TWO RECIPROCITIES ON HECKE ALGEBRAS PART 1 : ROBINSON'S RECIPROCITY

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Contents

1. Introduction

This paper is concerned with Representation Theory of algebras which are generalizations of group algebras such as Hecke algebras and their relatives. More precisely, we study behaviours of induction and restriction of simple modules and find the multitplicity reciprocity of their associated projctive indecomposable modules as direct summands in those functor images.

First, we recall Robinson Reciprocity on simple and projective modules in finite groups. Let G be a finite group and let H be a subgroup of G. Let $\Bbbk$ be a field. In [Rob89] Robinson proved the following reciprocity

Theorem 1 (G.R. Robinson). *Consider the group algebras of G and H over $\Bbbk$:*

$$A = \Bbbk G \text{ and } B = \Bbbk H. \tag{1}$$

On the (tensor) induction Ind *and the restriction* Res*:*

$$\mathrm{Ind} = A \otimes_B - : B\text{-mod} \to A\text{-mod} \text{ and } \mathrm{Res} : A\text{-mod} \to B\text{-mod}, \tag{2}$$

the following holds: For any simple A-module D and any simple B-module E, the multiplicity of $P(D)$ as a direct summand of $\mathrm{Ind}(E)$ *is equal to the multiplicity of $P(E)$ as a direct summand of* $\mathrm{Res}\, D$*. Here, $P(X)$ stands for the projective cover of X.*

In the case where $\Bbbk G$ is semisimple, the identity in the theorem is the intertwinning number identity from Frobenius Reciprocity. So, Robinson Reciprocity is a generalization in this sense. Robinson Reciprocity is useful to determine the Green correspondents of simple $\Bbbk G$-modules and $\Bbbk H$-modules since it is very important and essential to detect projective summands of the restriction of simple $\Bbbk G$-modules D to $\Bbbk H$ and the induction of simple $\Bbbk H$-modules E to $\Bbbk G$. see Kunugi[Kun00].

Next, roughly we state what we shall do in this paper. We shall make a generalization of this theorem in terms of (a) wider classes of algebras, (b) $\mathbb{Z}$-grading structures and (c) some truncation choices, such as block cutting (comming from ι in this paper). Our main result in this paper can be vaguely summarized as:

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Theorem 2. *In a suitable set up that A, B at (1) and* Ind, Res *at (2) above are replaced by suitable weakly symmetric $\mathbb{Z}$-graded selfinjective algebras and suitable graded functors respectively, a graded and possibly truncated analogue of Robinson Reciprocity holds.*

The precise statement is given at Theorem 5 together with its remark. Even without grading structures, Theorem 5 seems to be new (at least for the author). So, there is an application for finite dimensional Iwahori-Hecke algebras, symmetric cyclotomic Hecke algebras Malle-Mathas[MM98] and some classes of weakly symmetric cellular algebras as well.

Further, recent research developements of cyclotomic quiver Hecke algebras (also known as cyclotomic Khovanov-Lauda-Rouquier algebras) are very remarkable. (See [Bru13].) To check Theorem 5 applicable for those $\mathbb{Z}$-graded algebras, the existence of symmetrizing form is very important. See Webster,[Remark 3.19, Knot invariants and higher representation theory arXiv:1309.3796] , Shan-Varagnolo-Vasserot [SVV17, Prop.3.10] and Mathas-Hu [HM10, Cor.6.18]. So, our theorem in this paper applicable for this class.

2. Robinson's Reciprocity

Let A (resp. B) be a weakly symmetric $\Bbbk$-algebra as well as $\mathbb{Z}$-graded selfinjective so that $q^{d_A}\mathsf{D}A \cong A$ and $q^{d_B}\mathsf{D}B \cong B$ as graded left modules for some integers d_A and d_B where $\mathsf{D} = \mathrm{Hom}_{\Bbbk}(-, \Bbbk)$ and q stands for the shift functor for the graded module categories A-grmod and B-grmod *i.e.*

$$q : A\text{-grmod} \cong A\text{-grmod}$$

satisfying that for $M = \bigoplus_{d\in\mathbb{Z}} M_d \in A$-grmod, we have $(qM)_i = M_{i+1}$ for any $i \in \mathbb{Z}$.

We assume that there exists a (not necessarily unital) graded $\Bbbk$-algebra homomorphism $\iota : B \to A$ and there exist a graded exact induction functor Ind : B-grmod $\to A$-grmod and a graded exact restriction functor Res : A°-grmod $\to B^{\circ}$-grmod satisfying the following conditions (3) and (4):

$$\mathrm{Ind}\, X \cong A \cdot \iota X \text{ in } A\text{-grmod for any simple graded } B\text{-submodule } X \text{ of}_B B. \tag{3}$$

Here, the product in $A \cdot \iota X$ is taken inside A.

$$\mathrm{Res}(Y) \cong \iota(B^{\circ})Y \text{ in } B^{\circ}\text{-grmod for any } Y \text{ in } A^{\circ}\text{-grmod}. \tag{4}$$

Remark 3.
1. *To find graded algebras satisfying the assumptions, for example, one can take the case where B is a graded subalgebra of A, ι is its obvious inclusion, A is a right graded free B-module with finite homogeneous basis,* Ind *is the tensor functor $A \otimes_B -$ and* Res *is the obvious restriction.*
2. *Here, the terminologies "induction" and "restriction" should be treated as mere symbols for indicating an analogy of Robinson's original work.*

Let $\{E^{\gamma} \mid \gamma \in \Gamma\}$ (resp. $\{Q^{\gamma} \mid \gamma \in \Gamma\}$) be a complete set of representatives of isomorphism classes of simple left B-modules (resp. projective indecomposable left B-modules corresponding to E^{γ}'s). Here, we take the same convention on q^iD^{λ} and q^iP^{λ} for q^iE^{γ} and q^iQ^{γ}.

Definition 4 (Graded Multiplicities)**.** *Since we fixed $\{P^{\lambda}\}_{\lambda}$ and A-grmod is Krull-Schmidt, for any finite length graded A-module M the following Laurent polynomial $[M : P^{\lambda}](u)$ in $\mathbb{N}_0[u, u^{-1}]$ makes sense: First, we decompose M into two pieces*

$$M = M^{\lambda} \oplus M'$$

such that any direct summand of M' is not isomorphic to any q^iP^{λ}. So, there exists $[M : P^{\lambda}](u) = \sum_i a_iu^i$ in $\mathbb{N}_0[u, u^{-1}]$ such that M^{λ} is isomorphic to $\bigoplus_i a_i(q^iP^{\lambda})$ as graded modules.

Here, for $m \in \mathbb{N}$ mX stands for m times copies of X. We call $[M : P^\lambda](u)$ the graded multiplicity of P^λ as a direct summand of M.

Now, we can state one of the main results in this paper.

Theorem 5 (Graded and Possibly Truncated Robinson Reciprocity)**.** *For any $\lambda \in \Lambda$, $\gamma \in \Gamma$, we have the following identity:*

$$\mathrm{gdim}_u E^\gamma \cdot \mathrm{gdim}_u D^\lambda \cdot \left[\mathrm{Ind}\, E^\gamma : P^\lambda\right](u) = \mathrm{gdim}_u \mathsf{D}E^\gamma \cdot \mathrm{gdim}_u \mathsf{D}D^\lambda \cdot \left[\mathrm{Res}(\mathsf{D}D^\lambda) : \mathsf{D}Q^\gamma\right](u).$$

Suppose in addition that for any $\lambda \in \Lambda$ and $\gamma \in \Gamma$ with $\iota(\mathrm{Tr}_B(E^\gamma)) \neq \{0\}$ the representatives D^λ and E^γ satisfy

$$\mathrm{gdim}_u \mathsf{D}D^\lambda = \mathrm{gdim}_u D^\lambda \text{ and } \mathrm{gdim}_u \mathsf{D}E^\gamma = \mathrm{gdim}_u E^\gamma. \tag{5}$$

Here, $\mathrm{Tr}_R(M)$ stands for the graded trace of M in R. Then, for any $\lambda \in \Lambda$, $\gamma \in \Gamma$, the graded multiplicity of P^λ as a direct summand of $\mathrm{Ind}(E^\gamma)$ is equal to the graded multiplicity of projective indecomposable module $\mathsf{D}Q^\gamma$ as a direct summand of $\mathrm{Res}\,\mathsf{D}D^\lambda$.

Remark 6.
(1) *One can make degree shifts for $D^\lambda, P^\mu, E^\gamma, Q^\delta$ since* Ind *and* Res *are supposed to be graded functors.*
(2) *There is an important infinite family of algebras satifying (5): In the case where A and B are cyclotomic quiver Hecke algebras, one can find canonical selfdual modules $\mathsf{D}D^\lambda \cong D^\lambda$ and $\mathsf{D}E^\gamma \cong E^\gamma$ as graded modules via anti-automorphisms of A and B for any $\lambda \in \Lambda$ and $\gamma \in \Gamma$. (See* [Bru13, Lemma 3.5]. *)*
(3) *It would be interesting to find Lusztig's canonical/dual canonical or Kashiwara's lower global/upper global basis interpretation of the Reciprocity in Theorem 5.*

Corollary 7. *Let A be a symmetric cyclotomic Hecke algebra of type $G(r, 1, n)$ or an Iwahori-Hecke algebra with invertible parameters. Let B be a parabolic subalgebra of A. For any $\lambda \in \Lambda$, $\gamma \in \Gamma$, the multiplicity of P^λ as a direct summand of $\mathrm{Ind}(E^\gamma)$ is equal to the multiplicity of projective indecomposable module Q^γ as a direct summand of $\mathrm{Res}\, D^\lambda$.*

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