TWO RECIPROCITIES ON HECKE ALGEBRAS PART 2: MACKEY FORMULA FOR KAZHDAN-LUSZTIG CELLS

HYOHE MIYACHI

Contents

1. Introduction

The partition of a Coxeter group into Kazhdan-Lusztig cells is an important research subject. Those cells play an important role in the representation theory of Hecke algebras, Lie algebras, groups of Lie type, Cherednik algebras and Calogero-Moser cells.

There are notions on the induction and restriction of left cells, which were shown to be consistent with the induction and restriction of left cell representations. Those results on the induction and restriction on left cells were first discovered by Barbash and Vogan [BV83] on the connection with the theory of primitive ideals for universal enveloping algebras of Lie algebras, which corresponds to the case of the Iwahori-Hecke algebras of finite Weyl groups with equal parameters.

After a series of Lusztig's works on cells, using Lusztig's asymptotic Hecke algebras Gyoja [Gyo96] found an interpretation of left cell representation as unique lifts of projective indecomposable modules over the Iwahori-Hecke algebra in type B and D with equal and generic parameters in characteristic 2. This was extend to unequal parameter cases in [LM04] in a context of the canonical basis in the level 2 Fock space.

Mackey formulas for modules over Iwahori-Hecke algebras for Coxeter groups were studied by L. Jones [Jon90, 2.29]. Recently, Kuwabara, Wada and the author found Mackey formula in cyclotomic Hecke algebras of type $G(r,1,n)$ [KMW21].)

So, via Gyoja's homological interpretation, in type B and D including unequal parameter cases those projective modules (concludingly left cell modules over an appropriate coefficient ring) satisfy a Mackey system among parabolic subalgebras in terms of induction and restriction functors since those functors restrict to the categories of projective (=injective) modules and Mackey formula exists in the category of modules.

However, left cells are finer than left cell representations in the sense that different left cells might afford isomorphic left cell representations. In this paper, we fill those gaps and extend it to a wider class of Coxeter groups:

Theorem 1. *Mackey formula exists for Kazhdan-Lusztig left cells in Coxeter groups for any postive weight function among standard parabolic subgroups with respect to the induction and restriction which are studied by Barbarsh-Vogan*[BV83]*, Roichman*[Roi98]*, Geck*[Gec03]*, Lusztig*[Lus03, Chapter 9] *and Bonnafé*[Bon17, Chapter 8].

The precise statement is given at Theorem 9.

2. Left cells

2.1. **Kazhdan-Lusztig basis, left convex subsets and their modules.** Let (W, S) be a Coxeter system, *i.e.* S is the set of Coxeter generators for W. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function on W to the non negative integers. Here, we set $\ell(s) = 1$ for any $s \in S$. Let $L : W \to \mathbb{Z}$ be a *weight function*, *i.e.* a function satisfying $L(ww') = L(w) + L(w')$ for any $w, w' \in W$ as long as $\ell(ww') = \ell(w) + \ell(w')$. In this paper, we assume that for any $s, x, y \in S$, [1] $L(s) > 0$ and [2]

$$L(x) = L(y) \text{ whenever } x \text{ and } y \text{ are } W\text{-conjugate .} \tag{1}$$

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ where q is an indeterminate. For $s \in S$, we put $q_s := q^{L(s)} \in \mathcal{A}$. Let $\mathscr{H}_{\mathcal{A}}(W)$ be a generic Iwahori-Hecke alegebra corresponding to W, S with parameters $q_s, s \in S$. Thus, $\mathscr{H}_{\mathcal{A}}(W)$ has an $\mathcal{A}$-basis $\{T_w \mid w \in W\}$, and the multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (q_s - q_s^{-1})T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

And, it is known that T_w is invertible for any $w \in W$. By $\overline{q} := q^{-1}$ and $\overline{q^{-1}} := q$, we have a map $\bar{\cdot} : \mathcal{A} \to \mathcal{A}$. For $\sum_{w \in W} a_wT_w \in \mathscr{H}_{\mathcal{A}}(W)$ with $a_w \in \mathcal{A}$, define

$$\overline{\sum_{w \in W} a_wT_w} := \sum_{w \in W} \overline{a_w}(T_{w^{-1}})^{-1} \in \mathscr{H}_{\mathcal{A}}(W).$$

This $\overline{\cdot}$ is called the bar involution and is a ring automorphism of $\mathscr{H}_{\mathcal{A}}(W)$.

In this set up, we have the following theorem:

Theorem 2 (Kazhdan-Lusztig)**.** *There is a distinguished basis $\{C_w\}_{w \in W}$, called Kazhdan-Lusztig basis characterised by*

$$\overline{C_w} = C_w$$

and

$$C_w = T_w \bmod \sum_{w \in W,\ y < w} \mathbb{Z}[q^{-1}]T_y$$

for any $w \in W$. Here, in the condition of summention, $\leq$ is the Bruhat-Chevalley order.

The Kazhdan-Lusztig preorder $\leq_L$ is the relation on W induced by the following rule: $y \leq_L w$ if there is $s \in S$ such that the coefficient of C_y in the expantion of T_sC_w in terms of Kazhdan-Lusztig basis is non zero. We denote by $\sim_L$ the equivalence relation induced by $\leq_L$. And, the equivalence classes are called the left cells of W, L.

Following [Bon17, p.5 1.1.13], a subset Y of W is called a *left convex* subset if, for any $x, y \in C$ and any $z \in W$ with the property $x \leq_L z \leq_L y$, we have $z \in Y$. A left cell of W is left convex. And, a left convex subset of W is a union of left cells. So, convexity here is a sort of saturated

[1] This assumption reflects the author's current knowledges on restrictions of left cells. In his main sources [Lus03, p.51, 9.11] and [Bon17, p.118], $L > 0$ was assumed.

[2] The assumption (1) is natural since it is derived from the multiplication rule of $\mathscr{H}_{\mathcal{A}}(W)$.

property. Instead of left cells, we use a wider class, left convex subsets which include a single left cell.

One merit to deal with left convex subsets is to construct modules like Kazhdan-Lusztig cell modules. For a left convex subset Y of W, L, put $\mathcal{A}[Y]$ to be the quotient space:

$$\mathcal{A}[Y] := \frac{\displaystyle\bigoplus_{x\in W;\exists y\in Y;x\leq_L y} \mathcal{A}C_x}{\displaystyle\bigoplus_{x\in W-Y;\exists y\in Y;x<_L y} \mathcal{A}C_x}.$$

Then, $\mathcal{A}[Y]$ is a left $\mathscr{H}_{\mathcal{A}}(W)$-module. Moreover, when Y a left cell of W, $\mathcal{A}[Y]$ is the *left cell module* associated with Y in the sense of Kazhdan-Lusztig. Similar to this, we call $\mathcal{A}[Y]$ the *left convex module* ascociated with Y. For the later use, we write the image of C_x in the quotient $\mathcal{A}[Y]$ by $\underline{C}_x$.

2.2. **Parabolic subgroups.** For a subset I of S, we denote by W_I the parabolic subgroup generated by J, called a *standard* parabolic subgroup of W. For subsets I, J of S, we denote by $[W/W_I]$ (resp. $[W_J\backslash W/W_I]$) the minimal length (distinguished) coset representatives of W/W_I (resp. $W_J\backslash W/W_I$):

$$[W/W_I] = \{d \in W \mid d \text{ has the minimal length in } dW_I\},$$
$$[W_J\backslash W/W_I] = \{d \in W \mid d \text{ has the minimal length in } W_J dW_I\}.$$

Similarly, we can define $[W_J\backslash W]$, which is identical to the inverse set of $[W/W_J]$.

The following theorem on induction of left cells is due to the founder [BV83, Proposition 3.15] on the 1-parameter Iwahori-Hecke algebras associated with finite Weyl groups and to [Gec03], enhanced works on general Iwahori-Hecke algebras for Coxeter groups:

Theorem 3 (The induction theorem on left cells)**.** *Let Y be a left convex subset of W_I,$L|_{W_I}$. Then, $[W/W_I]\cdot Y$ (we denote it by $Y\uparrow_I^S$) is a left convex subst of W, L.*

Remark 4. *The convexity preservity was ensured by* [Bon17, 8.2.2]. *For any subset X of W_I, we have a bijection $[W/W_I]\times X \cong [W/W_I]\cdot X$. In particular, for any $a, b \in [W/W_I]$ and any $x, y \in Y$ with $x \neq y$, we have $ax \neq by$.*

Let $\mathrm{p}^I : W \to W_I$ and $\mathrm{r}^I : W \to [W_I\backslash W]$ be the unique maps such that $w = \mathrm{p}^I(w)\mathrm{r}^I(w)$. (see [Bon17, p.118].)

We denote by $\mathbb{Z}_{\geq 0}$ the set of non negative integers. A *multiset* is a pair (A, m_A) that A is a set and $m_A : A \to \mathbb{Z}_{\geq 0}\cup\{\infty\}$ is a map, called the multiplicity function. A as a set is called the underlying set of (A, m_A). Often, we simply write A instead of (A, m_A) if no confusion occurs. We write the join of A and B by $A \uplus B$, whose multiplicity function is $m_A + m_B$.

The following theorem on restrictions of left cells is due to the founder [BV83, Proposition 3.11] on 1-parameter finite Iwahori-Hecke algebras and to [Lus03, Chapter 9], enhanced works on more general Iwahori-Hecke algebras for Coxeter groups:

Theorem 5 (The restriction theorem on left cells)**.** *Let Y be a left convex subset of W,L. Then, $\biguplus_{y\in Y}\{\mathrm{p}^I(y)\}$ (we denote it by $Y\downarrow_I^S$) is a multiset whose underlying set is a left convex subset of W_I, $L|_{W_I}$.*

Remark 6. *Again, the convexity preservity was ensured by* [Bon17, 8.3.3]. *Note that $Y\downarrow_I^S$ is really happen to be a multiset in many cases. For example, if $I=\emptyset$, $Y\downarrow_I^S$ is $(\{1\}, |Y|\delta_{1x})$, namely $|Y|$ times multiply counted identity elements of W. So, there are some overlaps, which induce the multiplicities of identical left cells.*

And, further, we may replace S by J if $I \subset J \subset S$, and define $X \uparrow_I^J$ (resp. $X \downarrow_I^J$). And, the transitivities for those induction and restriction are known.

Example 7. *We assume in this example and only in this example that W is finite and $L(s) = 1$ for all $s \in S$. Put $\mathscr{H}_{\mathbb{C}(q)}(W') := \mathbb{C}(q) \otimes_{\mathcal{A}} \mathscr{H}_{\mathcal{A}}(W')$ for any standard parabolic subgroup W' of W. Note that $\mathscr{H}_{\mathbb{C}(q)}(W')$ is semisimple for all W' by* [GU89]. *Write $\mathbb{C}(q)[Y]$ for the base change of $\mathcal{A}[Y]$, which is a module over $\mathscr{H}_{\mathbb{C}(q)}(W)$. The naming for the restriction of cells is reasonable in the following sense: for a left convex subset Y of W, L, the restricted module $\mathrm{Res}^W_{W_I} \mathbb{C}(q)[Y] := \mathscr{H}_{\mathbb{C}(q)}(W_I) \otimes_{\mathscr{H}_{\mathbb{C}(q)}(W_I)} \mathbb{C}(q)[Y]$ is isomorphic to*

$$\bigoplus_{d \in [W_I \backslash W]} \mathbb{C}(q)[Y_{[d]}] \tag{2}$$

where

$$Y_{[d]} := \left\{ gd^{-1} \mid g \in Y \text{ and } \mathrm{r}^I(g) = d \right\}.$$

The multiplicity function of $Y_{[d]}$ is nothing but the characteristic function. Possibly, $Y_{[d]} = \emptyset$ might occur, in which case, we treat it as $\mathbb{C}(q)[Y_{[d]}] = \{0\}$. And, $Y_{[d]}$ is a disjoint union of left cells and $\mathbb{C}(q)[Y_{[d]}]$ has a decomposition:

$$\mathbb{C}(q)[Y_{[d]}] \cong \bigoplus_{Y' \subset Y_{[d]} : Y' \text{ is a left cell.}} \mathbb{C}(q)[Y'], \tag{3}$$

which means a direct sum of left cell modules. Moreover, for the induction of left cells has also a smilar meaning: For a left cell Y of $W_I, L|_{W_I}$, the direct sum of left cell modules which appear in the induction of left cell Y is isomorphic to the induced module of the left cell module $\mathbb{C}(q)[Y]$. See, [BV83].

Remark 8. *If modules are over $\mathcal{A}$ or W is infinite, it is difficult to obtain direct sum decompositions like (2) and (3) in the restriction of $\mathcal{A}[Y]$ to $\mathscr{H}_{\mathcal{A}}(W_J)$ for a left convex subset Y of W, L.*

Now, we can state our main result as follows:

Theorem 9 (Mackey formula for left cells)**.** *For any left convex subset Y of W_I, $L|_{W_I}$, we have the following multiset identity*:

$$Y \uparrow_I^S \downarrow_J^S = \biguplus_{t \in [W_J \backslash W / W_I]} \left(t(Y \downarrow^I_{I \cap t^{-1}Jt}) t^{-1} \right) \uparrow^J_{J \cap tIt^{-1}} \tag{4}$$

More precisely, for $(t, d) \in [W_J \backslash W / W_I] \times [W_{I \cap t^{-1}Jt} \backslash W_I] (\cong [W_J \backslash W])$, we have an equality on left convex subsets:

$$Y \uparrow_I^S (td)^{-1} \cap W_J = t(Yd^{-1} \cap W_{I \cap t^{-1}Jt}) t^{-1} \uparrow^J_{J \cap tIt^{-1}} . \tag{5}$$

Remark 10. *The statement at (5) says that only multiplications by at $\in [W/W_I]$ for the operation $\uparrow_I^S$ on Y contribute $Y \uparrow_I^S (td)^{-1} \cap W_J$.*

References

[Bon17] C. Bonnafé. *Kazhdan-Lusztig cells with unequal parameters*, volume 24 of *Algebra and Applications*. Springer, Cham, 2017.

[BV83] D. Barbasch and D. Vogan. Primitive ideals and orbital integrals in complex exceptional groups. *J. Algebra*, 80:350–382, 1983.

[Gec03] M. Geck. On the induction of Kazhdan-Lusztig cells. *Bull. London Math. Soc.*, 35(5):608–614, 2003.

[GU89] A. Gyoja and K. Uno. On the semisimplicity of Hecke algebras. *J. Math. Soc. of Japan*, 41(1):75–79, 1989.

[Gyo96] A. Gyoja. Cells and modular representations of Hecke algebras. *Osaka J. Math.*, 33:307–341, 1996.

[Jon90] Lenny K. Jones. Centers of generic Hecke algebras. *Trans. Amer. Math. Soc.*, 317(1):361–392, 1990.

[KMW21] T. Kuwabara, H. Miyachi, and K. Wada. On the Mackey formulas for cyclotomic Hecke algebras and categories $\mathcal{O}$ of rational Cherednik algebras. *Osaka J. Math.*, 58(1):103–134, 2021.

[LM04] B. Leclerc and H. Miyachi. Constructible characters and canonical bases. *J. Algebra*, 277(1):298–317, 2004.

[Lus03] G. Lusztig. *Hecke algebras with unequal parameters*, volume 18 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2003.

[Roi98] Yuval Roichman. Induction and restriction of Kazhdan-Lusztig cells. *Adv. Math.*, 134(2):384–398, 1998.

(H. Miyachi) Graduate School of Mathematics, Osaka Metropolitan University, Sugimoto, Sumiyoshi-ku, Osaka city, Osaka, Japan

Email address: miyachi @ omu.ac.jp