Forking in locally o-minimal structures

前園久智 (Hisatomo Maesono)
早稲田大学グローバルエデュケーションセンター
(Global Education Center, Waseda University)

概要

abstract Locally o-minimal structures are some local adaptation from o-minimal structures. They were treated, e.g. in [1], [2]. O-minimal structures are characterized by the notion of forking. We try analogous argument in locally o-minimal structures.

1. Introduction

First we recall some definitions.

Definition 1 A linearly ordered structure $M = (M, <, \cdots)$ is $o-minimal$ if every definable subset of M^1 is a finite union of points and intervals.

A linearly ordered structure $M = (M, <, \cdots)$ is $weakly\ o-minimal$ if every definable subset of M^1 is a finite union of convex sets.

Definition 2 Let $M = (M, <, \cdots)$ be a densely linearly ordered structure.

M is $locally\ o-minimal$ if for any $a \in M$ and any definable set $A \subset M^1$, there is an open interval $I \ni a$ such that $I \cap A$ is a finite union of points and intervals.

M is $strongly\ locally\ o-minimal$ if for any $a \in M$, there is an open interval $I \ni a$ such that whenever A is a definable subset of M^1, then $I \cap A$ is a finite union of points and intervals.

(We call the interval I "$SLOM-interval$" of a.)

M is $uniformly\ locally\ o-minimal$ if for any formula $\varphi(x, \bar{y})$ over $\emptyset$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of points and intervals for any $\bar{b} \in M^n$, where $\varphi(M, \bar{b})$ is the realization set of $\varphi(x, \bar{b})$ in M.

Example 3 The following examples are shown in [1] and [2].

($\mathbb{R}, +, <, \mathbb{Z}$) where $\mathbb{Z}$ is the interpretation of a unary predicate, and ($\mathbb{R}, +, <, \sin$) are (strongly) locally o-minimal structures.

Let a language $L = \{<\} \cup \{P_i : i \in \omega\}$ where P_i is a unary predicate. Let $M = (\mathbb{Q}, <^M, P_0^M, P_1^M, \dots)$ be the structure defined by $P_i^M = \{a \in M : a < 2^{-i}\sqrt{2}\}$. Then M is uniformly locally o-minimal, but it is not strongly locally o-minimal.

We recall some fundamental results of locally o-minimal structures.

Theorem 4 [1] *Weakly o-minimal structures are locally o-minimal.*

Theorem 5 [1] *Local o-minimality is preserved under elementary equivalence. But, strongly local o-minimality is not preserved under elementary equivalence.*

Theorem 6 [2] *Let M be strongly locally o-minimal. And let D be a definable set of M and $f : D \longrightarrow M$ a definable function.*

Then for any $a \in D$, there are open intervals $I \subset M$ containing a and $J \subset M$ containing $f(a)$ such that, by putting $f^* = f \cap (I \times J)$, the domain of f^* can be broken up into a finite union of points and open intervals, on each of which f^* is constant, strictly increasing and continuous, or strictly decreasing and continuous.

Theorem 7 [2] *Let M be strongly locally o-minimal. And let $a \in M^n$. Then the following results hold.*
1. Let $X_1, \cdots, X_m$ be definable subsets of M^n. Then there is an open box $B \ni a$ and a finite decomposition $\mathcal{P}$ of B into cells partitioning $X_1 \cap B, \cdots, X_m \cap B$.
2. Let $X \subset M^n$ be a definable set and $f : X \longrightarrow M$ a definable function. Then there is an open box $B \ni (a, f(a))$ such that for the restriction $f^* = f \cap B$, the domain of f^* admits a finite decomposition $\mathcal{P}$ into cells so that for any $Y \in \mathcal{P}$, $f^*|Y$ is continuous.

2. Characterization of strongly locally o-minimal structures by forking

O-minimal structures are usually argued in the monster model, that is, sufficiently large saturated model. But strongly local o-minimality is not elementary property. Thus we set some assumption and argue on it in the following.

Assumption

We consider a complete thoery T of a locally o-minimal structure whose language L is countable. T has an $\aleph_0-$saturated strongly locally o-minimal model.

Under this assumption, all $\aleph_0-$saturated models of T are strongly locally o-minimal. In particular, we argue in the monster model of T.

We recall some definitions.

Definition 8 A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with

$tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$.

A formula $\phi(\bar{x}, \bar{a})$ *forks* over A if $\phi(\bar{x}, \bar{a}) \vdash \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i)$ and each $\psi_i(\bar{x}, \bar{b}_i)$ divides over A.

We try some analogous argument developed in [3]. We can prove the next theorem.

Theorem 9 *Let $\mathcal{M}$ be a sufficiently large saturated strongly locally o-minimal structure and $a \in \mathcal{M}^k$.*

Then there is an open box $B \ni a$ satisfying that ;

For any $M_0 \prec \mathcal{M}$ such that M_0 contains the endpoints c of B, and for $p(x) \in S_k(M_0)$ the type of a over M_0 and $P = p(\mathcal{M})$,

if $\{X(ac) : a \in P\}$ is an M_0-definable family of closed and bounded subsets of B,

then $\{X(ac) : a \in P\}$ has the finite intersection property if and only if there is $d \in M_0$ such that $d \in X(ac)$ for every $a \in P$.

According to the argument in [3], we show some lemmas. First lemma is proved by the monotonicity theorem of strongly locally o-minimal structure.

Lemma 10 *Let $\mathcal{M}$ be a sufficiently large saturated strongly locally o-minimal structure. And let $p(x)$, $q(x) \in S_1(A)$ where A contains some endpoints of SLOM-intervals (of some realizations) of $p(x)$ and $q(x)$.*

Then either

(a) (i) all $A-$definable $f : p(\mathcal{M}) \longrightarrow q(\mathcal{M})$ are increasing, or

(ii) all $A-$definable $f : p(\mathcal{M}) \longrightarrow q(\mathcal{M})$ are decreasing.

(b) In case (i), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a > dcl(B) \cap p(\mathcal{M})$,
then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$,

In case (ii), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a < dcl(B) \cap p(\mathcal{M})$,
then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

In the lemma above, we just say that if there is a function f between $p(\mathcal{M})$ and $q(\mathcal{M})$, then f has these properties. There is no function between a cut (irrational) type and a noncut (rational) type.

By this lemma, they consider characteristic extensions of complete types in [3].

In the following, let $\mathcal{M}$ be a sufficiently large saturated strongly locally o-minimal structure.

Definition 11 Suppose $p(x_1, \cdots, x_n) \in S_n(A)$ where A contains some endpoints of SLOM-intervals (for realizations of p).

For $1 \leq i \leq n$, let $p_i(x_1, \cdots, x_i)$ be the restriction of p to the variables $x_1, \cdots, x_i$.

Fix some sequence $\eta = (\eta(1), \cdots, \eta(n))$ where each $\eta(i)$ is 1 or 0. And let $B \supset A$.

We define an extension $p^\eta_B \in S_n(B)$ of p. Choose a realization $(b_1, \cdots, b_n)$ of p^η_B inductively as follows ;

$b_1 \in p_1(\mathcal{M})$ and if $\eta(1) = 1$, then $b_1 > dcl(B) \cap p_1(\mathcal{M})$, while if $\eta(1) = 0$, then $b_1 < dcl(B) \cap p_1(\mathcal{M})$.

For some realization $b_1, \cdots, b_i$ of $p_i(x_1, \cdots, x_i)$, let b_{i+1} be a realization of $p_{i+1}(b_1, \cdots, b_i, x_{i+1})$ such that :

if $\eta(i+1) = 1$, then $b_{i+1} > dcl(B, b_1, \cdots, b_i) \cap p_{i+1}(b_1, \cdots b_i, \mathcal{M})$ and

if $\eta(i+1) = 0$, then $b_{i+1} < dcl(B, b_1, \cdots, b_i) \cap p_{i+1}(b_1, \cdots b_i, \mathcal{M})$.

Lemma 12 *Let $p(x_1, \cdots, x_n) \in S_n(A)$ and let $q(y) \in S_1(A)$ where A contains some endpoints of SLOM-intervals (for realizations of p and q).*

Then there is $\eta \in {}^n2$ as in the definition above such that ;

for any $B \supset A$ and any realization $\bar{a}$ of p^η_B, $dcl(\bar{a}A) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

Lemma 13 *Let $X(ac)$ be a closed and bounded subset of $\mathcal{M}^{n+1}$ defined over a tuple $ac \in \mathcal{M}^{l+m}$ where c is some endpoints of SLOM-intervals. Assume that $X(ac) \cap dcl(c) = \emptyset$.*

Let $p = \mathrm{tp}(a/c)$ and for $\eta \in {}^l2$, let $a_\eta \models p^\eta_{ac}$. And let $\pi : \mathcal{M}^{n+1} \longrightarrow \mathcal{M}^n$ be the projection on the first n coordinates.

Then the set $\pi\left(X(ac) \cap \bigcap_{\eta \in {}^l2} X(a_\eta c) \right)$ does not contain any element of $dcl(c) \cap \mathcal{M}^n$.

By means of these lemmas, we can prove Theorem 9.

And we can state the previous results of forking in the language of cover.

Corollary 14 *Let $X \subset \mathcal{M}^n$ be a closed and bounded set, definable over a model M_0 where M_0 contains some endpoints of SLOM-intervals.*

Then if $\{\varphi(\mathcal{M}, s) : s \in p(\mathcal{M})\}$ where $p(y) \in S_m(M_0)$ is a definable open cover of X, then it contains a finite subcover of X.

And there is some corollary.

Corollary 15 *Let $\varphi(x, ac)$ be a closed and bounded subset of $\mathcal{M}^1$ defined over $ac \in \mathcal{M}^n$ where c is some endpoints of SLOM-intervals.*

Assume that $\varphi(\mathcal{M}, ac)$ is nonalgebraic.

Then $\varphi(x, ac)$ forks over $\emptyset$.

Sketch of proof :

$\varphi(\mathcal{M}, ac)$ is a finite union of closed intervals, $I_1(ac) < \cdots < I_k(ac)$. For any i with $1 \le i \le k$, $I_i(ac) \cap dcl(c) = \emptyset$. So any $I_i(ac)$ is contained in the realization set of a complete nonalgebraic type $q_i \in S_1(c)$.

If $q_i \neq q_j$, then there is a formula $\psi_{ij}(x, c)$ such that $\psi_{ij}(x, c) \in q_i$ and $\neg\psi_{ij}(x, c) \in q_j$.

Claim. *$\xi(x, ac) := \varphi(x, ac) \wedge \bigwedge_{i \neq j} \neg\psi_{ij}(x, c)$ divides over $\emptyset$.*

For $tp(a/c) := p(x_1, \cdots, x_n)$ and $q_j(y) \in S_1(c)$, we take the appropriate η_j and a realization

a_{η_j} *of* $p^{\eta_j}_{ac}$ *in the lemma above. We write* $a_j := a_{\eta_j}$*. Moreover, by the same way, we take* $\{a_{jk} : k < \omega\}$ *inductively such that* a_{jk} *is a realization of* $p^{\eta_j}_{a_{j1}\cdots a_{jk-1}c}$*. The endpoints of* $I_j(ac)$ *are in* $dcl(ac) \cap q_j(\mathcal{M})$*. So the endpoints of* $I_j(a_{jk}c)$ *are in* $dcl(a_{jk}c) \cap q_j(\mathcal{M})$*.*

By the lemma above, the endpoints of $I_j(a_{jk}c)$ *lie above the endpoints of* $I_j(a_{ji}c)$ *for any* $i < k$*. Thus* $\{\xi(x, a_{jk}c) : k < \omega\}$ *is* $2-$*inconsistent.* ∎

3. Further problems

The argument of forking in o-minimal structures were applied to the characterization of definable groups in o-minimal structures afterward. For example, these characterization form the foundation of the argument of generic sets in definably compact groups, or groups with finitely satisfiable generics definable in o-minimal structures.

I will investigate whether definable groups in locally o-minimal structures are characterized by the argument of forking after this.

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