On non-conservative compressible magnetohydrodynamics

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1 Introduction

In this short note I would like to present the content of the talk given at the workshop. The talk was based on joint work with Eduard Feireisl from the Institute of Mathematics of the Academy of Sciences of the Czech Republic.

On full moon night you can see about 300 stars at the night sky. On a moonless night it becomes about 2000. However, it is estimated that there is about 200 billion trillion stars in the known Universe. For us the most important one is the Sun. Despite being one of many it is believed to have a great resemblance with a lot of other stars and serves as an important model object on many scales. The Sun is usually presented as divided into zones or layers based on different physical properties, energy transport mechanisms, and temperatures. The thickness of these layers, temperature and density vary greatly in magnitute and hence it seems to be a good subject for mathematical study and modelling.

Furthermore the behavior of the magnetic field of the Sun is quite interesting and not yet fully understood. Despite it being as "weak" as a fridge magnet it spans all the way out of our solar system. Quite interesting feature is that it changes its polarity every about 11 years. To cite Stanford University solar physicist Phil Scherrer on that in 2013 - " We still don't have a really self-consistent mathematical description of what's happening. And until you can model it, you don't really understand it - it's hard to really understand it. "

All of the above and more motivates our interest in modelling the behavior of plasma in the solar covective zone. To do so we describe the solar convective zone as a closed dissipative system, i.e., use a family of field equations accompanied with inhomogeneous boundary conditions describing the behavior of plasma in the solar convective zone.

Let us remind the reader of the following terminology used in this talk. We call a system *closed*, if there is no exchange of matter between the system and its surroundings. We call a system *isolated*, if it is closed and moreover there is no exchange of energy between the system and its surroundings. Finaly we call a system *open* if it exhanges both matter and energy with its surroundings. The above interactions can be imposed through boundary conditions. When talking about dissipative systems we imagine a thermodynamically open system that is often in the out of equilibrium regime.

2 System under consideration

Now let us state the variables that describe the state of a viscous, compressible, electrically and heat conducting fluid. Namely, mass density $\varrho = \varrho(t,x)$, the (absolute) temperature $\vartheta = \vartheta(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the magnetic field $\mathbf{B} = \mathbf{B}(t,x)$. The time evolution of the fluid is governed by the system of field equations of *compressible magnetohydrodynamics (MHD)*, see e.g. Weiss and Proctor [9]:

Equation of continuity:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0. \tag{2.1}$$

Momentum equation:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + (\boldsymbol{\omega} \times \varrho \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \mathbf{curl}_x \mathbf{B} \times \mathbf{B} + \varrho \nabla_x M. \tag{2.2}$$

Induction equation:

$$\partial_t \mathbf{B} + \mathbf{curl}_x(\mathbf{B} \times \mathbf{u}) + \mathbf{curl}_x(\zeta(\vartheta)\mathbf{curl}_x \mathbf{B}) = 0, \ \mathrm{div}_x \mathbf{B} = 0. \tag{2.3}$$

Internal energy balance:

$$\partial_t(\varrho e(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta)$$
$$= \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \zeta(\vartheta)|\mathbf{curl}_x \mathbf{B}|^2 - p(\varrho,\vartheta)\mathrm{div}_x \mathbf{u}. \tag{2.4}$$

The system is written in a general *rotating frame* frequently used in astro/geophysics, where $\boldsymbol{\omega} \times \varrho \mathbf{u}$ is the Coriolis force and the potential M,

$$M = G + \frac{1}{2}|\boldsymbol{\omega} \times x|^2,$$

includes the gravitational component $G = G(t,x)$ as well as the centrifugal force. Note that G may depend on the time t in the rotating frame if the source of gravitation is located outside the fluid domain. The term $\mathbf{curl}_x \mathbf{B} \times \mathbf{B}$ in (2.2) represents the action of the magnetic field on the fluid known as Lorentz force. The terms $\mathbf{curl}_x(\mathbf{B} \times \mathbf{u})$ and $\mathbf{curl}_x(\zeta(\vartheta)\mathbf{curl}_x \mathbf{B})$ appearing in (2.3) represent advection (induction) effects of the fluid on the magnetic field and diffusion of the magnetic field, respectively. Furthermore, $\zeta(\theta)$ is the coefficient of magnetic diffusion.

Since the motivation for this model comes from the convective layer of the sun let us note that the magnetic Reynolds number (the ratio of advective and diffusive forces) is estimated to be about 10^6 and hence the expected effect of the diffusion is quite small.

The induction equation (2.3) introduced above has been derived (after taking certain physical considerations into account) from the Maxwell's equations :

$$\mathrm{div}_x \mathbf{E} = \frac{1}{\varepsilon_0} \varrho_{\mathbf{E}},$$

$$\begin{aligned} \mathrm{div}_x \mathbf{B} &= 0, \\ \partial_t \mathbf{B} &= -\mathbf{curl}_x \mathbf{E}, \\ \frac{1}{c^2}\partial_t \mathbf{E} &= \mathbf{curl}_x \mathbf{B} - \mu_0 \mathbf{j}, \end{aligned} \tag{2.5}$$

where $\mathbf{E} = \mathbf{E}(t,x)$ denotes the electric field, $\varrho_{\mathbf{E}} = \varrho_{\mathbf{E}}(t,x)$ the electrostatic charge density and $\mathbf{j} = \mathbf{j}(t,x)$ the electric current density.

Here we would like to give a small warning about how simplification may lead to confusion. If one were to further simplify by assuming that the magnetic diffusivity coefficient ζ is constant, the induction equation (2.3) can be further reduced to the form

$$\partial_t \mathbf{B} - \zeta \Delta_x \mathbf{B} + \mathbf{curl}_x(\mathbf{B} \times \mathbf{u}) = 0, \ \mathrm{div}_x \mathbf{B} = 0. \tag{2.6}$$

As this system has quite a resemblance to the incompressible Navier-Stokes equations it is tempting to equip it with zero Dirichlet boundary conditions

$$\mathbf{B}|_{\partial\Omega} = 0, \tag{2.7}$$

and apply nowadays standard existence theory. Unfortunatelly, system (2.6), (2.7) is overdetermined. Indeed, compared to the incompressible Navier-Stokes the system (2.6) lacks the "pressure" term, that gives the so much needed degree of freedom to equip it with no-slip boundary conditions (2.7). As we shall present later on, admissible boundary conditions for (2.6) can describe either tangent or normal component only, never both parts at the same time.

We consider a *Newtonian fluid*, with the viscous stress tensor

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{2.8}$$

where the viscosity coefficients $\mu > 0$ and $\eta \geq 0$ are continuously differentiable functions of the temperature. Similarly, the heat flux obeys *Fourier's law*,

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta, \tag{2.9}$$

where the heat conductivity coefficient $\kappa > 0$ is a continuously differentiable function of the temperature.

As pointed out by Douglas O. Gough, a rich fluid behaviour is conditioned by a proper choice of boundary condtions and as we shall see the boundary conditions play a crucial role in the long time behaviour of the system. We suppose that the fluid occupies a bounded domain $\Omega \subset R^3$ with a smooth boundary,

$$\begin{aligned} \partial\Omega = \Gamma_D^{\mathbf{u}} \cup \Gamma_N^{\mathbf{u}} &= \Gamma_D^{\vartheta} \cup \Gamma_N^{\vartheta} = \Gamma_D^{\mathbf{B}} \cup \Gamma_N^{\mathbf{B}}, \\ \Gamma_D^{\mathbf{u}}, \Gamma_N^{\mathbf{u}}, \Gamma_D^{\vartheta}, &\Gamma_N^{\vartheta}, \Gamma_D^{\mathbf{B}}, \Gamma_N^{\mathbf{B}} \text{ compact}, \\ \Gamma_D^{\mathbf{u}} \cap \Gamma_N^{\mathbf{u}} = \emptyset, \ \Gamma_D^{\vartheta} &\cap \Gamma_N^{\vartheta} = \emptyset, \ \Gamma_D^{\mathbf{B}} \cap \Gamma_N^{\mathbf{B}} = \emptyset. \end{aligned} \tag{2.10}$$

Accordingly, each Γ_D^*, Γ_N^* ($* = \mathbf{u}, \vartheta, \mathbf{B}$) is either empty or coincides with a finite union of connected components of $\partial\Omega$. We impose the following boundary conditions:

Boundary velocity:

$$\mathbf{u}|_{\Gamma_D^{\mathbf{u}}} = 0, \tag{2.11}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0, \ [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0. \tag{2.12}$$

Boundary temperature/heat flux:

$$\vartheta|_{\Gamma_D^{\vartheta}} = \vartheta_B, \tag{2.13}$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\Gamma_N^{\vartheta}} = 0. \tag{2.14}$$

Boundary magnetic field:

$$\mathbf{B} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{b}_\tau, \tag{2.15}$$

$$\mathbf{B} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_\nu, \ [(\mathbf{B} \times \mathbf{u}) + \zeta \mathbf{curl}_x \mathbf{B}] \times \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = 0. \tag{2.16}$$

Clearly, the boundary conditions prescribed on Γ_D^* are of Dirichlet type, while those on Γ_N^* of Neumann type. The fluxes on Γ_N^* are set to be zero for the sake of simplicity. More general fluxes are shortly discussed in [4]. In accordance with (2.11), (2.12) there is no exchange of mass with the outer world and the system is therefore driven by the imposed boundary temperature and/or magnetic field.

2.1 Levinson dissipativity

The compressible MHD system admits a natural energy

$$E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) = \underbrace{\frac{1}{2}\varrho|\mathbf{u}|^2}_{\text{kinetic energy}} + \underbrace{\varrho e(\varrho, \vartheta)}_{\text{internal energy}} + \underbrace{\frac{1}{2}|\mathbf{B}|^2}_{\text{magnetic energy}} .$$

Our goal in the talk is to show that the system is dissipative in the sense of Levinson in terms of the total energy

$$\mathcal{E} = \int_\Omega E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \ \mathrm{d}x.$$

Specifically, there exists a universal constant $\mathcal{E}_\infty$ such that

$$\limsup_{\tau \to \infty} \int_\Omega E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \ \mathrm{d}x < \mathcal{E}_\infty \tag{2.17}$$

for *any* solution $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ of the compressible MHD system defined on (T, ∞). We point out that $\mathcal{E}_\infty$ depends only on the "data" but it is the same for any global trajectory. In particular, it is independent of the initial energy of the system.

The data are:

- the total mass m_0 of the fluid;
- the gravitational potential G;
- the rotation vector $\boldsymbol{\omega}$;
- the boundary temperature ϑ_B;
- the boundary tangential magnetic field $\mathbf{b}_\tau$, the boundary normal magnetic field b_ν.

We shall use the symbol $\|(\text{data})\|$ to denote the norm of the above data in suitable function spaces specified below.

At this point one might wonder whether the Levinson dissipativity is not too trivial and basically always given. To show otherwise, let us start with few observations and finish with an example of a systems with unbounded energy.

Besides the total mass m_0, there might be other conserved quantities. As shown e.g. by Bauer, Pauly, and Schomburg [3], see also Kozono and Yanagisawa [8], any vector field $\mathbf{b}$ defined on Ω admits a decomposition

$$\mathbf{b} = \nabla_x P + \mathbf{h} + \mathbf{curl}_x \mathbf{A}, \tag{2.18}$$

where

$$\mathbf{h} \in \mathcal{H}(\Omega) = \left\{ \mathbf{h} \in L^2(\Omega; R^3) \;\middle|\; \mathbf{curl}_x \mathbf{h} = \mathrm{div}_x \mathbf{h} = 0, \mathbf{h} \times \mathbf{n}|_{\Gamma^{\mathbf{B}}_D} = 0,\ \mathbf{h} \cdot \mathbf{n}|_{\Gamma^{\mathbf{B}}_N} = 0 \right\}. \tag{2.19}$$

It follows from the induction equation (2.3) and our choice of the boundary conditions (2.15), (2.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbf{B} \cdot \mathbf{h} \ \mathrm{d}x = 0 \text{ for any } \mathbf{h} \in \mathcal{H}(\Omega).$$

For the sake of simplicity, we shall assume

$$\int_\Omega \mathbf{B} \cdot \mathbf{h} \ \mathrm{d}x = 0 \text{ for all } \mathbf{h} \in \mathcal{H}(\Omega). \tag{2.20}$$

For notes on the general case and the space $\mathcal{H}(\Omega)$ see [4].

Next, to avoid development of rapidly rotating fluid in a rotationally symmetric domain, we impose

$$\Gamma^{\mathbf{u}}_D \neq \emptyset. \tag{2.21}$$

We impose a technical condition to control the boundary magnetic oscillations. First, we introduce the class of stationary magnetic boundary data.

Definition 2.1. [Stationary magnetic field]

We say that the boundary data b_ν, $\mathbf{b}_\tau$ are *stationary*, if there exists a continuously differentiable vector field $\mathbf{B}_B$ such that

$$\mathrm{div}_x \mathbf{B}_B = 0,\ \mathbf{curl}_x \mathbf{B}_B = 0 \text{ in } \Omega,\ \mathbf{B}_B \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{b}_\tau,\ \mathbf{B}_B \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_\nu.$$

Stationarity imposes certain restrictions to be satisfied by $\mathbf{b}_\tau$,

$$\mathrm{div}_\tau \mathbf{b}_\tau = 0 \text{ on } \Gamma_D^{\mathbf{B}},\ \text{where } \mathrm{div}_\tau \text{ denotes the tangential divergence,} \tag{2.22}$$

see Alexander and Auchmuty [1]. To establish the existence of a bounded absorbing set, we need an extra hypothesis imposed on the boundary data if the magnetic boundary field *is not* stationary, namely

$$\Gamma_D^{\mathbf{B}} \subset \Gamma_D^{\mathbf{u}}. \tag{2.23}$$

For more details, see [4]

In view of (2.17), one expects that the boundary conditions must allow the outflow of the thermal energy. We suppose the pressure $p = p(\varrho, \vartheta)$ and the internal energy $e = e(\varrho, \vartheta)$ are interrelated through *Gibbs' equation*

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \tag{2.24}$$

where $s = s(\varrho, \vartheta)$ is the entropy. Consequently, the internal energy balance (2.4) may be reformulated in the form of *entropy balance equation*

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\right)$$
$$= \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta} + \zeta(\vartheta)|\mathbf{curl}_x\mathbf{B}|^2\right), \tag{2.25}$$

see e.g. Weiss and Proctor [9]. The quantity on the right–hand side of (2.25) represents the entropy production rate, and, in accordance with the Second law of thermodynamics, it is always non–negative. Consequently, all forms of energy are eventually transformed to heat that must be allowed to leave through $\partial\Omega$. Thus, necessarily, we must assume that

$$\Gamma_D^{\vartheta} \neq \emptyset. \tag{2.26}$$

2.2 Known results

To the best of our knowledge, the present result is the first one addressing the problem of long–time behaviour of the compressible MHD system far from equilibrium. For details about related research please refer to [4] and the references therein.

3 Main hypothesis, weak solutions

Before introducing the concept of weak solution, let us state briefly the main hypotheses concerning the structural properties of the constitutive relations. The details can be found in [4] and the references therein.

3.1 Equation of state

The hypotheses imposed on the form of the equations of state are based on the Second law of thermodynamics enforced through Gibbs' relation (2.24) and the *hypothesis of thermodynamics stability*

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0. \tag{3.1}$$

We suppose the pressure obeys the *thermal equation of state*

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\vartheta), \ p_R(\vartheta) = \frac{a}{3}\vartheta^4, \ a > 0, \tag{3.2}$$

where p_M and p_R are the molecular and radiation pressure, respectively. Using radiation component is not only technically convenient but also relevant to problems in astrophysics, cf. Battaner [2]. The internal energy satisfies the *caloric equation of state*

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta), \ e_R(\varrho,\vartheta) = \frac{a}{\varrho}\vartheta^4. \tag{3.3}$$

The gas pressure components p_M and e_M satisfy the relation characteristic for monoatomic gases:

$$p_M(\varrho,\vartheta) = \frac{2}{3}\varrho e_M(\varrho,\vartheta). \tag{3.4}$$

Finally, we impose two technical but physically grounded hypotheses in the degenerate area

$$\frac{\varrho}{\vartheta^{\frac{3}{2}}} >> 1.$$

The first reflects the effect of the electron pressure in the degenerate area and the second one is the Third law of thermodynamics.

To summarize it follows that

$$\begin{aligned} p(\varrho,\vartheta) &\approx \varrho e(\varrho,\vartheta), \\ \varrho^{\frac{5}{3}} + \vartheta^4 &\lesssim p(\varrho,\vartheta) \lesssim \varrho^{\frac{5}{3}} + \vartheta^4 + 1, \\ 0 \leq \varrho s(\varrho,\vartheta) &\lesssim \vartheta^3 + \varrho\left(1 + [\log\varrho]^+ + [\log\vartheta]^+\right). \end{aligned} \tag{3.5}$$

Here and hereafter, the symbol $a \lesssim b$ means there is a positive constant c such that $a \leq cb$, the symbol $a \approx b$ is used to denote $a \lesssim b$ and $b \lesssim a$.

More detailed assumptions can be found in [4].

3.2 Diffusion and the transport coefficients

The transport coefficients appearing in (2.8), (2.9) satisfy

$$0 < \underline{\mu}\,(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}\,(1+\vartheta)\,,\ |\mu'(\vartheta)| \leq c \text{ for all } \vartheta \geq 0,$$
$$0 \leq \eta(\vartheta) \leq \overline{\eta}\,(1+\vartheta)\,, \tag{3.6}$$

and

$$0 < \underline{\kappa}\left(1+\vartheta^{\beta}\right) \leq \kappa(\vartheta) \leq \overline{\kappa}\left(1+\vartheta^{\beta}\right) \text{ for some } \beta > 6. \tag{3.7}$$

Finally, we suppose the coefficient of magnetic diffusion $\zeta = \zeta(\vartheta)$ is a continuously differentiable function of the temperature,

$$0 < \underline{\zeta}(1+\vartheta) \leq \zeta(\vartheta) \leq \overline{\zeta}(1+\vartheta),\ |\zeta'(\vartheta)| \leq c \text{ for all } \vartheta \geq 0. \tag{3.8}$$

Note that $\zeta(\theta) = \frac{1}{\mu_0\sigma(\theta)}$, where μ_0 is permeability of vacuum and $\sigma(\theta)$ is the electrical conductivity of the fluid.

3.3 Boundary data

It is convenient to assume the boundary data are given as restrictions of functions defined on Ω and for any $t \in R$. Specifically,

$$\vartheta_B = \tilde{\vartheta}|_{\Gamma_D^{\vartheta}},\ \mathbf{b}_\tau = \mathbf{B}_B \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}},\ b_\nu = \mathbf{B}_B \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}}$$

for suitable extensions $\tilde{\vartheta}$, $\mathbf{B}_B$. Accordingly, certain regularity of the boundary data and compatibility conditions are necessary, see [4] for details.

Finally, we introduce the space

$$H_{0,\sigma} = \left\{ \mathbf{b} \in L^2(\Omega;R^3) \ \Big|\ \mathbf{curl}_x\mathbf{b} \in L^2(\Omega;R^3),\ \mathrm{div}_x\mathbf{b} = 0, \mathbf{b}\times\mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = 0,\ \mathbf{b}\cdot\mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = 0 \right\}. \tag{3.9}$$

The space $H_{0,\sigma}$ is a Hilbert space with the norm $\|\mathbf{b}\|^2_{H_0} = \|\mathbf{curl}_x\mathbf{b}\|^2_{L^2(\Omega;R^3)} + \|\mathbf{b}\|^2_{L^2(\Omega;R^3)}$. The space $\mathcal{H}$ introduced in (2.19) is a closed subspace of $H_{0,\sigma}$, and the following version of Poincaré inequality

$$\|\mathbf{b}\|_{W^{1,2}(\Omega;R^3)} \lesssim \|\mathbf{curl}_x\mathbf{b}\|_{L^2(\Omega;R^3)} \text{ holds for all } \mathbf{b} \in \mathcal{H}^\perp \cap H_{0,\sigma}, \tag{3.10}$$

see Csató, Kneuss and Rajendran [5, Theorem 2.1].

The specific extensions of the boundary data are constructed in Section 3.1 in [4].

3.4 Weak solutions

As we are interested in the long-time behaviour when the system "forgets" its initial state, the choice of initial data plays no role in the analysis. Accordingly, it is convenient to introduce the concept of global in time weak solutions defined for $t \in (T,\infty)$.

Definition 3.1. (Global in time weak solutions)

A quantity $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ is termed *weak solution* of the compressible MHD system (2.1)–(2.4), with the boundary conditions (2.11) – (2.16) in the time–space cylinder $(T, \infty) \times \Omega$ if the following holds:

- **Equation of continuity.** $\varrho \in L^\infty_{\rm loc}(T,\infty; L^{\frac{5}{3}}(\Omega))$, $\varrho \geq 0$, and the integral identity

$$\int_T^\infty \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \ {\rm d}x \ dt = 0 \tag{3.11}$$

holds for any $\varphi \in C^1_c((T,\infty) \times \overline{\Omega})$. In addition, the renormalized version of (3.11)

$$\int_T^\infty \int_\Omega \Big(b(\varrho)\partial_t \varphi + b(\varrho)\mathbf{u} \cdot \nabla_x \varphi + \Big(b(\varrho) - b'(\varrho)\varrho \Big) {\rm div}_x \mathbf{u} \varphi \Big) \ {\rm d}x \, {\rm d}t = 0 \tag{3.12}$$

holds for any $\varphi \in C^1_c((T,\infty) \times \overline{\Omega})$ and any $b \in C^1(R)$, $b' \in C_c(R)$.

- **Momentum equation.**

$\varrho \mathbf{u} \in L^\infty_{\rm loc}(T,\infty; L^{\frac{5}{4}}(\Omega; R^3))$, $\mathbf{u} \in L^2_{\rm loc}(T,\infty; W^{1,2}(\Omega;R^3))$, $\mathbf{u}|_{\Gamma^{\mathbf{u}}_D} = 0$, $\mathbf{u} \cdot \mathbf{n}|_{\Gamma^{\mathbf{u}}_N} = 0$, and the integral identity

$$\begin{aligned}
&\int_T^\infty \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + (\varrho \mathbf{u} \times \boldsymbol{\omega}) \cdot \boldsymbol{\varphi} + p(\varrho,\vartheta) {\rm div}_x \boldsymbol{\varphi} \Big) \ {\rm d}x \\
&\quad = \int_0^\tau \int_\Omega \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \ {\rm d}x \, {\rm d}t - \int_T^\infty \int_\Omega \left(\mathbf{B} \otimes \mathbf{B} - \frac{1}{2}|\mathbf{B}|^2 \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \ {\rm d}x \, {\rm d}t \\
&\quad - \int_T^\infty \int_\Omega \varrho \nabla_x M \cdot \boldsymbol{\varphi} \ {\rm d}x \, {\rm d}t
\end{aligned} \tag{3.13}$$

for any $\boldsymbol{\varphi} \in C^1_c((T,\infty) \times \overline{\Omega}; R^3)$, $\boldsymbol{\varphi}|_{\Gamma^{\mathbf{u}}_D} = 0$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\Gamma^{\mathbf{u}}_N} = 0$.

- **Induction equation.** $\mathbf{B} \in L^\infty_{\rm loc}((T,\infty); L^2(\Omega; R^3))$,

$${\rm div}_x \mathbf{B}(\tau, \cdot) = 0 \text{ for any } \tau \in (T, \infty), \tag{3.14}$$

$$(\mathbf{B} - \mathbf{B}_B) \in L^2_{\rm loc}(T,\infty; H_{0,\sigma}(\Omega; R^3)). \tag{3.15}$$

The integral identity

$$\int_T^\infty \int_\Omega \Big(\mathbf{B} \cdot \partial_t \boldsymbol{\varphi} - (\mathbf{B} \times \mathbf{u}) \cdot \mathbf{curl}_x \boldsymbol{\varphi} - \zeta(\vartheta) \mathbf{curl}_x \, \mathbf{B} \cdot \mathbf{curl}_x \ \boldsymbol{\varphi} \Big) \ {\rm d}x \, {\rm d}t = 0 \tag{3.16}$$

holds for any $\boldsymbol{\varphi} \in C^1_c((T,\infty) \times \overline{\Omega}; R^3)$,

$$\boldsymbol{\varphi} \times \mathbf{n}|_{\Gamma^{\mathbf{B}}_D} = 0, \ \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\Gamma^{\mathbf{B}}_N} = 0. \tag{3.17}$$

- **Entropy inequality.** $\vartheta \in L^\infty_{\rm loc}(T,\infty; L^4(\Omega)) \cap L^2_{\rm loc}(T,\infty; W^{1,2}(\Omega))$, $\vartheta > 0$ a.a. in $(T,\infty)\times\Omega$, $\log(\vartheta) \in L^2_{\rm loc}(T,\infty; W^{1,2}(\Omega))$,

$$\vartheta|_{\Gamma^\vartheta_D} = \vartheta_B.$$

The integral inequality

$$\int_T^\infty \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\varphi \right) \,{\rm d}x\,{\rm d}t$$
$$\leq - \int_T^\infty \int_\Omega \frac{\varphi}{\vartheta} \left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta} + \zeta(\vartheta)|\mathbf{curl}_x\mathbf{B}|^2 \right) \,{\rm d}x\,{\rm d}t \tag{3.18}$$

holds for any $\varphi \in C^1_c((T,\infty)\times\overline{\Omega})$, $\varphi \geq 0$, $\varphi|_{\Gamma^\vartheta_D} = 0$.

- **Ballistic energy inequality.** The inequality

$$\int_T^\infty \partial_t\psi \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + \frac{1}{2}|\mathbf{B}|^2 - \tilde\vartheta\varrho s(\varrho,\vartheta) - \mathbf{B}_B\cdot\mathbf{B} \right) \,{\rm d}x\,{\rm d}t$$
$$- \int_T^\infty \psi \int_\Omega \frac{\tilde\vartheta}{\vartheta} \left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta} + \zeta(\vartheta)|\mathbf{curl}_x\mathbf{B}|^2 \right) \,{\rm d}x\,{\rm d}t$$
$$\geq \int_T^\infty \psi \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\tilde\vartheta + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\tilde\vartheta + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\tilde\vartheta \right) \,{\rm d}x\,{\rm d}t$$
$$+ \int_T^\infty \psi \int_\Omega \Big(\mathbf{B}\cdot\partial_t\mathbf{B}_B - (\mathbf{B}\times\mathbf{u})\cdot\mathbf{curl}_x\mathbf{B}_B - \zeta(\vartheta)\mathbf{curl}_x\,\mathbf{B}\cdot\mathbf{curl}_x\mathbf{B}_B \Big) \,{\rm d}x\,{\rm d}t$$
$$- \int_T^\infty \psi \int_\Omega \varrho\nabla_x M\cdot\mathbf{u} \,{\rm d}x\,{\rm d}t \tag{3.19}$$

holds for any $\psi \in C^1_c(T,\infty)$, $\psi \geq 0$, and any $\tilde\vartheta \in C^1([T,\infty)\times\overline{\Omega})$, $\tilde\vartheta > 0$, $\tilde\vartheta|_{\Gamma^\vartheta_D} = \vartheta_B$.

In our definition we use *ballistic energy inequality* instead of the more common energy inequality. The reason is that given our boundary conditions (2.13) we have no control over the flux on Γ^ϑ_D that would appear in any energy inequality.

Remark 3.2. Note carefully that the ballistic energy inequality (3.19) remains valid if we replace $\mathbf{B}_B$ by any other extension $\widetilde{\mathbf{B}}$ such that

$${\rm div}_x\widetilde{\mathbf{B}} = 0,\ \widetilde{\mathbf{B}}\times\mathbf{n}|_{\Gamma^{\mathbf{B}}_D} = \mathbf{B}_B\times\mathbf{n}|_{\Gamma^{\mathbf{B}}_D},\ \widetilde{\mathbf{B}}\cdot\mathbf{n}|_{\Gamma^{\mathbf{B}}_N} = \mathbf{B}_B\cdot\mathbf{n}|_{\Gamma^{\mathbf{B}}_N},$$

respectively. Indeed the difference $\mathbf{B}_B - \widetilde{\mathbf{B}}$ becomes an eligible test function for the weak formulation of the induction equation (3.16), (3.17).

The existence of global–in–time weak solutions as well as the weak–strong uniqueness property for any finite energy initial data was shown in [6] under more restrictive hypotheses on the boundary data.

4 Main result - Bounded absorbing set

Finally we can state the main result of this talk. More precise statement as well as its proof can be found in [4].

4.1 Existence of bounded absorbing set

It is convenient to suppose that both the potential G as well as the boundary data ϑ_B, $\mathbf{b}_\tau$, and b_ν are defined for all $t \in R$. Accordingly, we introduce

$$\begin{aligned}\|(\text{data})\| = m_0 + m_0^{-1} + \|G\|_{W^{1,\infty}(R\times\Omega)} + |\boldsymbol{\omega}| + \|\vartheta_B^{-1}\|_{L^\infty(R\times\partial\Gamma_D^\vartheta)} \\ + \sup_{t\in R}\|\vartheta_B(t,\cdot)\|_{W^{2,\infty}(\Gamma_D^\vartheta)} + \sup_{t\in R}\|\partial_t\vartheta_B\|_{W^{1,\infty}(\Gamma_D^\vartheta)} \\ + \sup_{t\in R}\|\mathbf{b}_\tau(t,\cdot)\|_{W^{2,\infty}(\Gamma_D^{\mathbf{B}};R^3)} + \sup_{t\in R}\|b_\nu\|_{W^{2,\infty}(\Gamma_N^{\mathbf{B}})} \\ + \sup_{t\in R}\|\partial_t\mathbf{b}_\tau(t,\cdot)\|_{W^{1,\infty}(\Gamma_D^{\mathbf{B}};R^3)} + \sup_{t\in R}\|\partial_t b_\nu\|_{W^{1,\infty}(\Gamma_N^{\mathbf{B}})}.\end{aligned} \tag{4.1}$$

In what follows, we shall use the symbol $c(\|(\text{data})\|)$ to denote a generic positive function of $\|(\text{data})\|$ bounded for bounded arguments.

Theorem 4.1. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, the boundary of which admits the decomposition* (2.10)*. Let the pressure p be related to the internal energy e through the equations of state* (3.2), (3.3), (3.4), *where more detailed assumptions can be found in [4]. Let the transport coefficients μ, η, κ, and ζ be continuously differentiable functions of the temperature satisfying* (3.6)–(3.8)*. Finally, let $G \in BC^1(R \times \overline{\Omega})$ and let the boundary data belong to the class specified in [4], where*

$$\Gamma_D^{\mathbf{u}} \neq \emptyset,\ \Gamma_D^{\vartheta} \neq \emptyset. \tag{4.2}$$

If, in addition, the boundary magnetic field $b_\nu(\tau_n,\cdot)$, $\mathbf{b}_\tau(\tau_n,\cdot)$ is not stationary for some sequence of times $\tau_n \to \infty$, we assume

$$\Gamma_D^{\mathbf{B}} \subset \Gamma_D^{\mathbf{u}}. \tag{4.3}$$

Then there exists a positive constant $\mathcal{E}_\infty(\|(\text{data})\|)$ depending only on the amplitude of the data specified in (4.1) *such that following holds. For any weak solution $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ of the compressible MHD system in $(T,\infty)\times\Omega$ satisfying*

$$\int_\Omega \varrho \ \mathrm{d}x = m_0,\ \int_\Omega \mathbf{B}\cdot\mathbf{h} \ \mathrm{d}x = 0 \text{ for all } \mathbf{h} \in \mathcal{H}(\Omega),$$

there exists a time $\tau > 0$ *such that*

$$\int_\Omega E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \ \mathrm{d}x < \mathcal{E}_\infty \text{ for all } t > T + \tau.$$

Moreover, the length of the time τ *depends only on* $\|(\mathrm{data})\|$ *and on the "initial energy"*

$$\mathcal{E}_T \equiv \limsup_{t \to T-} \int_\Omega E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \ \mathrm{d}x.$$

A similar result in the context of the Rayleigh–Bénard convection problem was shown in [7]. The influence of the magnetic field on the fluid motion, however, requires essential modifications of the proof presented in [7]. In particular, we construct a two component extension of the boundary magnetic field, where the first component is solenoidal, irrotational and satisfies the Neumann boundary condition, while the second one is solenoidal and small in a suitable Lebesgue norm in Ω with bounded rotation, see [4].

As mentioned previously, the result claimed in Theorem 4.1 cannot hold if the fluid system is thermally isolated, meaning

$$\Gamma_D^\vartheta = \emptyset. \tag{4.4}$$

Indeed it follows from the entropy inequality (3.18) that the total entropy

$$\tau \mapsto \int_\Omega \varrho s(\varrho, \vartheta)(\tau, \cdot) \ \mathrm{d}x$$

is a non–decreasing function, which precludes the existence of a bounded absorbing set depending solely on the data specified in (4.1).

What is more, let us show that (4.4) may give rise to trajectories with unbounded energy.

4.2 Closed systems that generate energy

Let us suppose that there is a global–in–time solution $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ and a sequence of times $\tau_n \to \infty$ such that

$$\sup_{n \geq 1} \int_\Omega E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau_n, \cdot) \ \mathrm{d}x < \infty. \tag{4.5}$$

It follows from the constitutive restrictions (3.5) that boundedness of total energy implies boundedness of the total entropy $\int_\Omega \varrho s(\varrho, \vartheta)(\tau_n, \cdot) \ \mathrm{d}x$. However, by virtue of the no–flux boundary conditions induced by (4.4), the total entropy is a non–decreasing function of the time. Consequently, we may infer that

$$\int_T^\infty \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) \ \mathrm{d}x \, \mathrm{d}t < \infty. \tag{4.6}$$

In particular, it follows from (4.6) that

$$\int_{\tau_n}^{\tau_n+1} \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) \ \mathrm{d}x \, \mathrm{d}t \to 0 \text{ as } n \to \infty. \tag{4.7}$$

Consequently, introducing the time shifts

$$\mathbf{u}_n = \mathbf{u}(\cdot + \tau_n),\ \mathbf{B}_n = \mathbf{B}(\cdot + \tau_n),$$

we deduce

$\mathbf{u}_n \to 0$ in $L^2(0,1; W^{1,2}(\Omega; R^3))$, $\mathbf{B}_n \to \widehat{\mathbf{B}}$ in $L^2(0,1; L^2(\Omega; R^3))$ and weakly in $L^2(0,1; W^{1,2}(\Omega; R^3))$.

As the limits satisfy the induction equation (2.3), we conclude

$$\widehat{\mathbf{B}} = \widehat{\mathbf{B}}(x),\ \mathrm{div}_x \widehat{\mathbf{B}} = 0,\ \mathbf{curl}_x \widehat{\mathbf{B}} = 0 \text{ in } \Omega,\ \widehat{\mathbf{B}} \times \mathbf{n}|_{\Gamma^{\mathbf{B}}_D} = \mathbf{b}_\tau,\ \widehat{\mathbf{B}} \cdot \mathbf{n}|_{\Gamma^{\mathbf{B}}_N} = b_\nu. \tag{4.8}$$

Thus the boundary data b_ν, $\mathbf{b}_\tau$ must be stationary in the sense of Definition 2.1. Otherwise, the conclusion contradicts (4.5).

Using the necessary condition for stationarity (2.22) we obtain the following result.

Theorem 4.2. [Systems with unbounded energy]
Suppose

$$\Gamma^{\mathbf{u}}_D \neq \emptyset,\ \Gamma^{\vartheta}_D = \emptyset,$$

and b_ν, $\mathbf{b}_\tau$ *independent of* t,

$$\Gamma^{\mathbf{B}}_D \neq \emptyset,\ \textit{where } \mathbf{b}_\tau \cdot \mathbf{n} = 0,\ \mathrm{div}_\tau \mathbf{b}_\tau \not\equiv 0.$$

Then

$$\int_\Omega E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \ \mathrm{d}x \to \infty \ \textit{as } \tau \to \infty$$

for any global–in–time weak solution to the compressible MHD system.

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