On Coloring of Fraïssé Limits

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We want to generalize some results in Ramsey theory. The results presented in this article are not final ones. Research is currently ongoing to obtain better results. Some of the results were obtained in a joint work with Kota Takeuchi.

Let L be a finite language and K a class of (isomorphism types of) finite L-structures. For a finite L-structure A, we write $A \in K$ if the isomorphism type A belongs to K. In this paper, we assume K has the following property:

- Hereditary property: $A_0 \subset A \in K \Rightarrow A_0 \in K$;
- Free amalgamation property: $A \subset B \in K$, $A \subset C \in K \Rightarrow B \oplus_A C \in K$;

K has the Fraïssé limit M, which is a countable structure. We assume that M has the triviality:

- Triviality: For any $p(x) \in S(A)$ with A finite and for any tuples $\bar{a}, \bar{b} \in p(M)$ with the same length, $\bar{a} \cong_A \bar{b}$ holds.

Our main interest is the case $L = \{R, U_0, \dots, U_{n-1}\}$, where R is a binary predicate symbol for graph edges and U_i's are unary predicate symbols. By the elimination of quantifiers for M, and by the free amalgamation, the triviality above is equivalent to:

- Let $A \in K$ and $a, b \in K$. If $U_i(a) \leftrightarrow U_i(b)$ $(i < n)$, then there is no edge between a and b.

From now on we work on M.

Example 1. $L = \{R, U_0, \dots, U_{n-1}\}$. The following example of K satisfy our requirements:

1. K is the set of all finite n-partite graphs.

2. K is the set of all finite triangle free n-partite graphs.

We work in a K-generic M. $A, B, \ldots$ are used to denote finite subsets o M.

Definition 2. . We say that B and C are free over A (in symbol $B \downarrow_A C$) if $AB \cap AC = A$ and if every edge in ABC is either in AB or in AC. For a finite sets $I = \{B_i\}_{i<n}$, if $B_i \downarrow_A \bigcup_{j \neq i} B_j$ holds for all i, we say I is a free set over A.

Remark 3. For each $i < n$, we assume $A \subset B_i$. Let $I = \{B_i\}_{i<n}$ be a free set over A. Let $p(x) \in S(A)$ and let $p_i(x) \in S(B_i)$ be an extension of $p(x)$ $(i < n)$. Then there is $d \models \bigcup_{i<n} p_i(x)$. Moreover, for any $e \models \bigcup_{i<n} p_i(x)$, $Id \cong Ie$.

Proof. By QE, each p_i is a quantifier free type. Write p_i as $p_i = p_i(x; A, B_i \backslash A)$ to explicitly indicate its parameters. Let $b \models p(x)$, and consider the type $\Gamma(X_0, \ldots, X_{n-1})$:

$$\bigcup_{i<n} p_i(b, A, X_i) \cup \bigcup_{i<j<n} \{ \text{ no edges exist between } X_i \text{ and } X_j\}.$$

This set is consistent by free amalgamation. Let $(C_i)_{i<n}$ realize Γ. Then $AC_0, \ldots, AC_{n-1} \cong B_0, \ldots, B_{n-1}$. So there is an automorphism σ over A sending each AC_i to B_i. Then $d := \sigma(b)$ realizes $\bigcup_{i<n} p_i(x)$. Moreover clause is clear. □

Definition 4. Let $A \subset B$. Let $p_i^*(x) \in S(B)$ $(i < n)$ be non-algebraic types. We say that $\{p_i^* : i < n\}$ is in general position over A if

(*) For all $ADd \in K$, where $D \subset \bigcup_{i<n} p_i^*(x)$ and $d \in \bigcup_{i<n} p_i(M)$, where $p_i = p_i|A$ $(i < n)$, then we can find $d^* \in \bigcup_{i<n} p_i^*(M)$ such that $d \cong_{AD} d^*$.

In the above, if $A = \emptyset$, we simply say that $\{p_i^* : i < n\}$ is in general position. We also say that q^* is p^*-general, if $\{p^*, q^*\}$ is in general position.

Lemma 5. *(Existence of general position) Let $p \neq q \in S(\emptyset)$. Let $p^* \supset p$ and $q^* \supset q$ be two non-algebraic types in $S(A)$. Then p^* and q^* are in general position, if any of the following are true:*

1. *There is an edge between $p^*(M)$ and $q^*(M)$.*

2. There exists no edge connecting $p(M)$ and $q(M)$.

3. $q^*(x)$ is the free extension to the domain A, i.e., $q^*(x) = q(x) \cup \{\neg R(x, a) : a \in A\}$.

Proof. 1. Let $D_0 D_1 d \in K$, where $D_0 \subset p^*(M)$ and $D_1 \subset q^*(M)$ are finite sets. By symmetry, we can assume $d \in q(M)$. We need to find $e \in q^*(M)$ such that $d \cong_{D_0 D_1} e$.

Let $d_1 \in q^*(M) \setminus D_1$. Then we have $D_1 d \cong D_1 d_1$, by the triviality.

Claim A. *For each $b \in D_0$, we can choose $b' \in p^*(M)$ such that $bD_1 d \cong b' D_1 d_1$.*

For simplicity, we assume $R(b, d)$ holds, since the other case is easier. Let $r(x) = \mathrm{qftp}(b/AD_1)$ and $s(x, y) = p^*(x) \cup q^*(y) \cup \{R(x, y)\}$. The set $s(x, y)$ is consistent, since there is an edge connecting $p^*(M)$ and $q^*(M)$. Since $D_1 \mathop{\smile\!\!\!\!\!\!\;|}_A d_1$, by Remark 3, we can find b' realizing $r(x) \cup s(x, d_1)$. This b' satisfies

(i) $b \cong_{AD_1} b'$ (in particular $b' \models p^*$);

(ii) $bD_1 d \cong b' D_1 d_1$. (This also follows from Remark 3.)

Therefore the claim was established. Now let $t_b(x, y) = \mathrm{qftp}(b', d_1/AD_1)$. Since D_0 is a free set over AD_1, by (i), the set

$$\bigcup_{b \in D_0} t_b(b, y)$$

is consistent, and is realized in M, say by e. Clearly, $e \models q^*$. Moreover, by (ii) and by the choice of e, we have $bD_1 d \cong bD_1 e$ for all $b \in D_0$. Hence $D_0 D_1 d \cong D_0 D_1 e$, by the triviality.

2. Trivial.

3. This follows from 1 and 2. □

Proposition 6. *Let $A \subset B$ and let $p_i^*(x) \in S(B)$ $(i < n)$. We assume $\{p_i^* : i < n\}$ is in general position over A. Then we can find a generic substructure $N \supset A$ such that $p_i(N) = p_i^*(M)$ for all $i < n$, where $p_i = p_i^* | A$.*

Proof. Let $F^* \subset M \setminus (\bigcup_{i<n} p_i(M))$ be the maximum set with $F^* \mathop{\smile\!\!\!\!\!\!\;|}_A B$. Notice that $F^* \supset A$. We claim that $N = (\bigcup_{i<n} p_i^*(M)) F^*$ is such a substructure. Let $DFd \in K$, wehre $D \subset \bigcup_{i<n} p_i(M)$, $F \subset F^*$ and $d \in M$. For our purpose, it is sufficient to show the existence of $e \in N$ such that $DFd \cong DFe$.

Case 1. $d \in \bigcup_{i<n} p_i(M)$. By the definition of general position over A, we can find $d' \in p^*(M)$ such that $Dd \cong_A Dd'$. Choose a copy F' of F such that

$$DFd \cong_A DF'd'.$$

By the free amalgamation property, we can assume:

$$F' \underset{ADd'}{\downarrow} B.$$

Clearly, $B \cap (ADd') = B \cap (AD)$. Also, $d' \notin F'$ holds. Hence we know that $F' \downarrow_{AD} B$. From this and $F \downarrow_{AD} B$, we have $F'B \cong_{AD} FB$. Let σ be an elementary mapping over AD such that $\sigma(F'B) = FB$. Then, $DFd \cong_A DF'd' \cong_A DF\sigma(d')$. Since $\sigma|B = id_B$, $e := \sigma(d')$ realizes p^*.

Case 2. $d \notin \bigcup_{i<n} p_i(M)$. By moving d over ADF, we can assume

$$d \underset{ADF}{\downarrow} B.$$

Since $B \cap (ADF) = A$, we have $d \downarrow_A B$. So, $d \in F$. Then $e = d$ has the desired property. □

Corollary 7. *1. Let $p^*(x), q^*(x) \in S(A)$. Suppose that p^* and q^* are in general position (ove $\emptyset$). Then there is a generic substructure N such that $p(N) = p^*(M)$ and $q(N) = q^*(M)$, where $p = p^*|\emptyset$, $q = q^*|\emptyset$.*

2. Let $A \subset B$ be finite subsets of M. Let $p(x) \in S(\emptyset)$ and let $q_i(x)$ $(i < n)$ be types over A extending p. Further, for each $i < n$, let $q_i^*(x) \in S(B)$ be an extension of $q_i(x)$. Then there is a generic substructure $N \subset M$ containing A such that $q_i(N) = q_i^*(M)$ for all $i < n$.

Proof. 1 is clear. 2 follows from the triviality.

□

Proposition 8 (Vertex Coloring)**.** *Let $c : M \to n$ be a vertex coloring. For any $A \subset_{fin} M$, we can find $N \subset M$ such that $N \cong_A M$ and that c is A-locally monochromatic on N, i.e., for any $a, b \in N$, $a \cong_A b$ implies $c(a) = c(b)$.*

Proof. Let $p(x) \in S(A)$. We show that there is a generic substructure $N \subset M$ with $A \subset N$ such that $p(N)$ is monochromatic for the coloring c. If this is shown, then by an iterated application of this argument, we can find N with the full condition. For the same reason, we can also assume that c is 2-coloring, i.e., $n = 2 = \{0, 1\}$.

Case 1: There is a finite $B \supset A$ and a non-algebraic $q \in S(B)$ extending p such that $c = 0$ on $q(M)$.

By the second item of Corollary 7, we can find a generic substructure $N \supset A$ with $p(N) = q(M)$. N has the desired property.

Case 2: There is no $q \supset p$ with $q(M)$ monochromatic. Let $\{a_i\}_{i\in\omega}$ be an enumeration of M. Then, we can inductively define b_i $(i \in \omega)$ such that, for each n,

1. $a_{\le n} \cong_A b_{\le n}$;
2. $c(b_n) = 0$, if $b_n \models p$.

$N := \{b_i\}_{i\in\omega}$ satisfies our requirements. □

From now on, G is a generic graph with triviality.

Corollary 9. *Let $f : [G]^2 \to n$ be a finite coloring for the two element sets. Then, for any finite $A \subset G$, there is a generic subgraph $H \supset A$ such that for any $h \in H$ there is a finite $B \subset H$ with $A \subset B$ such that the coloring $f(h, x)$ is B-locally monochromatic.*

Proof. Let $G = \{g_i\}_{i\in\omega}$ be an enumeration. By induction on i, using the above proposition, we choose elements h_i and generic subgraphs H_i such that

1. $g_0 \dots g_i \cong_A h_0 \dots h_i$;
2. $Ah_0 \dots h_i \subset H_i$;
3. $G \supset H_0 \supset \dots \supset H_i$;
4. $f(h_i, x)$ is $Ah_0 \dots h_i$-locally monochromatic on H_i.

Then, $H = \{h_i\}_{i\in\omega}$ has the desired property. □

Theorem 10 (Main Theorem)**.** *Let $f : R(G) \to n$ be any edge coloring on G. There is a generic subgraph $H \subset G$ such that for each tuple (p, q) of types over $\emptyset$, there is an $n_{p,q} < n$ such that for all $a \in p(H)$ and for almost all $b \in q(H)$, if $R(a, b)$ then $f(\{a, b\}) = n_{p,q}$. (almost all = all but finitely many)*

Proof. We work on fixed types p and q. (If this was done, then we can continue the process until all pairs of types are considered.) Also we can assume $n = 2$.

Case 1: There is a generic subgraph G', a finite set $A \subset G'$, and a non-algebraic extension $p^* \in S(A)$ of p such that for any $a \in p^*(G')$ there is a p^*-general $q^* \in S(A)$ extending q such that for all but finitely many $b \in q^*(G')$, if $R(a, b)$ then $f(a, b) = 0$.

We consider the mapping $a \mapsto q^*$ as a finite coloring on $p^*(G')$. Owing to Proposition 8, by taking a subgraph again, this coloring is A-locally monochromatic. In other words, we can assume q^* is a fixed type (not depending on the choice of a). By the first item of Corollary 7, choose a generic subgraph H such that $p(H) = p^*(G')$ and $q(H) = q^*(G')$. Then, for all $a \in p(H)$ and for almost all $b \in q(H)$, if there is an edge between a and b, then we have $f(a, b) = 0$.

Case 2: For all generic subgraphs G', and for all finite $A \subset G'$ and $p^* \in S(A)$ with $p^* \supset p$, there is $a \in p^*(G')$ for which every p^*-general extension $q^* \in S(A)$ of q has an infinite set $I_{q^*} \subset q^*(G')$ such that $R(a, b)$ and $f(a, b) = 1$ for all $b \in I_{q^*}$.

In this case we claim the following:

Claim A. *For all generic subgraphs G', and for all finite $A \subset G'$ and $p^* \in S(A)$ with $p^* \supset p$, there is a generic subgraph $G'' \subset G'$ containing A and its element $a \in p^*(G'')$ such that $f(a, b) = 1$ for all but finitely many $b \in q(G'')$ with $R(a, b)$.*

If this claim is proved, then we can find a desired subgraph as follows. The argument here is a modification of that in the proof of Corollary 9. Let $G = \{g_i\}_{i \in \omega}$ be an enumeration. Then we can inductively find $h_i \in G$ and a generic subgraph H_i with $H_0 \supset \cdots \supset H_i \ni h_i$ such that

1. $g_0 \dots g_i \cong h_0 \dots h_i$;
2. if $g_i \models p$, then $f(h_i, b) = 1$ for almost all $b \in q(H_i)$.

When finding h_i and H_i, we can use Claim A for $G' = H_{i-1}$, $A = \{h_0, \dots, h_{i-1}\}$, and $p^*(x)$ asserting $g_0 \dots g_i \cong h_0 \dots h_{i-1} x$.

Now we concentrate on showing Claim A. So suppose G', A and p^* are given. By Corollary 9, we can choose a generic subgraph $H \subset G'$, $A \subset H$, satisfying the following:

- For every $h \in H$, there is a finite set B such that the function $f(h, x)$ is B-locally monochromatic.

For this H, applying the case assumption, choose $a \in p^*(H)$ and I_{q^*}'s with the property as described there. By the choice of H, there is a finite $B \subset H$ with $Aa \subset B$ such that $f(a, x)$ is B-locally monochromatic. For every $q^* \in S(A)$ that is p^*-general (and extending q), there is a non-algebraic extension $q^{**} \in S(B)$ for which $I_{q^*} \cap q^{**}(H) \neq \emptyset$. By $f(a, x)$ being B-locally monochromatic, this means that $f(a, b) = 1$ for all $b \in q^{**}(H)$. Take a generic subgraph $G'' \subset H$ containing Aa such that

$$(q^{**}|Aa)(G'') = q^{**}(H)$$

holds for all p^*-general types $q^* \in S(A)$. (Here we used Corollary 7.) Now notice that $(q^{**}|Aa)(x)$ is equivalent to $q^*(x) \cup \{R(a, x)\}$. Notice also that if q^* is not p^*-general then $q^*(x) \cup \{R(a, x)\}$ has no solutions. Thus we know that G'' has the required property.

$\square$

References

[1] K. Takeuchi, and A. Tsuboi, Infinite subgraphs with monochromatic edges. (preprint)

[2] A. Tsuboi, Nonstandard methods for finite structures, MLQ, first published: 18 September 2020 https://doi.org/10.1002/malq.202000024

[3] Maurice Pouzet and Norbert Sauer, Edge partitions of the Rado graph, COMBINATORICA 16 (4) (1996) 505–520.

[4] Graham, Ronald L., Bruce L. Rothschild, and Joel H. Spencer. Ramsey theory. Vol. 20. John Wiley and Sons, 1990.

[5] Kechris, Alexander S., Vladimir G. Pestov, and Stevo Todorcevic. "Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups." Geometric and Functional Analysis 15.1 (2005): 106-189.

[6] Nešetřil, Jaroslav, and Vojtěch Rödl. "The partite construction and Ramsey set systems." Discrete Mathematics 75.1 (1989): 327-334.

[7] Ramsey, FP. "On a Problem of Formal Logic." Proc. Lond. Math. Soc. (2) 30 (1930): 264-286.

[8] Solecki, Sławomir. "Direct Ramsey theorem for structures involving relations and functions." Journal of Combinatorial Theory, Series A 119.2 (2012): 440-449.

[9] Thé, Van, Lionel Nguyen. "A survey on structural Ramsey theory and topological dynamics with the Kechris-Pestov-Todorcevic correspondence in mind." Selected Topics in Combinatorial Analysis, Zb. Rad. (Beogr.), 17. 25 (2015), 189-207.