discovered for compressible flows until 2010.

From the analogy between incompressible and compressible flows, the commutability between the divergence and advection forms with the aid of the continuity is satisfied including time derivative terms for compressible flows.

$$(Div.)_\phi \equiv \frac{\partial \rho \phi}{\partial t} + \frac{\partial \rho u_j \phi}{\partial x_j}, \tag{56}$$

$$(Adv.)_\phi \equiv \rho \frac{\partial \phi}{\partial t} + \rho u_j \frac{\partial \phi}{\partial x_j}, \tag{57}$$

$$(Div.)_\phi = (Adv.)_\phi + \phi (Cont.) \tag{58}$$

Therefore, the skew-symmetric form which is secondary conservative a priori should be the average of the divergence and advection forms.

$$\frac{1}{2}(Div.)_\phi + \frac{1}{2}(Adv.)_\phi = \frac{1}{2}\left(\frac{\partial \rho\phi}{\partial t} + \rho\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\left(\frac{\partial \rho u_j \phi}{\partial x_j} + \rho u_j \frac{\partial \phi}{\partial x_j}\right) \tag{59}$$

However, the equation above includes a couple of time derivative terms, and it seems impractical for time marching. This is probably the reason for the overlook of the proper skew-symmetric form for compressible flows. Fortunately, the author found in 2010 [5] that the couple of time derivative terms can be transformed into a single form by using the Leibniz rule.

$$\frac{1}{2}\left(\frac{\partial \rho\phi}{\partial t} + \rho\frac{\partial \phi}{\partial t}\right) = \frac{\partial \rho\phi}{\partial t} - \frac{\phi}{2}\frac{\partial \rho}{\partial t} = \rho\frac{\partial \phi}{\partial t} + \frac{\phi}{2}\frac{\partial \rho}{\partial t} = \sqrt{\rho}\frac{\partial \sqrt{\rho}\phi}{\partial t} \tag{60}$$

The couple of spatial derivative term can also be transformed into a single form.

$$\frac{1}{2}\left(\frac{\partial \rho u_j \phi}{\partial x_j} + \rho u_j \frac{\partial \phi}{\partial x_j}\right) = \frac{\partial \rho u_j \phi}{\partial x_j} - \frac{\phi}{2}\frac{\partial \rho u_j}{\partial x_j} = \rho u_j \frac{\partial \phi}{\partial x_j} + \frac{\phi}{2}\frac{\partial \rho u_j}{\partial x_j} = \sqrt{\rho u_j}\frac{\partial \sqrt{\rho u_j}\phi}{\partial x_j} \tag{61}$$

By using the 4 variants of time and the 4 variants of spatial derivative terms, there are at least 16 variants of the skew-symmetric form. Among the variants, the author selected the canonical form of the skew-symmetric form for compressible flows as follows.

$$(Skew.)_\phi \equiv \sqrt{\rho}\frac{\partial \sqrt{\rho}\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \rho u_j \phi}{\partial x_j} + \rho u_j \frac{\partial \phi}{\partial x_j}\right) \tag{62}$$

$$= \frac{1}{2}(Div.)_\phi + \frac{1}{2}(Adv.)_\phi \tag{63}$$

This form is easily integrable even with an explicit time marching method. The secondary conservation property of the skew-symmetric forms is demonstrated as follows.

$$\phi(Skew.)_\phi = \frac{\partial\, \rho\phi^2/2}{\partial t} + \frac{\partial\, \rho u_j \phi^2/2}{\partial x_j} \tag{64}$$

3.2 Fully conservative convection schemes for compressible flows

Then, the fully conservative convection schemes for compressible flows are constructed. Here, a spatio-temporal regular grid is used where all variables are located at the same discrete points.

Based on the analytical relations and the discrete operators, the fully discretized and fully conservative convection schemes for compressible flows are constructed. The continuity is discretized as follows.

$$(Cont. - FR2) \equiv \frac{\delta_1 \rho}{\delta_1 t} + \frac{\delta_1 g_j}{\delta_1 x_j} = 0, \tag{65}$$

where $g_j \equiv \overline{\overline{\rho}^{1t}}^{1x_j} \overline{\overline{u_j}^{1t}}^{1x_j}$ is the numerical mass flux. The convection schemes for divergence, advection, and skew-symmetric forms are defined respectively as follows.

$$(Div. - FR2)_\phi \equiv \frac{\delta_1 \rho\phi}{\delta_1 t} + \frac{\delta_1\, g_j\, \overline{\hat{\phi}}^{1x_j}}{\delta_1 x_j} \tag{66}$$

$$(Adv. - FR2)_\phi \equiv \overline{\rho}^{1t} \frac{\delta_1 \phi}{\delta_1 t} + \frac{1}{2}\left(\overline{\phi}^{1t} - \hat{\phi} \right) \frac{\delta_1 \rho}{\delta_1 t} + \overline{g_j \frac{\delta_1 \hat{\phi}}{\delta_1 x_j}}^{1x_j} \tag{67}$$

$$(Skew. - FR2)_i \equiv \overline{\sqrt{\rho}}^{1t} \frac{\delta_1 \sqrt{\rho}\phi}{\delta_1 t} + \frac{1}{2}\left(\frac{\delta_1\, g_j\, \overline{\hat{\phi}}^{1x_j}}{\delta_1 x_j} + \overline{g_j \frac{\delta_1 \hat{\phi}}{\delta_1 x_j}}^{1x_j} \right) \tag{68}$$

where $\hat{\phi} \equiv \frac{\overline{\sqrt{\rho}\phi}^{1t}}{\overline{\sqrt{\rho}}^{1t}}$ is the special temporal interpolation for compressible flow equations (density weighted equations), which is required for the commutability and the conservation properties. The commutability between the divergence and advection forms is demonstrated by using the identities of Eqs.(23) and (20).

$$(Div. - FR2)_\phi = (Adv. - FR2)_\phi + \hat{\phi}\,(Cont. - FR2) \tag{69}$$

The skew-symmetric form is the average of the divergence and advection forms.

$$(Skew. - FR2)_\phi = \frac{1}{2}(Div. - FR2)_\phi + \frac{1}{2}(Adv. - FR2)_\phi \tag{70}$$

The divergence form is primary conservative a priori. The secondary conservation property of the skew-symmetric form is demonstrated by using the identities of Eqs.(24) and (21).

$$\hat{\phi}(Skew. - FR2)_\phi = \frac{\delta_1 \rho \phi^2/2}{\delta_1 t} + \frac{\delta_1\, g_j\, \widetilde{\hat{\phi}\,\hat{\phi}}^{1x_j}/2}{\delta_1 x_j} \tag{71}$$

Therefore, these forms are fully discretized and fully conservative convection schemes for compressible flows.

The spatial order of accuracy for compressible flow schemes can be increased by the same way as explained in section 2.3.

4 Full conservative convection schemes for flows on moving grid

Finally, the fully conservative convection schemes for flows on moving grid are introduced. Here, an ALE (Arbitrary Lagrangian and Eulerian) type moving grid is used. In the ALE simulations a numerical grid in a physical space (t, x_1, x_2, x_3) is mapped into a simple numerical grid in a computational space $(\tau, \xi^1, \xi^2, \xi^3)$.

The continuity and the transport equations (Eqs. (48) and (54)) in a physical space are transformed into those in a computational space.

$$(Cont.) \equiv \frac{1}{J}\frac{\partial J\rho}{\partial \tau} + \frac{1}{J}\frac{\partial J\rho V^j}{\partial \xi_j} = 0, \tag{72}$$

$$\frac{1}{J}\frac{\partial J\rho\phi}{\partial \tau} + \frac{1}{J}\frac{\partial J\rho V^j\phi}{\partial \xi_j} = \frac{1}{J}\frac{\partial J\alpha^j_k F_{\phi k}}{\partial \xi_j} + \rho S_\phi \tag{73}$$

where $V^j \equiv \alpha^j_i u_i + \frac{\partial \xi^j}{\partial t}$ and $\alpha^j_i \equiv \frac{\partial \xi^j}{\partial x_i}$. $J \equiv \varepsilon_{ijk}\frac{\partial x_i}{\partial \xi^1}\frac{\partial x_j}{\partial \xi^2}\frac{\partial x_k}{\partial \xi^3}$ is the Jacobian of the transformation.

Note that the transformed equations are Jacobian and density weighted, while the original equations are density weighted. Therefore, the analogy between different weights should be available for the scheme construction.

4.1 Forms of convection term in flow equations on moving grid

From the analogy, the forms of the convection term in flow equations on moving grid are defined as follows.

$$(Div.)_\phi \equiv \frac{1}{J}\frac{\partial J\rho\phi}{\partial \tau} + \frac{1}{J}\frac{\partial J\rho V^j\phi}{\partial \xi_j}, \tag{74}$$

$$(Adv.)_\phi \equiv \frac{1}{J}J\rho\frac{\partial \phi}{\partial \tau} + \frac{1}{J}J\rho V^j\frac{\partial \phi}{\partial \xi_j}, \tag{75}$$

$$(Skew.)_\phi \equiv \frac{1}{J}\sqrt{J\rho}\frac{\partial \sqrt{J\rho}\,\phi}{\partial \tau} + \frac{1}{J}\frac{1}{2}\left(\frac{\partial J\rho V^j\phi}{\partial \xi_j} + J\rho V^j\frac{\partial \phi}{\partial \xi_j}\right) \tag{76}$$

The commutability and the secondary conservation property are satisfied in the same manner as those for the original compressible flow equations.

$$(Div.)_\phi = (Adv.)_\phi + \phi(Cont.), \tag{77}$$

$$(Skew.)_\phi = \frac{1}{2}(Div.)_\phi + \frac{1}{2}(Adv.)_\phi, \tag{78}$$

$$\phi(Skew.)_\phi = \frac{1}{J}\frac{\partial\, J\rho\phi^2/2}{\partial \tau} + \frac{1}{J}\frac{\partial\, J\rho V^j\phi^2/2}{\partial \xi^j} \tag{79}$$

4.2 Fully conservative convection schemes for flows on moving grid

In the present ALE type simulations, a spatio-temporal regular grid is used in a computational space. As we have already known the fully discretized and fully conservative convection schemes for compressible flows, we can easily extend them for flows on moving grid through the analogy.

The continuity is discretized as follows.

$$(Cont. - FMR2) \equiv \frac{1}{\overline{J}^{1\tau}} \frac{\delta_1 J\rho}{\delta_1 \tau} + \frac{1}{\overline{J}^{1\tau}} \frac{\delta_1 (Jg)^j}{\delta_1 \xi^j} = 0, \tag{80}$$

where $(Jg)^j \equiv \overline{\overline{\rho}^{1\tau}}^{1\xi^j} \left[\overline{\left(J\alpha_i^j\right)}^{1\tau} \overline{\overline{u_i}^{1\tau}}^{1\xi^j} + J\frac{\partial \xi^j}{\partial t} \right]$ is the numerical mass flux in the computational space. The convection schemes for divergence, advection, and skew-symmetric forms are defined respectively as follows.

$$(Div. - FMR2)_\phi \equiv \frac{1}{\overline{J}^{1\tau}} \frac{\delta_1 J\rho\phi}{\delta_1 \tau} + \frac{1}{\overline{J}^{1\tau}} \frac{\delta_1 \, (Jg)^j \, \overline{\hat{\phi}}^{1\xi^j}}{\delta_1 \xi^j} \tag{81}$$

$$(Adv. - FMR2)_\phi \equiv \frac{1}{\overline{J}^{1\tau}} \overline{J\rho}^{1\tau} \frac{\delta_1 \phi}{\delta_1 \tau} + \frac{1}{\overline{J}^{1\tau}} \frac{1}{2} \left(\overline{\phi}^{1\tau} - \hat{\phi} \right) \frac{\delta_1 J\rho}{\delta_1 \tau} + \frac{1}{\overline{J}^{1\tau}} \overline{(Jg)^j \frac{\delta_1 \hat{\phi}}{\delta_1 \xi^j}}^{1\xi^j} \tag{82}$$

$$(Skew. - FMR2)_i \equiv \frac{1}{\overline{J}^{1\tau}} \overline{\sqrt{J\rho}}^{1\tau} \frac{\delta_1 \sqrt{J\rho}\, \phi}{\delta_1 \tau} + \frac{1}{\overline{J}^{1\tau}} \frac{1}{2} \left(\frac{\delta_1 \, (Jg)^j \, \overline{\hat{\phi}}^{1\xi^j}}{\delta_1 \xi^j} + \overline{(Jg)^j \frac{\delta_1 \hat{\phi}}{\delta_1 \xi^j}}^{1\xi^j} \right) \tag{83}$$

where $\hat{\phi} \equiv \frac{\overline{\sqrt{J\rho}\phi}^{1\tau}}{\overline{\sqrt{J\rho}}^{1\tau}}$ is the special temporal interpolation for flows on moving grid (Jacobian and density weighted equations), which is required for the commutability and the conservation properties. The commutability between the divergence and advection forms is demonstrated by using the identities of Eqs.(23) and (20).

$$(Div. - FMR2)_\phi = (Adv. - FMR2)_\phi + \hat{\phi}\, (Cont. - FMR2) \tag{84}$$

The skew-symmetric form is the average of the divergence and advection forms.

$$(Skew. - FMR2)_\phi = \frac{1}{2}(Div. - FMR2)_\phi + \frac{1}{2}(Adv. - FMR2)_\phi \tag{85}$$

The divergence form is primary conservative a priori. The secondary conservation property of the skew-symmetric form is demonstrated by using the identities of Eqs.(24) and (21).

$$\hat{\phi}(Skew. - FMR2)_\phi = \frac{1}{\overline{J}^{1\tau}} \frac{\delta_1 J\rho\phi^2/2}{\delta_1 \tau} + \frac{1}{\overline{J}^{1\tau}} \frac{\delta_1 \, (Jg)^j \, \widetilde{\hat{\phi}\hat{\phi}}^{1\xi^j} /2}{\delta_1 \xi^j} \tag{86}$$

Therefore, these forms are fully discretized and fully conservative convection schemes for flows on moving grid.

5 Summary

Fully conservative convection schemes preserve primary and secondary conservation quantities simultaneously in discrete sense. The simulations with the convection schemes are now available for incompressible flows [3, 4], compressible flows [5], and flows on moving grid [6].

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